THE ASYMMETRIC BALOG-SZEMERÉDI-GOWERS THEOREM

BEN GREEN

Abstract. Lecture notes on the asymmetric Balog-Szemerédi-Gowers theorem.

1. Introduction

This note arose from a lecture course on the work of Bateman and Katz [1] on capsets. The asymmetric Balog-Szemerédi-Gowers theorem, which we will state below, is an important ingredient of their work. Midway through giving the course, the breakthrough work of Croot-Lev-Pach [3] and Ellenberg-Gijswijt [4] became available. I immediately abandoned the discussion of the work of Bateman and Katz, which was, in any case, reaching a particularly unpleasant point in the argument. I am making this portion of the notes available since the result is of independent interest. I thank Aled Walker and Freddie Manners for helpful remarks.

If A, B are two finite subsets of an abelian group then we write $E(A, B)$ for the number of solutions to $a_1 + b_1 = a_2 + b_2$ with $a_1, a_2 \in A$ and $b_1, b_2 \in B$.

Theorem 1.1 (Asymmetric BSG). *Let $C > 1 > \eta > 0$ be real constants, and suppose that $N > N_0(C, \eta)$. Suppose that A, B are subsets of an abelian group with $|B| = N$ and $|A| \leqslant N^C$. Suppose that $E(A, B) \geqslant N^{-\eta}|A||B|^2$. Then there are sets H, Λ such that*

1. *We have $|B \cap H| \geqslant N^{-\tilde{\eta}}|B|$;*
2. *We have $|A \cap (H + \Lambda)| \geqslant N^{-\tilde{\eta}}|A|$;*
3. *$|H||\Lambda| \leqslant N^{\tilde{\eta}}|A|$;*
4. *$|H + H| \leqslant N^{\tilde{\eta}}|H|$.*

Here, $\tilde{\eta} \ll \frac{C}{\log(1/\eta)}$, with the implied constant being absolute.

2000 *Mathematics Subject Classification.* Primary .

Thus if $E(A,B)$ is close to the maximum possible size of $|A||B^2|$ then, roughly, B is close to a set H with small doubling and A is close to a union of translates of H.

This theorem appears in [5, Theorem 2.35]. Tao and Vu remark that the method of proof was inspired by arguments of Bourgain [2], but there is certainly no proposition in [2] which could be said to resemble Theorem 1.1. The argument we give here is the same as the one in Tao and Vu, but we have endeavoured to make the presentation self-contained and hence a little easier to follow and to keep track of the exponents.

2. A lemma on symmetry sets

It is convenient to introduce a piece of notation. If S is a subset of an abelian group, write $\mathrm{Sym}_\delta(S)$ for the set of all $d \in S - S$ with at least $\delta|S|$ representations as $s_1 - s_2$.

Lemma 2.1. *Suppose that $S \subset \mathrm{Sym}_\delta(A)$. Then there are at least $\frac{1}{2}\delta^2|S|^2$ pairs $(s_1, s_2) \in S \times S$ with $s_1 - s_2 \in \mathrm{Sym}_{\delta^2/2}(A)$.*

Proof. We have

$$\sum_{a \in A}\sum_{s \in S} 1_A(a+s) \geqslant \delta|A||S|.$$

By Cauchy-Schwarz,

$$\sum_{a \in A}\sum_{s_1,s_2 \in S} 1_A(a+s_1)1_A(a+s_2) \geqslant \delta^2|A||S|^2.$$

Swapping the order of summation, we have

$$\sum_{(s_1,s_2)\in S\times S}\sum_{a \in A} 1_A(a+s_1)1_A(a+s_2) \geqslant \delta^2|A||S|^2,$$

and so

$$\sum_{a \in A} 1_A(a+s_1)1_A(a+s_2) \geqslant \frac{1}{2}\delta^2|A|$$

for at least $\frac{1}{2}\delta^2|S|^2$ pairs $(s_1,s_2) \in S \times S$. But for such a pair (s_1, s_2), there are $\geqslant \frac{1}{2}\delta^2|A|$ pairs $(a_1, a_2) \in A \times A$ such that $a + s_1 = a_1$, $a + s_2 = a_2$, and hence certainly $s_1 - s_2 = a_1 - a_2$. $\square$

