The Baire Category Theorem **BJG May 2009**

THE THEOREM. The Baire Category theorem is a very innocent-looking statement which is, furthermore, not too hard to prove. It can, however, be used to obtain some rather interesting results. It requires one piece of terminology that we did not introduce in the main part of the course: a subset of a metric space X is said to be *dense* if it intersects every open set $U \subseteq X$.

Theorem 1 (Baire Category)**.** *Suppose that X is a complete metric space and that $A_1, A_2, A_3, \ldots$ is a sequence of dense open sets. Then $\bigcap_{n=1}^{\infty} A_n$ is nonempty.*

Proof. In this proof we will frequently use the fact that an open ball $B_\varepsilon(x)$ contains the *closure* of another open ball, for example $\overline{B_{\varepsilon/2}(x)}$. Pick some open ball whose closure $\overline{B_{\varepsilon_1}(x_1)}$ is contained in A_1. Since A_2 is open and dense, $B_{\varepsilon_1}(x_1) \cap A_2$ contains the closure $\overline{B_{\varepsilon_2}(x_2)}$ of some open ball. Since A_3 is open and dense, $B_{\varepsilon_2}(x_2) \cap A_3$ contains the closure $\overline{B_{\varepsilon_3}(x_3)}$ of some open ball. Carry on in this fashion, obtaining nested open balls $B_{\varepsilon_n}(x_n)$ such that $B_{\varepsilon_n}(x_n) \cap A_{n+1}$ contains $\overline{B_{\varepsilon_{n+1}}(x_{n+1})}$ for $n = 1, 2, \ldots$. The construction may clearly be arranged so that $\varepsilon_n \to 0$, in which case the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy. Since X is complete, there is $x \in X$ such that $x_n \to x$. Now for each n the centres x_j, $j \geqslant n$, all lie in $B_{\varepsilon_n}(x_n)$ by construction, and so x lies in the closure $\overline{B_{\varepsilon_n}(x_n)}$ and hence in A_n. It follows that $\bigcap_{n=1}^{\infty} A_n$ contains the point x and, in particular, is nonempty. □

Remark. A weak version of the axiom of choice has been used in this argument, and this is known to be necessary.

One may take complements to obtain the following equivalent formulation of the result.

Theorem 2 (Baire Category, again)**.** *Suppose that X is a complete metric space and that $F_1, F_2, F_3, \ldots$ are closed sets, none containing any open ball $B_\varepsilon(x)$. Then $\bigcup_{n=1}^{\infty} F_n$ is not the whole of X.*

Proof. Set $A_j = X \setminus F_j$; then A_j is open and dense, and we may apply the previous formulation. □

There is a language associated with this formulation of Baire category. We say that a subset of some metric space X is *meagre* (or a set of *first category* if it cannot be written as a countable union of sets S whose closures $\overline{S}$ do not contain any open ball (that is, as a countable union of sets which are *nowhere dense*). The Baire category theorem then states that a complete metric space X is not meagre.

CONTINUOUS, NOWHERE DIFFERENTIABLE FUNCTIONS. We now give the classic application of the Baire category theorem, which is the following result.

Theorem 3. *There is a continuous function* $f : [0,1] \to \mathbb{R}$ *which is not differentiable at any point in* $(0,1)$.

In fact we shall show, in a certain sense, that "most" continuous functions are nowhere differentiable. Indeed the plan of the proof is to view the continuous functions $C[0,1]$ as a complete metric space X, and then to show that the functions which are differentiable at some point in $(0,1)$ form a meagre subset of it.

We have already stated how $X = C[0,1]$ may be viewed as a metric space: the distance $d(f,g)$ between two continuous functions $f,g : [0,1] \to \mathbb{R}$ is defined to be

$$\|f - g\|_\infty := \sup_{x\in[0,1]} |f(x) - g(x)|.$$

It is easy to check that this distance function does define a metric on X.

Proposition 1. *X is a complete metric space.*

Proof. This is, essentially, a theorem you already know in disguise: the result that a uniform limit of continuous functions is continuous. Here is a sketch proof. Suppose that $(f_n)_{n=1}^\infty$ is a Cauchy sequence. This means that $d(f_n, f_m) \to 0$ as $n, m \to \infty$. In particular the sequence $(f_n(x))_{n=1}^\infty$ is a Cauchy sequence of reals for each fixed $x \in [0,1]$, and therefore it tends to a limit which we call $f(x)$. We must now show (i) that f is continuous, and (ii) that $d(f_n, f) \to 0$ as $n \to \infty$. To establish (i), let $x \in [0,1]$ and suppose that $\varepsilon > 0$. Choose n large enough that $d(f_n, f_m) \leqslant \varepsilon/3$ for all $m \geqslant n$. Since f_n is continuous, there is some $\delta > 0$ such that $|f_n(x) - f_n(y)| \leqslant \varepsilon/3$ whenever $|x - y| \leqslant \delta$. For such a y we have

$$\begin{aligned} |f_m(x) - f_m(y)| &\leqslant |f_m(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f_m(y)| \\ &\leqslant 2d(f_m, f_n) + |f_n(x) - f_n(y)| \\ &\leqslant 2\varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Letting $m \to \infty$, it follows that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $|x - y| \leqslant \delta$, whence f is continuous at x. (ii) is rather easy: if n is large enough then $|f_n(x) - f_m(x)| \leqslant \varepsilon$ for all $x \in [0,1]$ and for all $m \geqslant n$. Now let $m \to \infty$ to obtain that $|f_n(x) - f(x)| \leqslant \varepsilon$ for all $x \in [0,1]$. □

