Proof. Write $\pi_i : \mathbb{C}^d \to V_i$ for the projection induced by the direct sum decomposition $\mathbb{C}^d = \bigoplus_{i=1}^r V_i$, thus $x = \sum_{i=1}^r \pi_i(x)$ for all x. Define a map $\phi : \mathbb{C}^d \to \mathbb{C}^d$ by $\phi(x) = \sum_{i=1}^r \pi_i^* \pi_i(x)$. Note that if $x \in V_i$ and $y \in V_j$ then

$$\langle \phi(x), y\rangle = \langle \pi_i^*\pi_i(x), y\rangle = \langle \pi_i(x), \pi_i(y)\rangle = 1_{i=j}\langle x, y\rangle. \tag{2.6}$$

In particular, ϕ is not identically zero. We claim that ϕ is G-equivariant, that is to say

$$\phi(gx) = g\phi(x). \tag{2.7}$$

To prove this, it suffices by bilinearity to show that

$$\langle \phi(gx), y\rangle = \langle g\phi(x), y\rangle \tag{2.8}$$

whenever $x \in V_i, y \in V_j$. Since g permutes the V_i, we have $gx \in V_{\sigma_g(i)}$, $g^{-1}y \in V_{\sigma_{g^{-1}}(j)}$ for some mutually inverse permutations $\sigma_g, \sigma_{g^{-1}}$. Therefore, by (2.6), we have

$$\langle \phi(gx), y\rangle = 1_{\sigma_g(i)=j}\langle gx, y\rangle,$$

whilst

$$\langle g\phi(x), y\rangle = \langle \phi(x), g^{-1}y\rangle = 1_{i=\sigma_{g^{-1}}(j)}\langle x, g^{-1}y\rangle = 1_{\sigma_g(i)=j}\langle gx, y\rangle.$$

This establishes (2.8) and thus ϕ is indeed G-equivariant. By Schur's lemma and the irreducibility of G, we must have $\phi(x) = \lambda x$ for some non-zero scalar $\lambda \in \mathbb{C}$. Therefore if $x \in V_i$ and $y \in V_j$ with $i \neq j$ we have, from (2.6),

$$\lambda\langle x, y\rangle = \langle \phi(x), y\rangle = 1_{i=j}\langle x, y\rangle = 0.$$

It follows that V_i is indeed orthogonal to V_j. $\square$