3. Proof of asymmetric Balog-Szemerédi-Gowers

Set $J := \lfloor \frac{1}{10}\log(1/\eta)\rfloor$; this will be fixed throughout the proof. Set $\delta_1 := \frac{1}{2}N^{-\eta}$ and define δ_j inductively by $\delta_{j+1} := \frac{1}{2}\delta_j^2$, and note that by the choice of J we have

$$\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_{J+1} \geqslant N^{-\sqrt{\eta}}$$

provided that $N > N_0(\eta)$ is large enough (since δ_j behaves like $N^{-2^j\eta}$). For the first part of the proof, we write $X \gtrapprox Y$ to mean that $X \geqslant N^{-O(\sqrt{\eta})}Y$. Different instances of the notation may entail different constants $O()$ in this notation.

We are going to construct a sequence $B = B_0, B_1, B_2, \ldots, B_J$ of sets having the following two properties, for all $j = 1, 2, \ldots, J$:

(1) Every $x \in B_j$ is a difference of two elements of A in at least $\delta_j|A|$ (and hence $\gtrapprox |A|$) ways;
(2) $B_j \subset B_{j-1} - B_{j-1}$, and moreover every $x \in B_j$ is a difference of two elements of B_{j-1} in $\gtrapprox \frac{|B_{j-1}|^2}{|B_j|}$ ways.

Regarding point (2), note that the lower bound $\frac{|B_{j-1}|^2}{|B_j|}$ is what one would expect if $B_j = B_{j-1} - B_{j-1}$, each difference being represented the same number of times. Thus (2), which we think of as a fairly routine kind of "regularisation", is expressing that fact that B_j is most of $B_{j-1} - B_{j-1}$, and in a uniform way.

The construction will be inductive. Suppose that B_j has been constructed. Then we will first construct a set $B_{j+1}^{\mathrm{prelim}}$ satisfying (1), then refine it to a set B_{j+1} which additionally satisfies the regularisation property (2). The construction of B_1^{prelim} from $B_0 = B$ is special (and is the only place we use the assumption that $E(A, B) \geqslant N^{-\eta}|A||B|^2$), so we handle that first. Writing $r_A(x)$ for the number of ways in which x is a difference of two elements of A, the assumption implies that

$$\sum_{b,b' \in B_0} r_A(b - b') \geqslant N^{-\eta}|B_0|^2|A|.$$

Since $r_A(x) \leqslant |A|$ for all x, it follows immediately that there are at least $\delta_1|B_0|^2$ pairs (b, b') for which $r_A(b - b') \geqslant \delta_1|A|$. Define B_1^{prelim} to be the set of all $b - b'$, over all such pairs (b, b'). Then (1) is satisfied.

For $j \geqslant 1$, the construction of $B_{j+1}^{\mathrm{prelim}}$ from B_j goes via Lemma 2.1. Since B_j satisfies (1), that lemma gives us a set of at least $\delta_{j+1}|B_j|^2 \gtrapprox |B_j|^2$ pairs $(b, b') \in B_j \times B_j$ such that $b - b' \in \mathrm{Sym}_{\delta_{j+1}}(A)$. Write $B_{j+1}^{\mathrm{prelim}}$ for the set of all these differences $b - b'$. For $x \in B_{j+1}^{\mathrm{prelim}}$, write

$r(x) = r_{B_j}(x)$ for the multiplicity with which x occurs as a difference $b - b'$ with $b, b' \in B_j$. Thus

$$\sum_{x \in B_{j+1}^{\text{prelim}}} r(x) \gtrapprox |B_j|^2.$$

Evidently $r(x) \leqslant |B_j|$, so we may split this into dyadic ranges:

$$\sum_{m \geqslant 0} \sum_{x \in B_{j+1}^{\text{prelim}} : r(x) \sim 2^{-m}|B_j|} r(x) \gtrapprox |B_j|^2. \tag{3.1}$$

