Dynamical Systems: Lecture 1

Naima Hammoud

Feb 21, 2017

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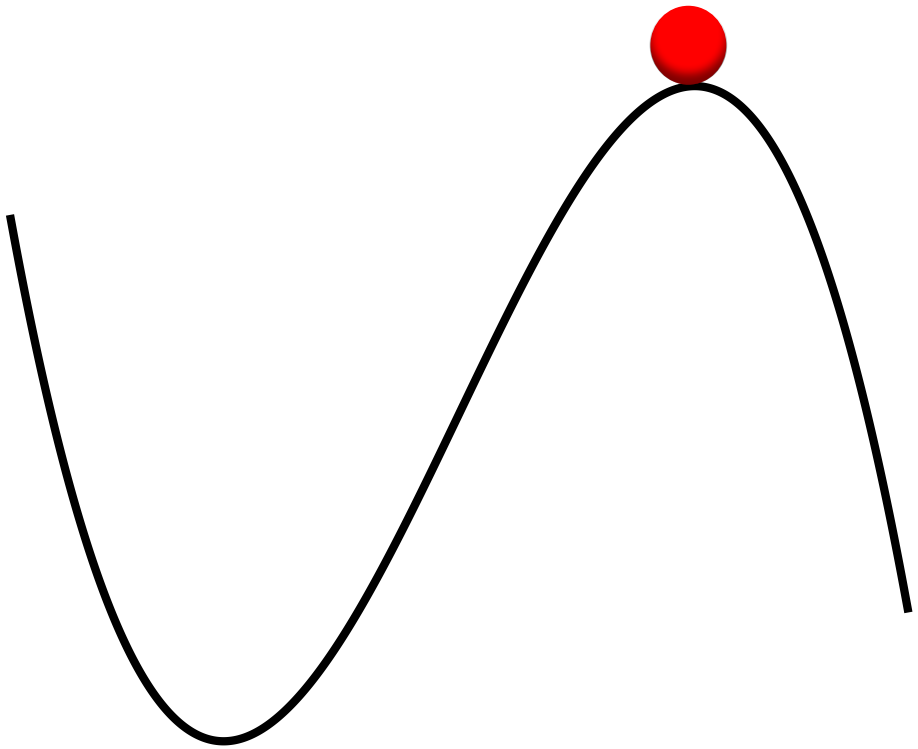

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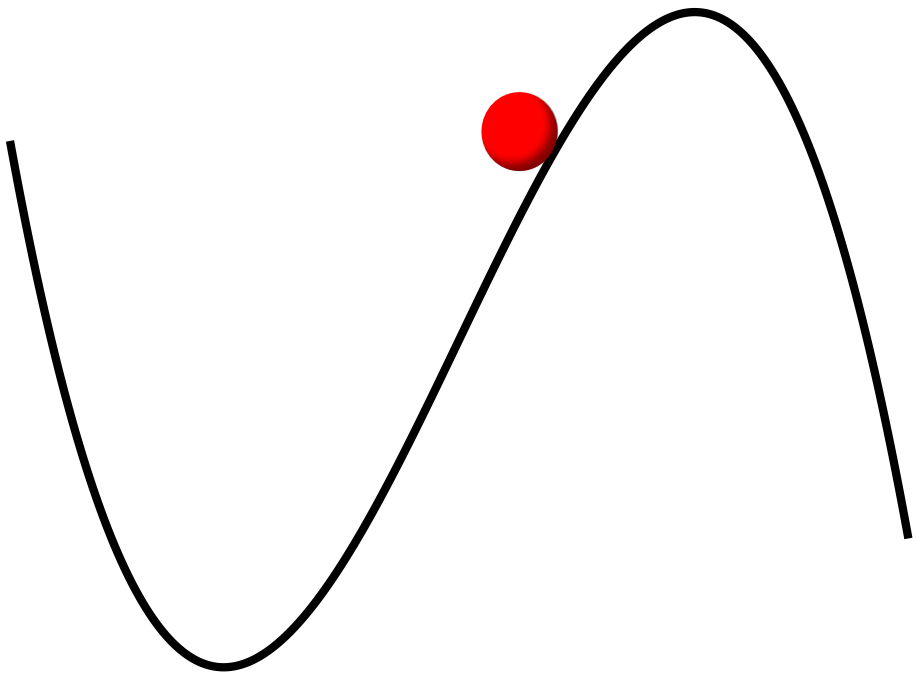

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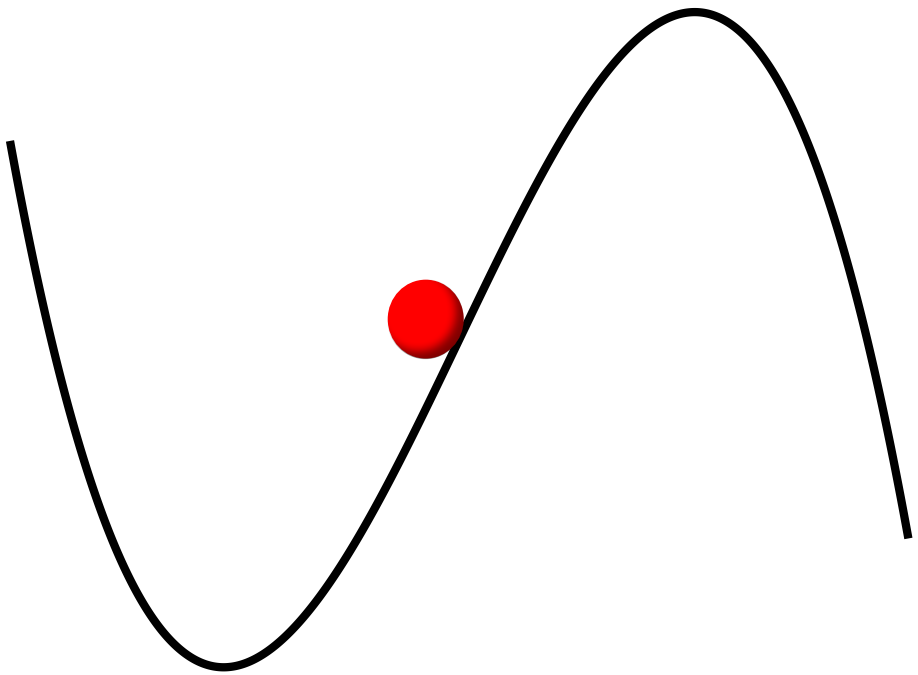

