Dynamical Systems: Lecture 1

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What is dynamics?

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- A system can settle to an equilibrium.
- It can repeat itself in cycles.
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What is dynamics?

Dynamics is the study of systems that evolve in time.

- A system can settle to an equilibrium.
- It can do something very complex.
- It can repeat itself in cycles.
What is dynamics?
History of dynamics

• Subject began mid 1600s when Newton invented **differential equations**
• Newton combined his laws of motion and gravitation to explain Kepler’s laws
• Newton solved the two-body problem
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• Many scientists tried to extend Newton’s methods to solve the three-body problem, but always to a dead end
• Breakthrough came with Poincaré in late 1800s who emphasized qualitative rather than quantitative questions
• Instead of asking the exact positions of planets at all times, Poincaré asked, “Is the solar system stable? Or will planets fly off to infinity?”
• In the 1950s, the invention of computers allowed scientists to find numerical solutions to equations
History of dynamics

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• Solutions to Lorenz’s equations never settled down to equilibrium or a periodic state, instead they continued to oscillate in an irregular manner
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• Solutions were completely unpredictable: changing the starting point changed the outcomes completely. This is what we refer to as **chaos**.
History of dynamics

• In 1963, Lorenz discovered the chaotic motion of a strange attractor.

• Solutions to Lorenz’s equations never settled down to equilibrium or a periodic state, instead they continued to oscillate in an irregular manner.

• Solutions were completely unpredictable: changing the starting point changed the outcomes completely. This is what we refer to as **chaos**.

• But, there was structure in chaos!
History of dynamics

• The 1970s marked the boom of chaos:
  • Feigenbaum discovered universal laws that govern the transition from regular to chaotic behavior
History of dynamics

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  - Feigenbaum discovered universal laws that govern the transition from regular to chaotic behavior
  - Mandelbrot popularized fractals
How do we study such dynamical systems?

• We need some terminology
• There are two types of dynamical systems

- differential equations
- iterated maps (difference equations)
How do we study such dynamical systems?

- We need some terminology
- There are two types of **dynamical systems**

  - differential equations
  - iterated maps (difference equations)

- We will start by analyzing problems using differential equations.
- Next week we will study some examples using iterated maps, which will lead us to chaotic solutions.
Differential Equations: defining the derivative

• Suppose you have a variable $x$ which varies with time $t$ ($x$ could be the position of an object at time $t$)

• The position at time $t$ will be denoted by $x(t)$

• Suppose you know the position at time $t$ and you want to calculate it at another time $t'$. So, you have $x(t)$ and you want $x(t')$

• To do this you need to know the velocity at which the object moved between times $t$ and $t'$

• The velocity will be denoted by:

$$\dot{x} = \frac{dx}{dt}$$

This is called a derivative
Differential Equations: defining the derivative

- A derivative of a quantity (say position) $x$ with respect to time is the variation of $x$ with time, where both $x$ and time are continuous.

- The discrete version is denoted by $\frac{\Delta x}{\Delta t}$.

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Differential Equations: defining the derivative

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- The discrete version is denoted by
  \[
  \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}
  \]

\((t_1, x_1)\) \hspace{1cm} \((t_2, x_2)\)
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In this expression, both time and position are discrete.
Differential Equations: the derivative (graphically)
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position $x$

time $t$

$(t_1, x_1)$

$(t_2, x_2)$
Differential Equations: the derivative (graphically)

\[ \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \]

position \( x \)

(\( t_2, x_2 \))

(\( t_1, x_1 \))

time \( t \)
Differential Equations: the derivative (graphically)

\[ \Delta x / \Delta t = \frac{x_2 - x_1}{t_2 - t_1} \approx \text{slope} \]
Differential Equations: the derivative (graphically)

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Differential Equations: the derivative (graphically)

\[ \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \approx \text{slope} \]

\[ \text{slope} = \frac{dx}{dt} \]
In a dynamical system, we try to model a real life problem mathematically.

A dynamical system will show how some quantity varies with time.

A one-dimensional dynamical system has ONE equation representing the variation of ONE variable with time.

Examples:

\[ \dot{x} = x + 1 \]

\[ \dot{x} = 3x - 2 \]

\[ \dot{x} = -5x^2 + \sin(x) \]
• A two-dimensional dynamical system has TWO equations which are usually coupled, and which represent the variation of TWO variables with time.

Example: \[ \dot{x}_1 = 3x_2 - 2x_1 \]
\[ \dot{x}_2 = x_1 + 4x_2 \]

• We will only consider autonomous systems, i.e. systems that do not explicitly depend on time

• We won’t consider something like \[ \dot{x} = x + t \]
A two-dimensional dynamical system has TWO equations which are usually **coupled**, and which represent the variation of TWO variables with time.