We claim that the number of relevant dyadic ranges is only logarithmic in N, and in particular $\lessapprox 1$ (provided $N_0(C, \eta)$ is big enough). For this, very crude bounds suffice. Indeed, since $B_j \subset \mathrm{Sym}_{\delta_j}(A)$ we have

$$|B_j| \cdot \delta_j |A| \leqslant |A|^2,$$

whence

$$|B_j| \leqslant \frac{|A|}{\delta_j} \lessapprox N^C. \tag{3.2}$$

Thus the contribution from the mth dyadic range is crudely bounded by $2^{-m}|B_{j+1}^{\text{prelim}}||B_j| \leqslant 2^{-m}|B_j|^3$, which is a small fraction of the right hand side of (3.1) for some $m = O(C \log N)$, which is $\lessapprox 1$ provided N is large enough. It follows that there is an m such that

$$\sum_{x \in B_{j+1}^{\text{prelim}} : r(x) \sim 2^{-m}|B_j|} r(x) \gtrapprox |B_j|^2. \tag{3.3}$$

Let B_{j+1} be the set of such x. Since $B_{j+1} \subset B_{j+1}^{\text{prelim}}$, (1) is satisfied. Furthermore we have

$$|B_{j+1}| \cdot 2^{-m}|B_j| \gtrapprox |B_j|^2,$$

and so for $x \in B_{j+1}$ we have

$$r(x) \sim 2^{-m}|B_j| \gtrapprox \frac{|B_j|^2}{|B_{j+1}|},$$

which is property (2). This completes the inductive construction of the sets $B_1, B_2, B_3, \ldots$ and we now move on to the rest of the proof.

Now comes the key idea: since $|B_j| \lessapprox N^C$ (by (3.2)), it follows from the pigeonhole principle that there is *some* $j \leqslant J$ such that $|B_{j+1}| \lessapprox N^{O(C/J)}|B_j|$. It is now convenient to introduce the notation $X \gtrsim Y$ to mean $X \geqslant N^{-O(C/J)}Y$, where again the $O()$ notation is allowed to change from line to line. Note that if $N_0(C, \eta)$ is large enough then

$X \gtrapprox Y$ implies that $X \gtrsim Y$, so this new notion is even rougher than the old, as the notation might suggest. In particular,

$$|B_{j+1}| \lesssim |B_j|.$$

If $x \in B_{j+1}$, we once again write $r(x)$ for the number of ways to write $x = b - b'$ with $b, b' \in B_j$. Then by (2) above we have $r(x) \gtrapprox |B_j|^2/|B_{j+1}|$, and therefore

$$E(B_j, B_j) \geqslant \sum_x r(x)^2 \gtrapprox \frac{|B_j|^4}{|B_{j+1}|} \gtrsim |B_j|^3.$$

By the usual Balog-Szemerédi-Gowers theorem, there is a set $H_0 \subset B_j$ with

$$|2H_0| \lesssim |H_0|$$

and $|H_0| \gtrsim |B_j|$. Set $H := H_0 - H_0$. Then, by Plünnecke's inequality, $|2H| \lesssim |H|$. Taking x_j to be any element of $-H_0$, so that $H_0 \subset H + x_j$, we have

$$|B_j \cap (H + x_j)| \geqslant N^{-O(C/J)}|B_j|. \tag{3.4}$$

To control B in terms of H, we work backwards down the sequence $B_j, B_{j-1}, \dots, B_0$.