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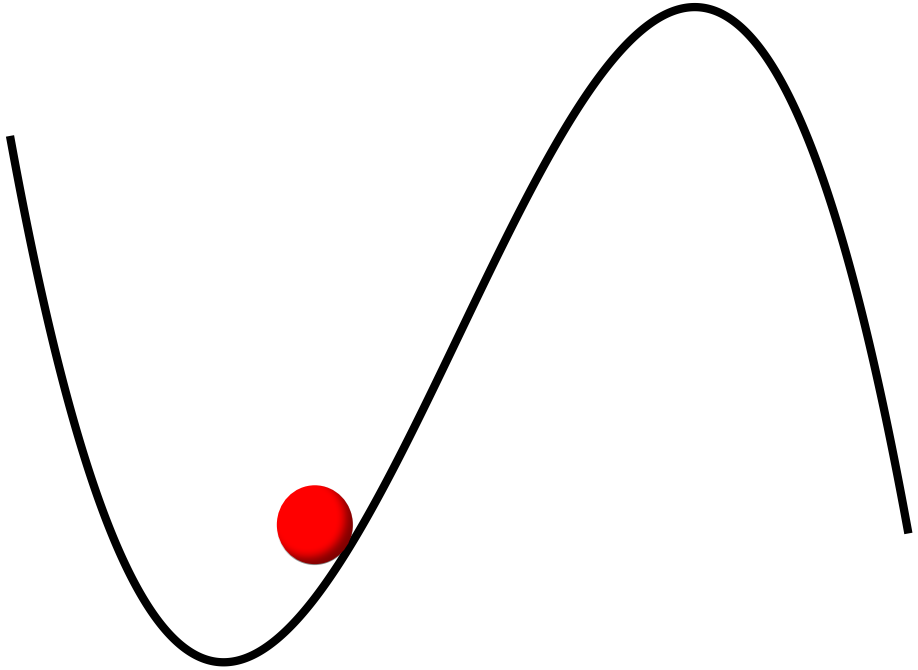

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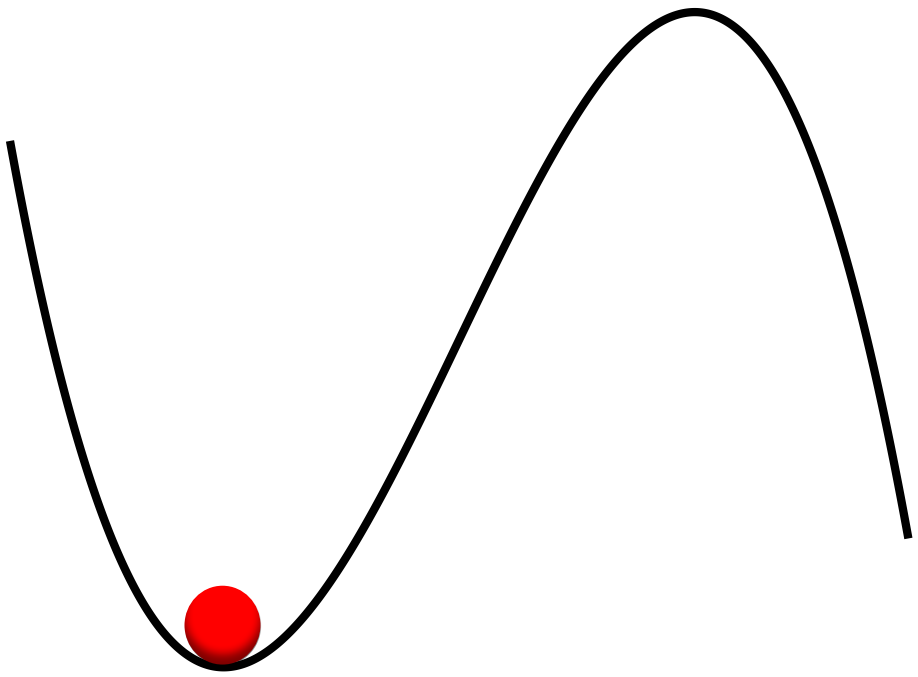

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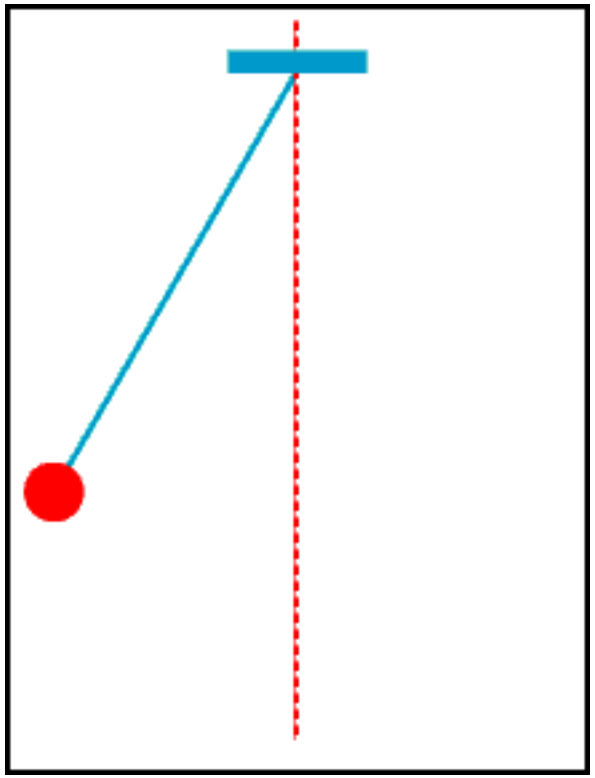

What is dynamics?

Dynamics is the study of systems that evolve in time

a system can settle to an equilibrium

It can repeat itself in cycles

It can do something very complex

What is dynamics?

History of dynamics

- Subject began mid 1600s when Newton invented **differential equations**
- Newton combined his laws of motion and gravitation to explain Kepler's laws
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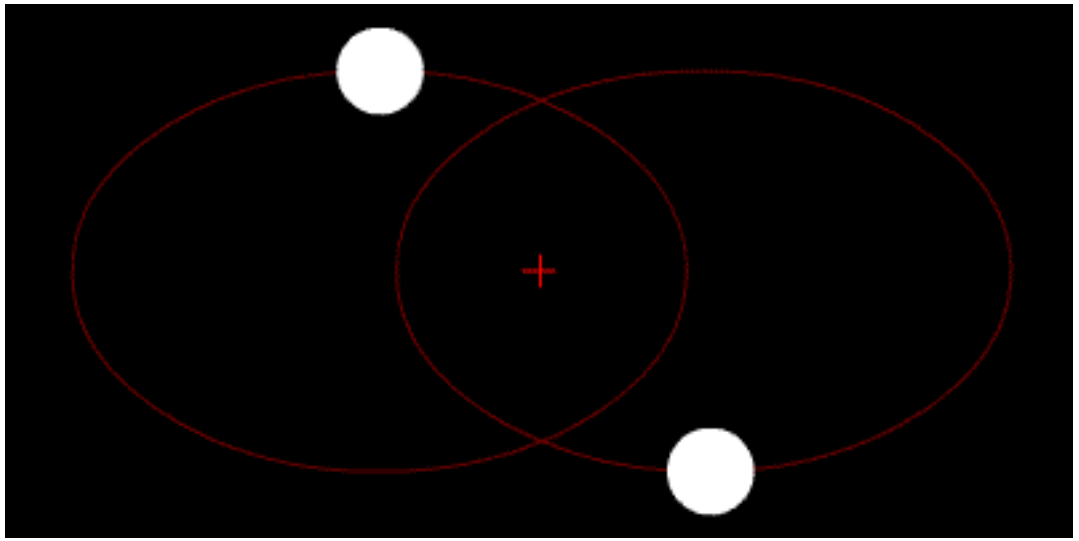