**Example:**

\[
\begin{align*}
\dot{x}_1 &= 3x_2 - 2x_1 \\
\dot{x}_2 &= x_1 + 4x_2
\end{align*}
\]

- We will only consider **autonomous** systems, i.e. systems that do not **explicitly** depend on time.
- We won’t consider something like \( \dot{x} = x + t \) (non-autonomous)
A Geometric Way of Thinking: Flows on the Line

• The idea is NOT to solve the equation
• Instead, we want to think geometrically
• We begin with a one-dimensional dynamical system

\[ \dot{x} = f(x) \]

\( x(t) \) is a real-valued function of time \( t \)

\( f(x) \) is smooth and also real-valued
\[ \dot{x} = f(x) \]

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$x(t)$ is a real-valued function of time $t$

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\( x(t) \) is a real-valued function of time \( t \)

\( f(x) \) is smooth and also real-valued
• Let’s take an example: \( \dot{x} = f(x) = \sin x \)
• The idea is to analyze how \( x(t) \) behaves
• If you know calculus, you can solve this system exactly and you get

\[
t = -\log |\csc x + \cot x| + \text{constant}
\]

• We don’t even have \( x(t) \), we have \( t(x) \) instead, and inverting is impossible!
A Geometric Way of Thinking: Flows on the Line

• Instead of solving exactly, we will plot $f(x)$ against $x$, i.e. $\dot{x}$ against $x$
A Geometric Way of Thinking: Flows on the Line

\[ f(x) = \dot{x} \]
A Geometric Way of Thinking: Flows on the Line

• Instead of solving exactly, we will plot $f(x)$ against $x$, i.e. $\dot{x}$ against $x$

• Then we want to find the **fixed points**, which correspond to points where the system isn’t varying with time, i.e. points where

\[ \dot{x} = 0 \]
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There are two kinds of fixed points: **stable** and **unstable**.
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The **flow** goes **towards** stable points and **away** from unstable points.
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• There are two kinds of fixed points: **stable** and **unstable**

• The **flow** goes **towards** stable points and **away** from unstable points

• This is determined by the sign of $\dot{x}$
A Geometric Way of Thinking: Flows on the Line

\[ f(x) = \dot{x} \]
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Stable vs. Unstable
Romans and their hot baths!

flat ceiling will always drip!

curved ceiling may protect your back!
Simple Example

• Let’s start with a simple example

\[ \dot{x} = x^2 - 1 \]

• First, we must plot this system
Plot: $x^2 - 1$ for $x = -2$ to $2$.
Simple Example

\[ f(x) = \dot{x} \]

\[ \dot{x} = x^2 - 1 \]
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Population Growth

• Suppose you have a species and you’re interested in how it will grow in time or if it could possibly die out.

• The species has population $N(t)$ at time $t$.

• We assume the population grows at a steady rate $r > 0$. 

Population Growth

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- This system can be modeled by:

$$\dot{N} = rN$$
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• The species has population $N(t)$ at time $t$
• We assume the population grows at a steady rate $r > 0$
• This system can be modeled by:

\[
\dot{N} = rN
\]

• In this simple model, the population will grow indefinitely
Population Growth

In the model \( \dot{N} = rN \)

- We did not specify what may cause the population to die
- We did not consider how the presence of a **carrying capacity** may hinder growth
- We also did not consider what happens if a disease breaks out
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To model the effects of overpopulation, demographers usually assume that the per-capita growth-rate \( \dot{N}/N \) decreases as \( N \) becomes sufficiently large
Population Growth

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To model the effects of overcrowding, demographers usually assume that the per-capita growth-rate $\dot{N}/N$ decreases as $N$ becomes sufficiently large.

When $N$ is small, growth rate is $r$; when $N$ is larger than a carrying capacity $K$, growth rate is negative (death rate higher than birthrate).
Population Growth

\[ \frac{\dot{N}}{N} \]

\[ \dot{N}/N \]

\[ r \]

\[ K \]

\[ N \]
Population Growth: the logistic equation

This leads to the logistic model for population growth

\[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]

How to solve this?
1. \( N \geq 0 \): a negative population makes no sense
2. We plot the function to find the fixed points
3. We find the values of these fixed points and analyze what they mean
Here I chose $r=0.3$, $K=5$. 

WolframAlpha.com: 
plot: $0.3 \times (1 - x/5)$, $(x,0.5,5)$
Population Growth: the logistic equation

\[ \dot{N} = rN \left(1 - \frac{N}{K}\right) \]
Population Growth: the logistic equation

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Calculating the fixed points:

\[ \ddot{N} = 0 \]
Population Growth: the logistic equation

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Calculating the fixed points:

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Population Growth: the logistic equation

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Calculating the fixed points:

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\[ rN = 0 \]

\[ \implies N = 0 \]
Population Growth: the logistic equation

\[ \dot{N} = rN \left( 1 - \frac{N}{K} \right) \]

Calculating the fixed points:

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\[ rN = 0 \]

\[ \Rightarrow N = 0 \]

\[ \left( 1 - \frac{N}{K} \right) = 0 \]

\[ \Rightarrow N = K \]