To prove the theorem, we need only establish that the subset of X consisting of functions which are differentiable at some point $x \in (0,1)$ is meagre. In other words, it suffices to exhibit the set of functions differentiable at some point as a countable union of closed sets A_n, none of which contains an open ball in X.

Define A_n to consist of all functions $f \in X$ for which there exists some $x \in (0,1)$ such that "the slope of f near x is bounded by n", that is to say

$$|\frac{f(x) - f(y)}{x - y}| \leqslant n$$

for all y with $0 < |x - y| \leqslant 1/n$. It is clear from the definition of differentiability that all functions differentiable at some point in $(0,1)$ lie in one of these sets A_n, and so it suffices to show that each A_n is closed and does not contain any open ball in X.

Proof that A_n is closed. Take a sequence $(f_j)_{j=1}^{\infty}$ in A_n with $f_j \to f$ (where $f \in X$). The aim, of course, is to show that $f \in A_n$. Now for each j, there is some point x_j such that the slope of f_j near x_j is bounded by n, that is to say

$$|\frac{f_j(x) - f(y)}{x_j - y}| \leqslant n$$

whenever $0 < |x_j - y| \leqslant 1/n$. By the Bolzano-Weierstrass theorem there is a subsequence of the x_j tending to some limit x; to ease notation, let us relabel so that the whole sequence x_j tends to x. Suppose that $0 < |x - y| < 1/n$. Then $|x_j - y| \leqslant 1/n$ for j sufficiently large and so (by continuity and the fact that $f_j \to f$) we have

$$|\frac{f(x) - f(y)}{x - y}| = \lim_{j \to \infty} |\frac{f_j(x_j) - f(y)}{x_j - y}| \leqslant n.$$

Since f is continuous, the same is true when $|x - y| = 1/n$ as well. □

Proof that A_n contains no open balls in X. Why does A_n not contain an open ball? The point is that close to any continuous function f there is a function whose slope is very badly behaved indeed. One might think here of the sequence of functions $f_n(x) = n^{-1} \sin n^2 x$ on $[0,1]$, which tend uniformly to the constant function 0 but whose derivatives blow up as $n \to \infty$.

An arbitrary continuous function f is a slightly tricky thing to work with, but for every $\varepsilon > 0$ there is, within a distance ε of f, a *piecewise linear* function $p : [0,1] \to \mathbb{R}$. By this we mean a function for which there is some finite collection $0 = a_0 < a_1 < \cdots < a_k = 1$ of points such that $p|_{[a_i, a_{i+1}]}$ is linear for each $i = 0, \ldots, k-1$. The existence of such a p is an easy consequence of fact that f, being continuous on the closed interval $[0,1]$, is *uniformly continuous*: for all $\varepsilon > 0$ there is some $\delta > 0$ such that $|f(x) - f(y)| \leqslant \varepsilon$ whenever $|x - y| \leqslant \delta$. Now simply take any sequence (a_i) with $|a_{i+1} - a_i| \leqslant \delta$, and define p to equal f at the points a_i and to be linear in between.

It follows that if A_n contains some ball $B_\varepsilon(f)$ in X then it also contains some ball $B_{\varepsilon'}(p)$ where p is piecewise linear (and $\varepsilon' = \varepsilon/2$, say). Let us suppose, then, that A_n contains some ball $B_\varepsilon(f)$ where now f is piecewise linear. The point about a piecewise linear function, such as f, is that its "slopes are bounded" by some quantity M (the maximum of the slopes of the linear segments of f). Now define a continuous function $g : [0,1] \to \mathbb{R}$ by setting $g(j/M') = \pm 1$, where the plus sign is chosen if j is even and the minus sign if j is odd, and defining g to be linear on each segment $[j/M', (j+1)/M']$. Here, M' is a very large integer which we will shortly specify precisely. The function $f + \frac{1}{2}\varepsilon g$ satisfies $d(f, g) < \varepsilon$ but its slope around any point x is at least $\frac{1}{2}\varepsilon M' - n$. This is, of course, strictly larger than N if M' is chosen large enough, for example if we take $M' := \lceil 2(M+n)/\varepsilon \rceil$. □

Let us conclude with the following classic problem. (*Hint: why have I mentioned this here?*)

Problem 1. Suppose that $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function with the following property: for all $x \in \mathbb{R}^+$, the sequence $f(x), f(2x), f(3x), \dots$ tends to 0. Prove that $\lim_{t\to\infty} f(t) = 0$.