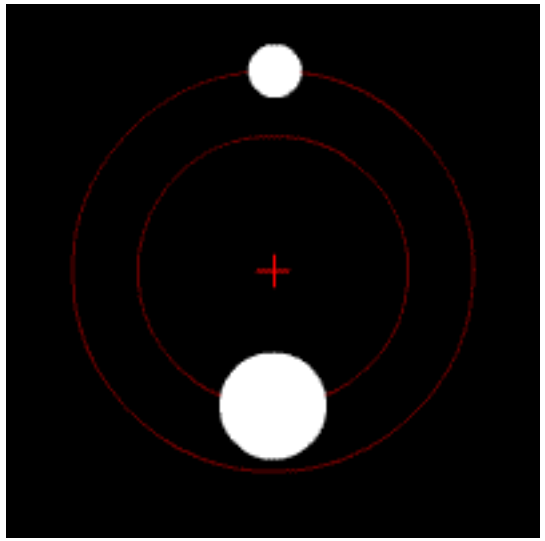

History of dynamics

- Subject began mid 1600s when Newton invented **differential equations**
- Newton combined his laws of motion and gravitation to explain Kepler's laws
- Newton solved the two-body problem
- Many scientists tried to extend Newton's methods to solve the three-body problem, but always to a dead end
- Breakthrough came with Poincaré in late 1800s who emphasized qualitative rather than quantitative questions
- Instead of asking the exact positions of planets at all times, Poincaré asked, "Is the solar system **stable**? Or will planets fly off to infinity?"
- In the 1950s, the invention of computers allowed scientists to find numerical solutions to equations

History of dynamics

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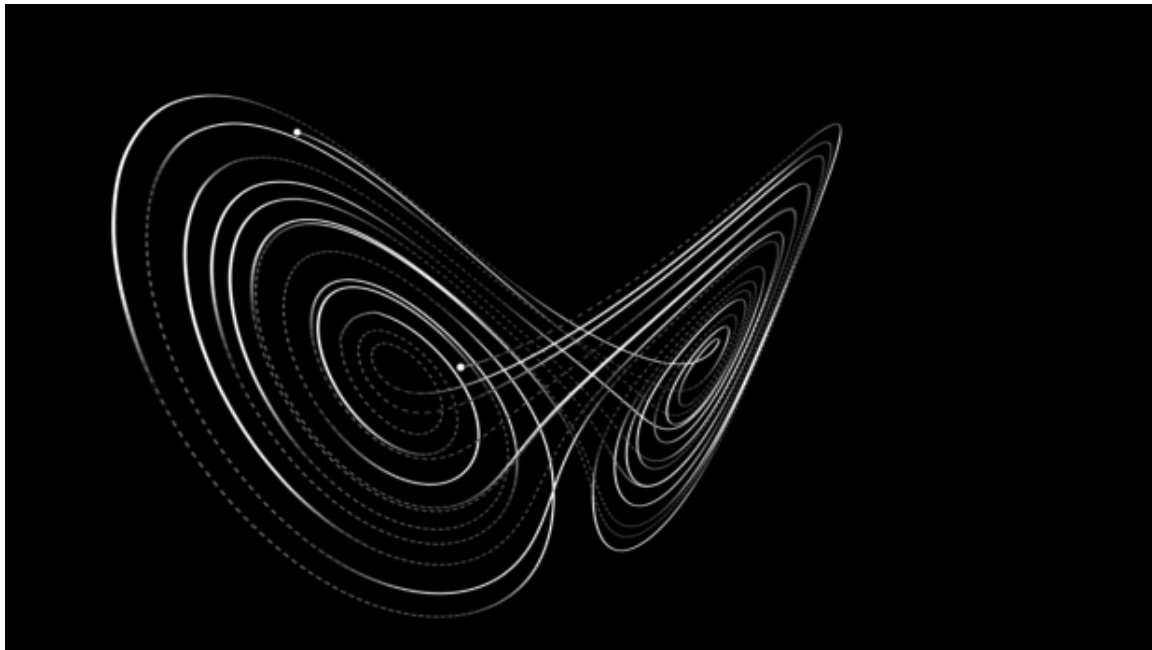

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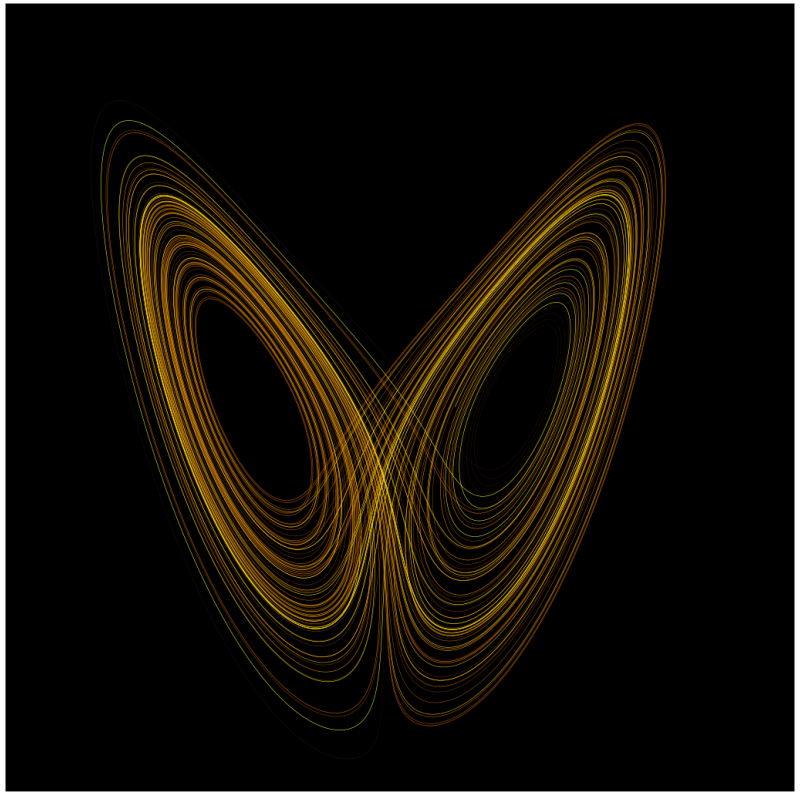
- But, there was structure in chaos!