The number of pairs $(b, b') \in B_{j-1} \times B_{j-1}$ with $b - b' \in H + x_j$ is $\sum_{x \in H + x_j} r(x)$. By (2) and (3.4), this is

$$\gtrapprox \frac{|B_{j-1}|^2}{|B_j|} \cdot |B_j \cap (H + x_j)| \gtrapprox N^{-O(C/J)}|B_{j-1}|^2.$$

By pigeonhole, there is some $x_{j-1} := x_j + b'$ such that

$$|B_{j-1} \cap (H + x_{j-1})| \gtrapprox N^{-O(C/J)}|B_{j-1}|.$$

We proceed inductively in this manner, obtaining at the end some x_0 such that

$$|B_0 \cap (H + x_0)| \gtrapprox N^{-O(C/J)}|B_0| \tag{3.5}$$

or, in shorthand (and recalling that $B_0 = B$),

$$|B \cap (H + x_0)| \gtrsim |B|.$$

An important point must be made here: in the course of this inductive process, the implicit constant in the $\gtrapprox$ notation is modified J times. Thus the $N^{-O(\sqrt{\eta})}$ concealed by this notation must be replaced by $N^{-O(J\sqrt{\eta})}$. However we do have $J\sqrt{\eta} = O(C/J)$ and so (3.5) is valid.

We now have items (1) and (4) of Theorem 1.1. It remains to relate A to $H + \Lambda$, and thus establish (2) and (3).

To this end, note that if $h \in H_0$ then, since $H_0 \subset B_j$, h is a difference of two elements of A in at least $\delta_j|A| \gtrsim |A|$ ways. Thus

$$\sum_{h \in H_0} \sum_{a \in A} 1_A(a+h) \gtrsim |A||H_0|.$$

Extending the sum over all $h \in H$ (and using the fact that $|H_0| \gtrsim |H|$), we have

$$\sum_{h \in H} \sum_{a \in A} 1_A(a+h) \gtrsim |A||H|,$$

or in other words

$$\sum_{a \in A} |A \cap (H+a)| \gtrsim |A||H|.$$

Therefore there is a set $A' \subset A$, $|A'| \gtrsim |A|$, with $|A \cap (H+a)| \gtrsim |H|$ for all $a \in A'$. Let $\Lambda^{\mathrm{prelim}}$ be a maximal subset of A' with the property that the translates $H + \lambda$ are pairwise disjoint. Then on the one hand we have

$$|A| \geqslant \sum_{\lambda \in \Lambda^{\mathrm{prelim}}} |A \cap (H+\lambda)| \gtrsim |\Lambda^{\mathrm{prelim}}||H|. \tag{3.6}$$

On the other hand, if $a \in A'$ then, by maximality, there is some λ with $(H+a) \cap (H+\lambda) \neq \emptyset$. This means that $A' \subset H - H + \Lambda^{\mathrm{prelim}}$. Now by the Ruzsa covering lemma, $H - H = 2H_0 - 2H_0$ is covered by $\lesssim 1$ translates of $H = H_0 - H_0$, and so there is some translate Λ of $\Lambda^{\mathrm{prelim}}$ such that

$$|A \cap (H+\Lambda)| \geqslant |A' \cap (H+\Lambda)| \gtrsim |A|. \tag{3.7}$$

Items (2) and (3) of Theorem 1.1 follow from (3.7) and (3.6) respectively. This concludes the proof.

References

[1] M. Bateman and N. .H. Katz, *New bounds on cap sets,* J. Amer. Math. Soc. **25** (2012), no. 2, 585–613.

[2] J. Bourgain, *Estimates on exponential sums related to the Diffie-Hellman distributions,* Geom. Funct. Anal. **15** (2005), no. 1, 1–34.

[3] E. Croot, V. Lev and P. Pach, *Progression-free sets in $\mathbb{Z}_4^n$ are exponentially small*, arxiv.org/abs/1605.01506

[4] J. Ellenberg and D. Gijswijt, *On large subsets of $\mathbb{F}_q^n$ with no 3-term arithmetic progression*, arxiv.org/abs/1605.09223

[5] T. Tao and V. Vu, *Additive combinatorics,* Cambridge studies in advanced mathematics **105**, CUP 2006.

Mathematical Institute, Radcliffe Observatory Quarter, Woodstock Road, Oxford OX2 6GG, England

E-mail address: ben.green@maths.ox.ac.uk