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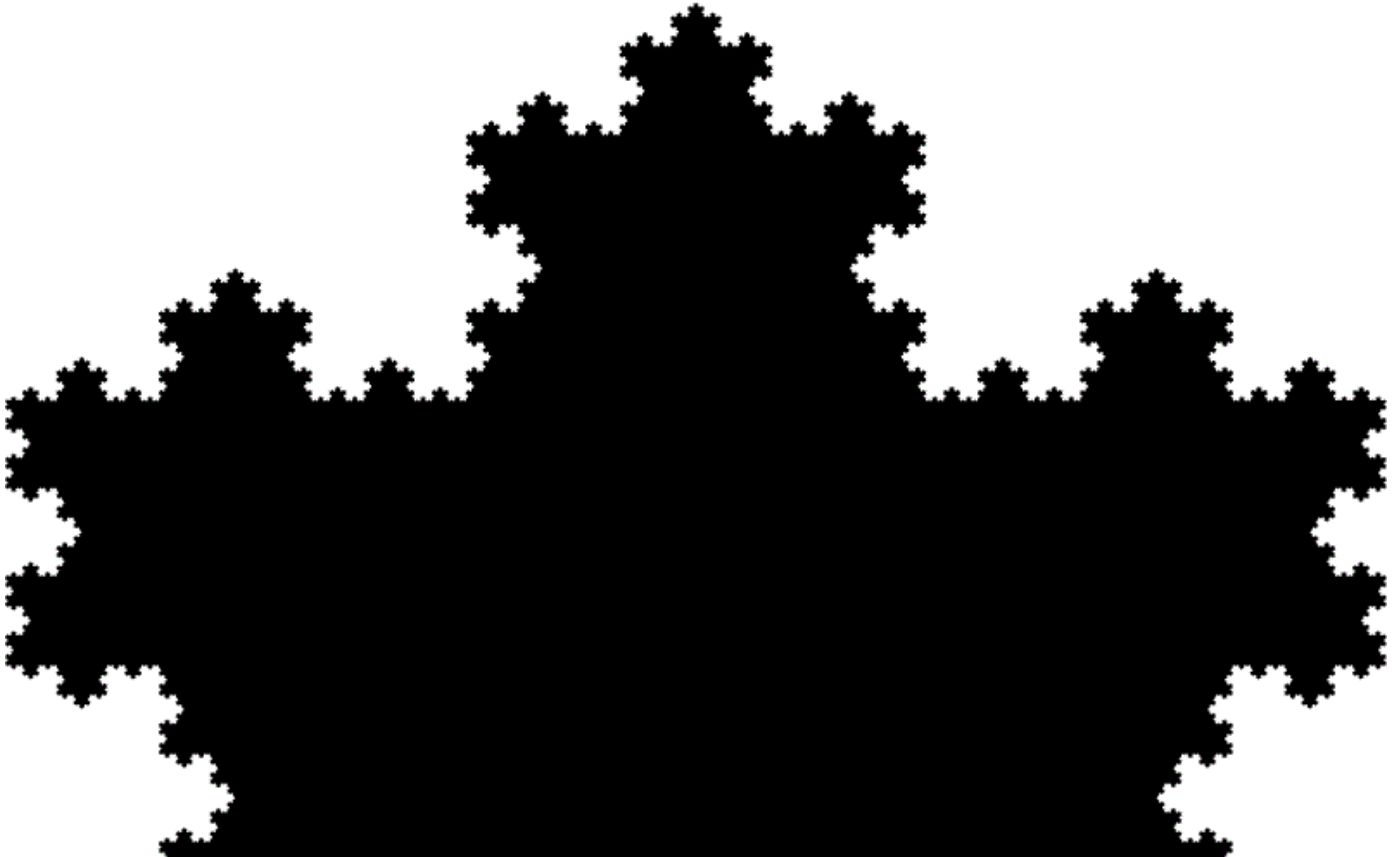
  - Mandelbrot popularized fractals



How do we study such dynamical systems?

- We need some terminology
- There are two types of **dynamical systems**

differential equations

iterated maps
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- There are two types of **dynamical systems**

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- We will start by analyzing problems using differential equations.
- Next week we will study some examples using iterated maps, which will lead us to chaotic solutions

Differential Equations: defining the derivative

- Suppose you have a variable ***x*** which varies with time ***t*** (***x*** could be the position of an object at time ***t***)
- The position at time ***t*** will be denoted by ***x(t)***
- Suppose you know the position at time ***t*** and you want to calculate it at another time ***t'***. So, you have ***x***(***t***) and you want ***x***(***t'***)
- To do this you need to know the velocity at which the object moved between times ***t*** and ***t'***
- The velocity will be denoted by:

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$$

This is called a **derivative**

Differential Equations: defining the derivative

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- A derivative of a quantity (say position) $\boldsymbol{x}$ with respect to time is the variation of $\boldsymbol{x}$ with time, where both $\boldsymbol{x}$ and time are continuous

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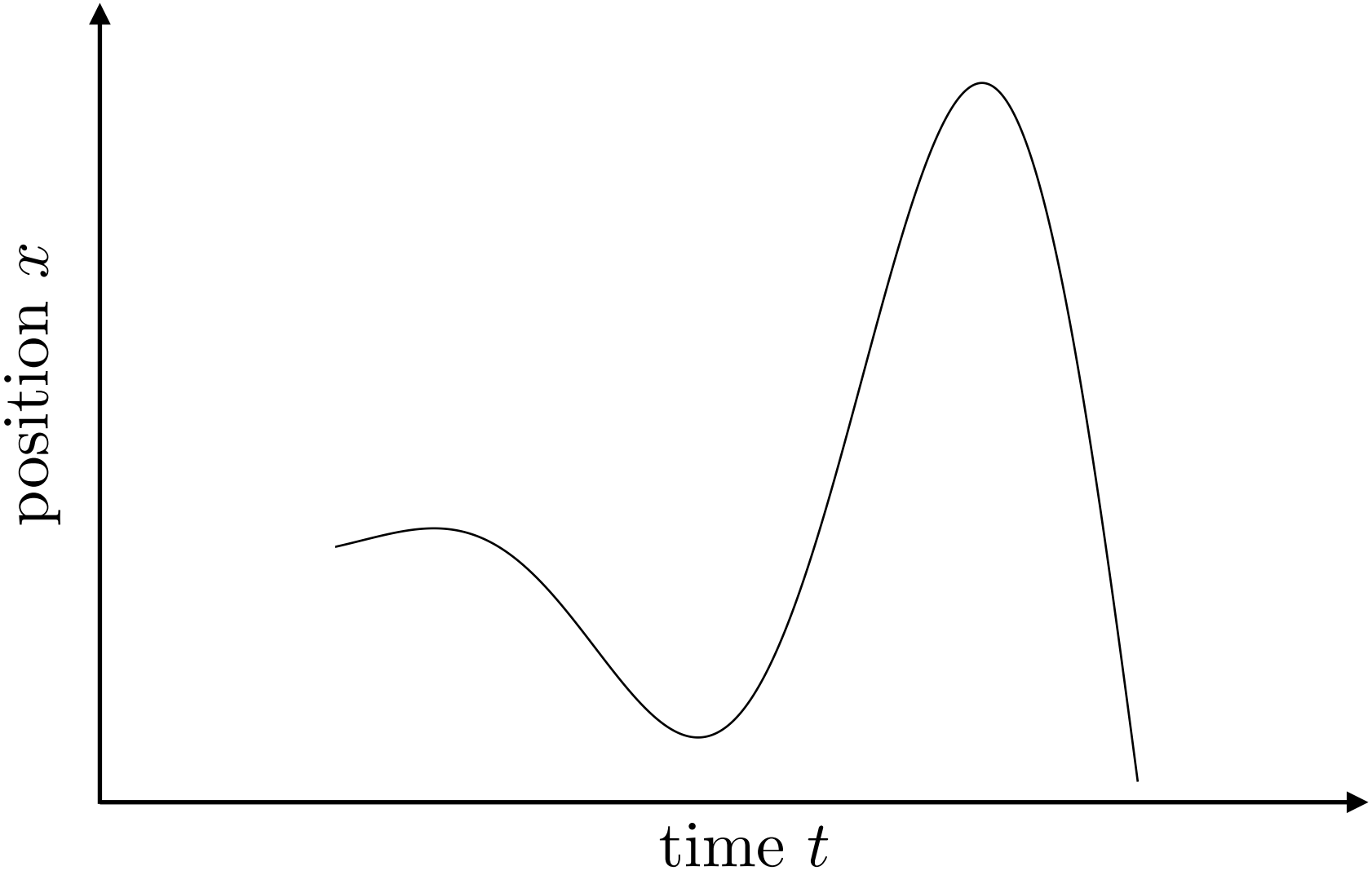
In this expression both time and position are discrete

Differential Equations: the derivative (graphically)



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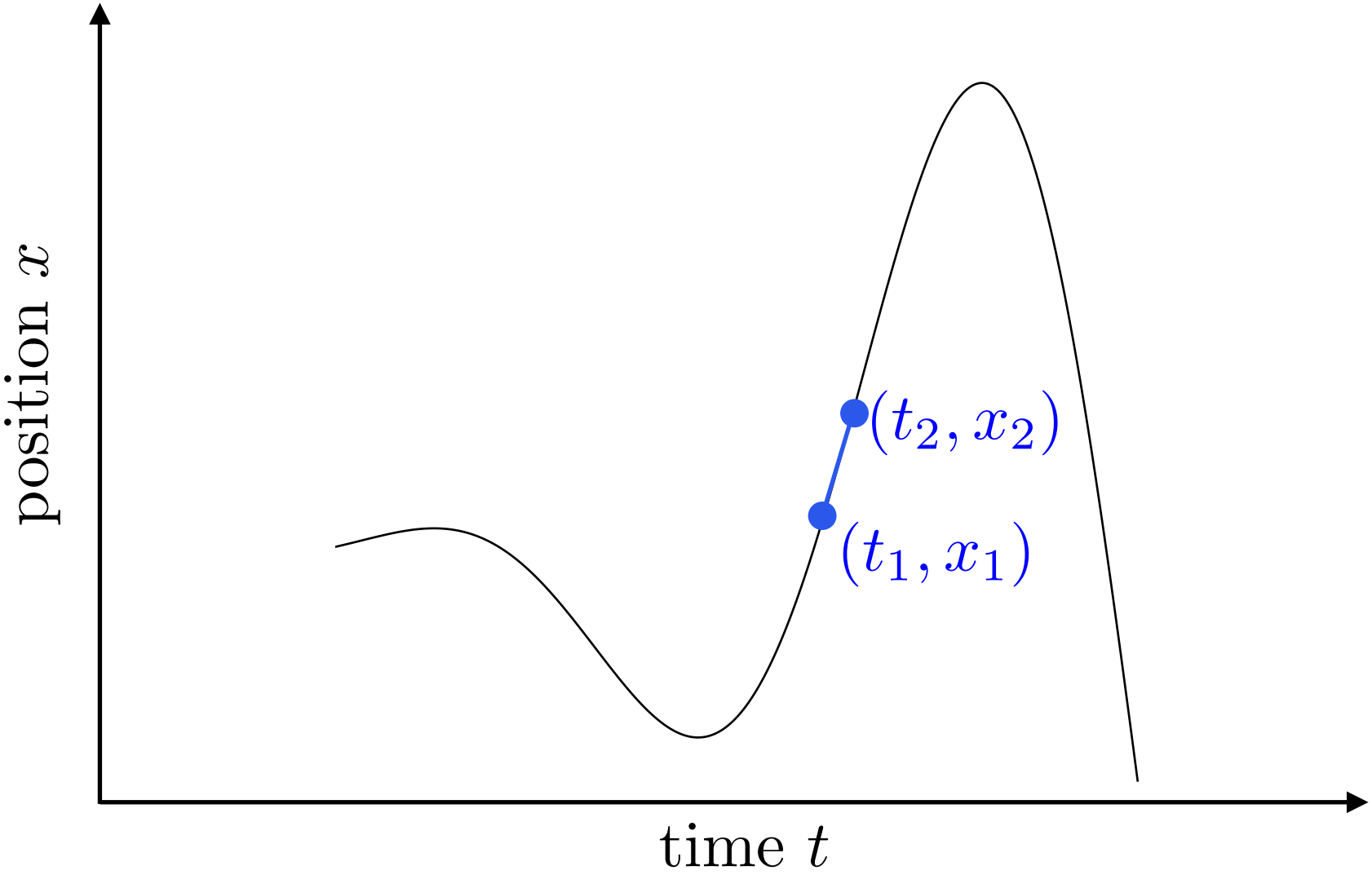

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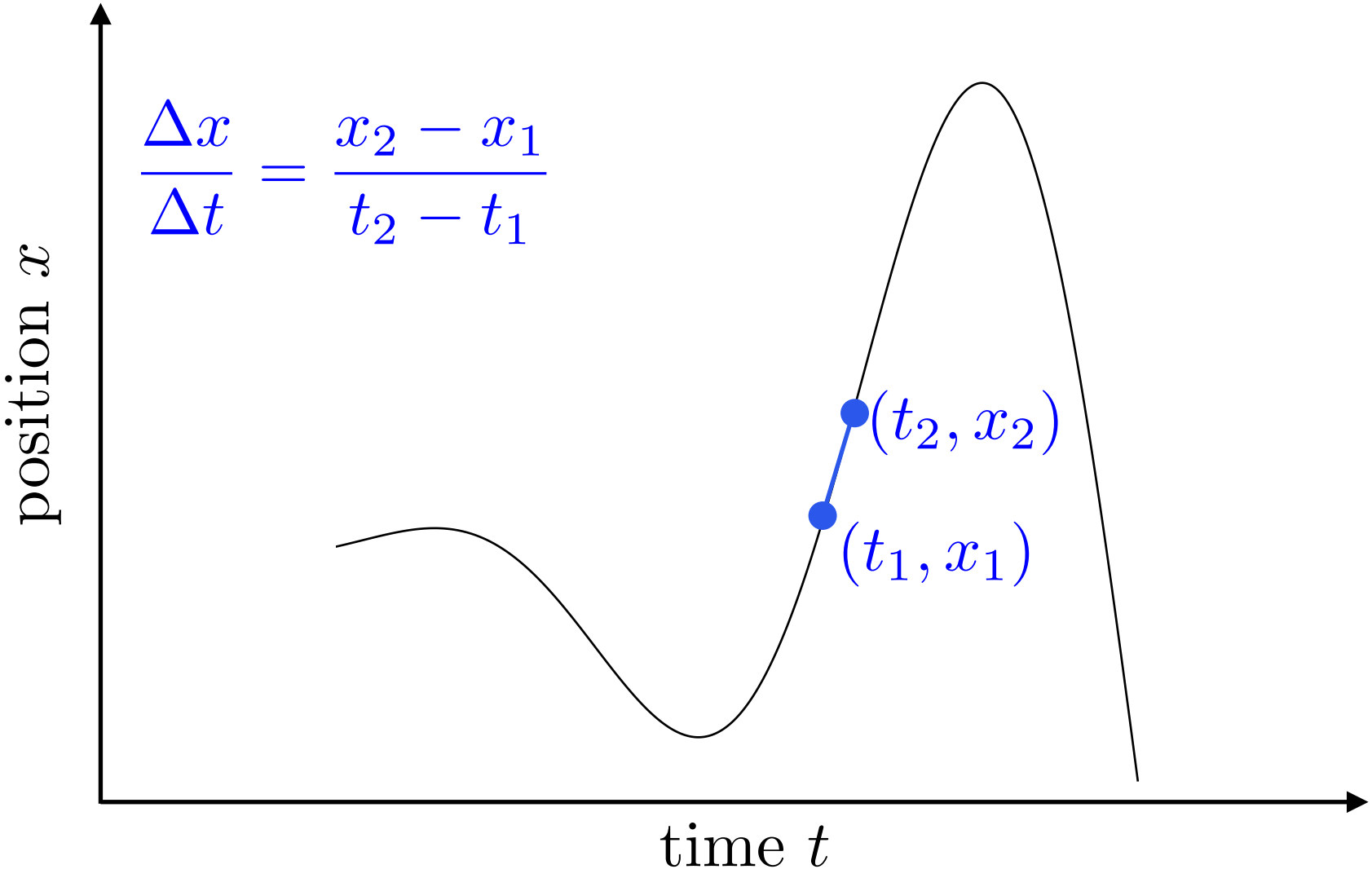

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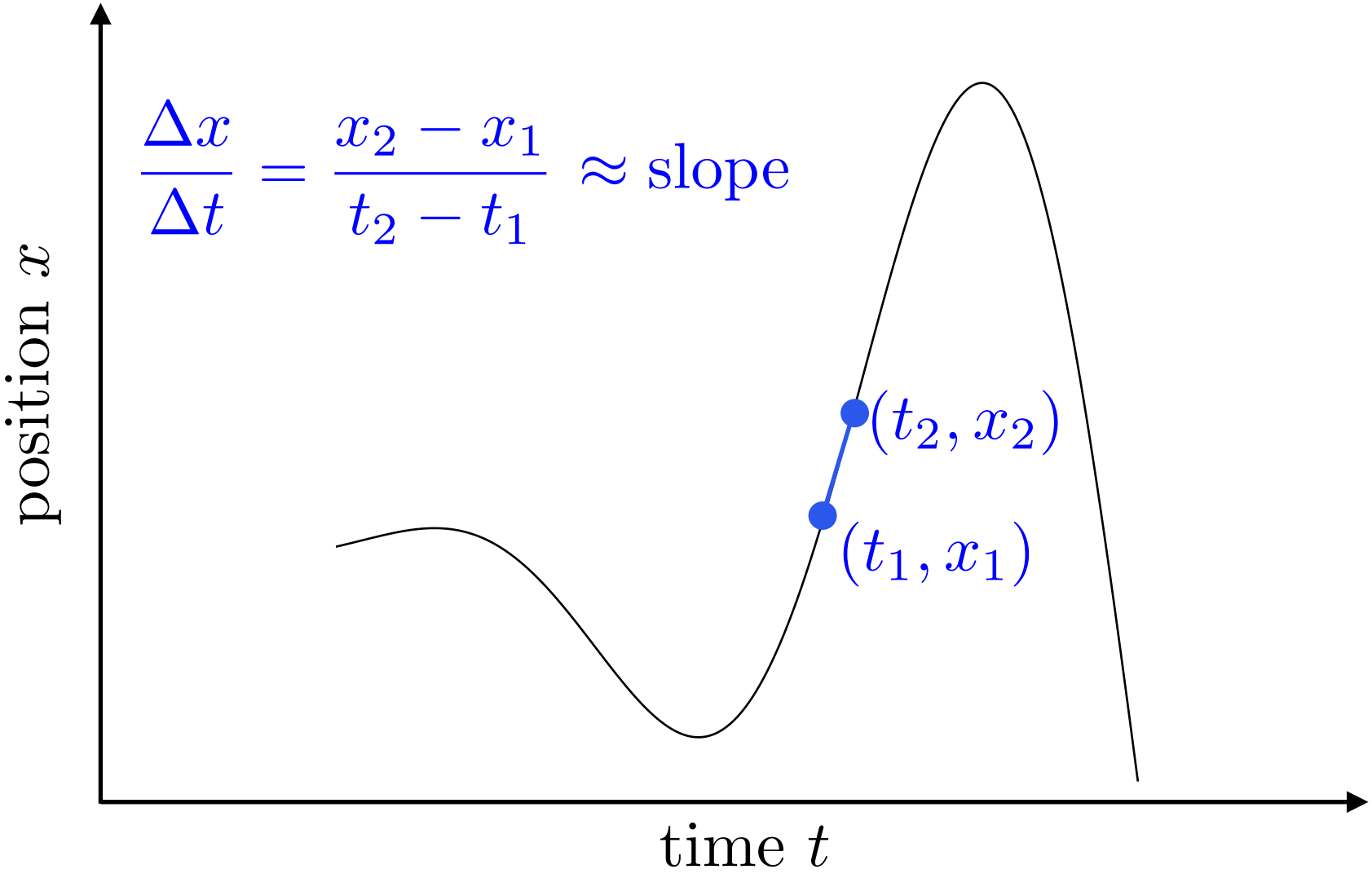

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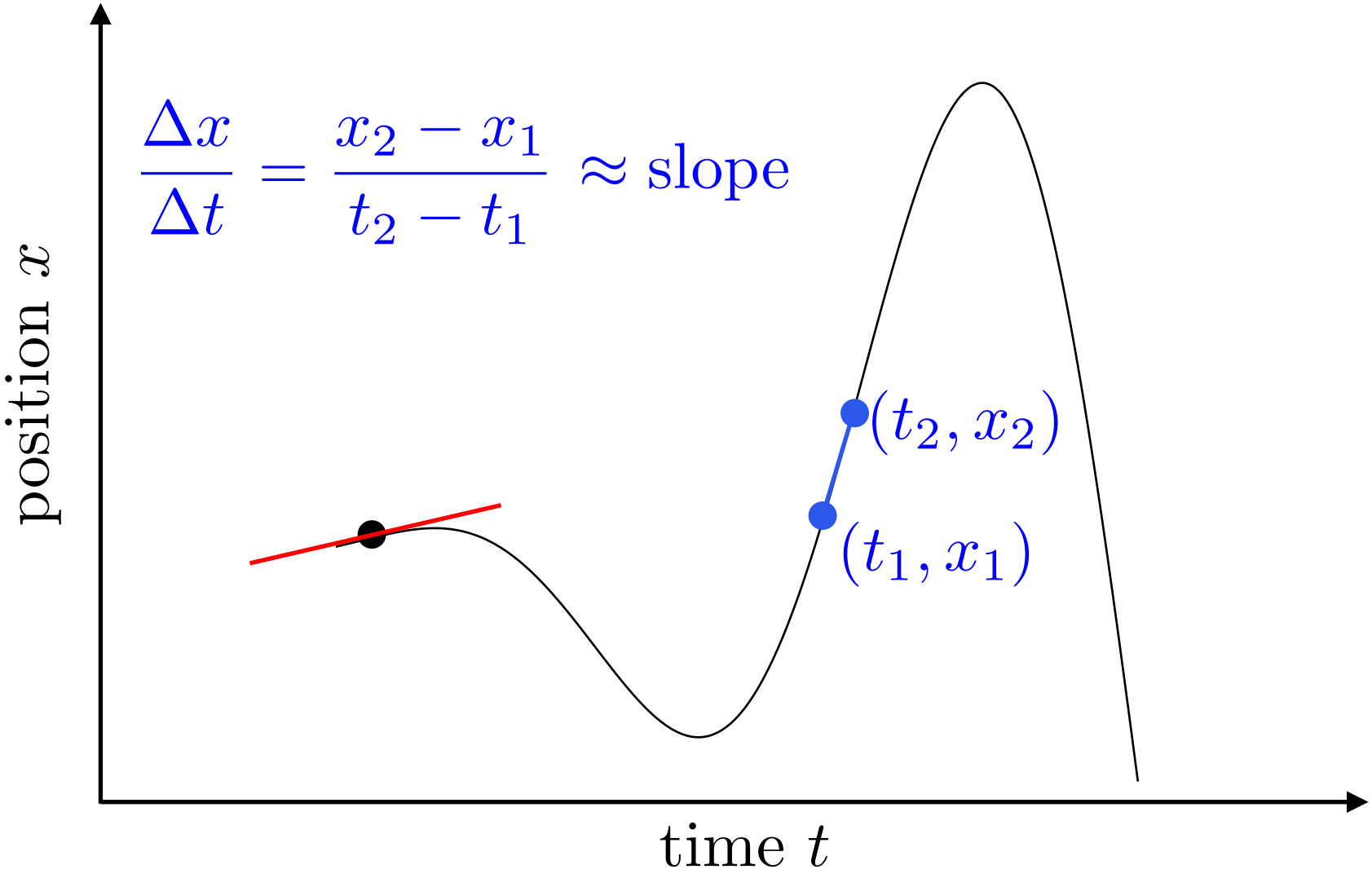

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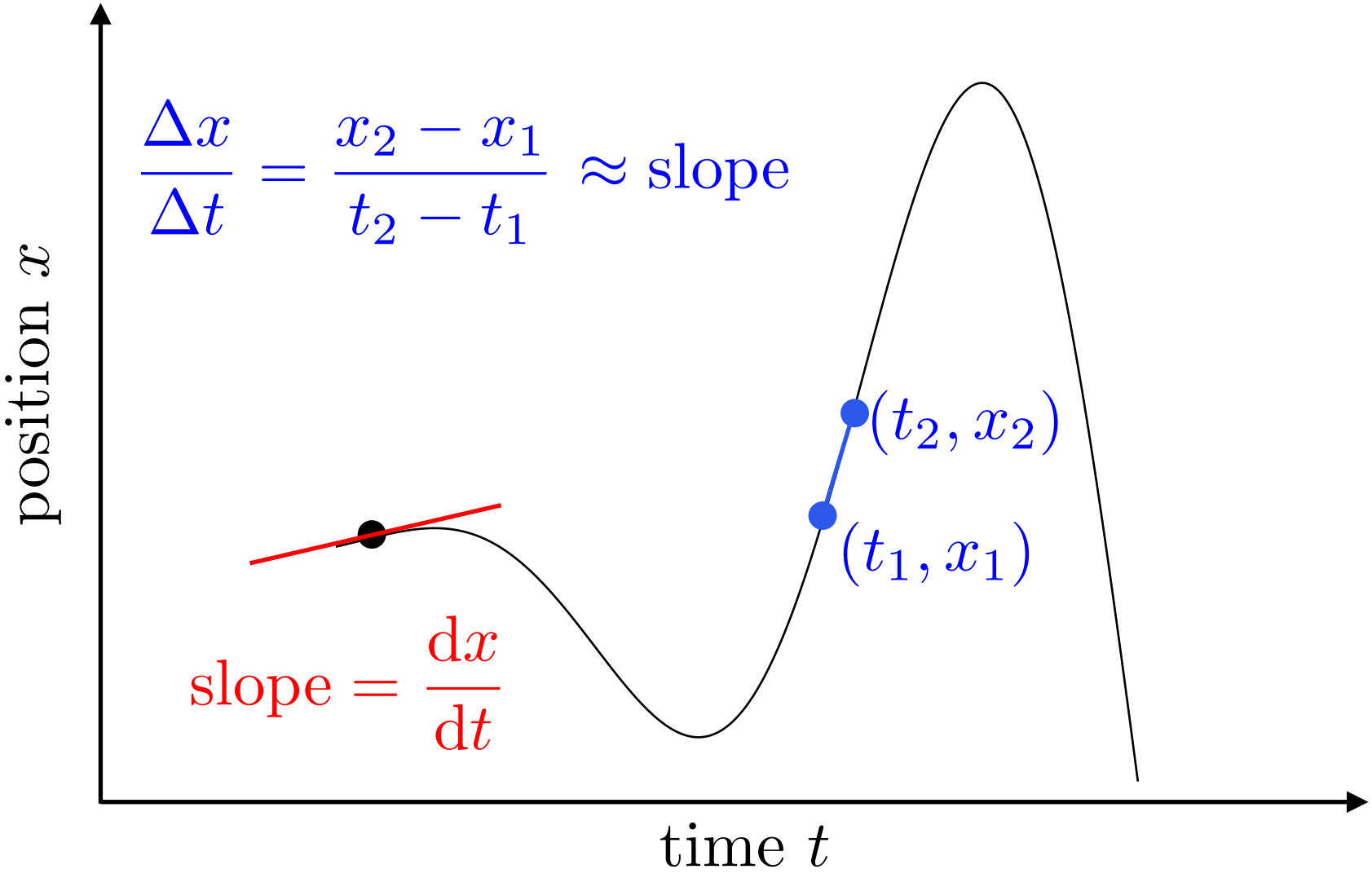

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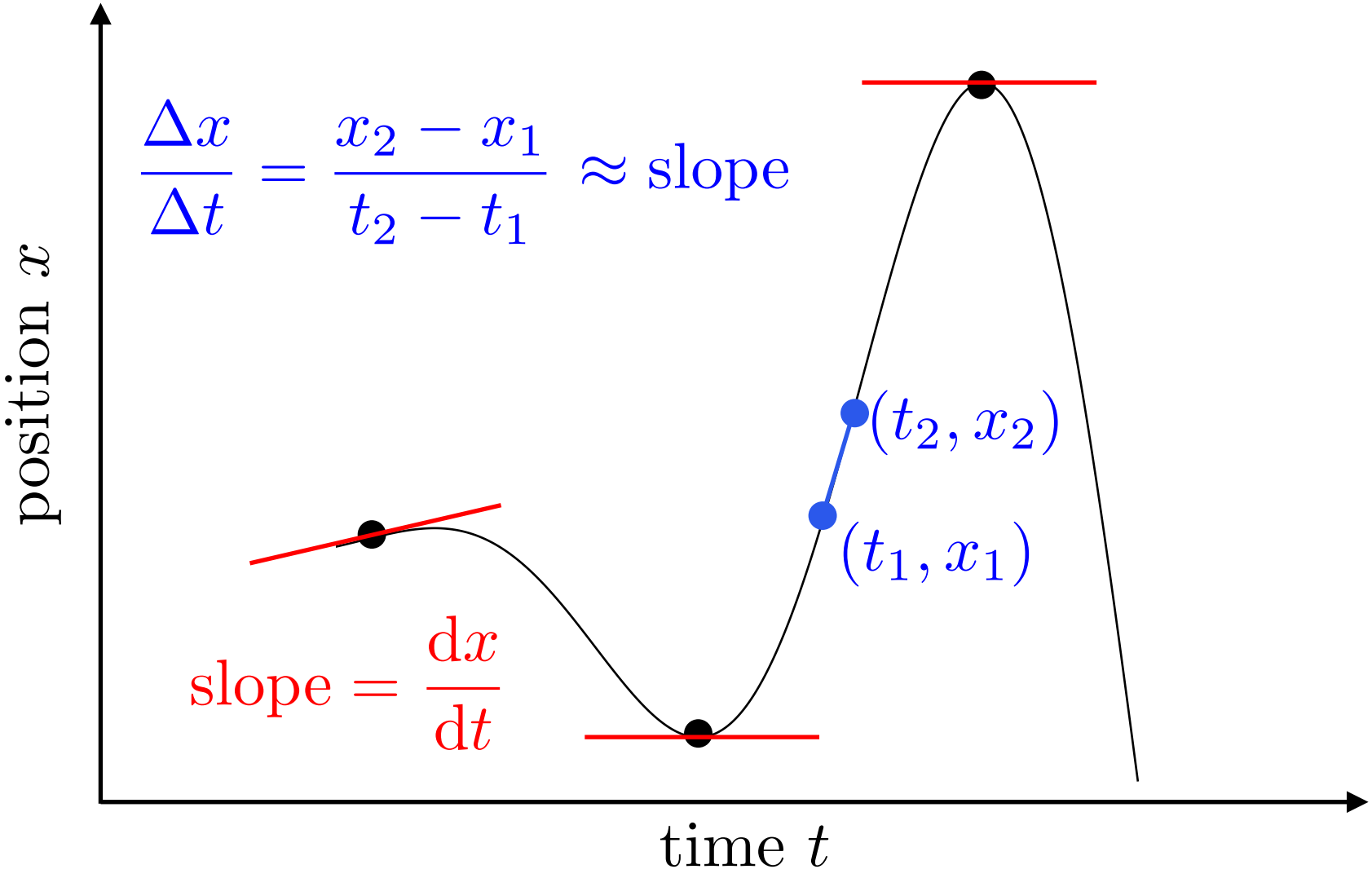

Representation of a dynamical system: 1D

- In a dynamical system, we try to model a real life problem mathematically
- A dynamical system will show how some quantity varies with time
- A one-dimensional dynamical system has ONE equation representing the variation of ONE variable with time.

Examples:

$$\dot{x} = x + 1$$

$$\dot{x} = 3x - 2$$

$$\dot{x} = -5x^2 + \sin(x)$$

Representation of a dynamical system: 2D

- A two-dimensional dynamical system has TWO equations which are usually **coupled**, and which represent the variation of TWO variables with time.

Example: $$\dot{x}_1 = 3x_2 - 2x_1$$
$$\dot{x}_2 = x_1 + 4x_2$$

- We will only consider **autonomous** systems, i.e. systems that do not **explicitly** depend on time
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- We won't consider something like $\dot{x} = x + t$ **non-autonomous**

A Geometric Way of Thinking: Flows on the Line

- The idea is NOT to solve the equation
- Instead, we want to **think geometrically**
- We begin with a one-dimensional dynamical system

$$\dot{x} = f(x)$$

$x(t)$ is a real-valued function of time t

$f(x)$ is **smooth** and also real-valued

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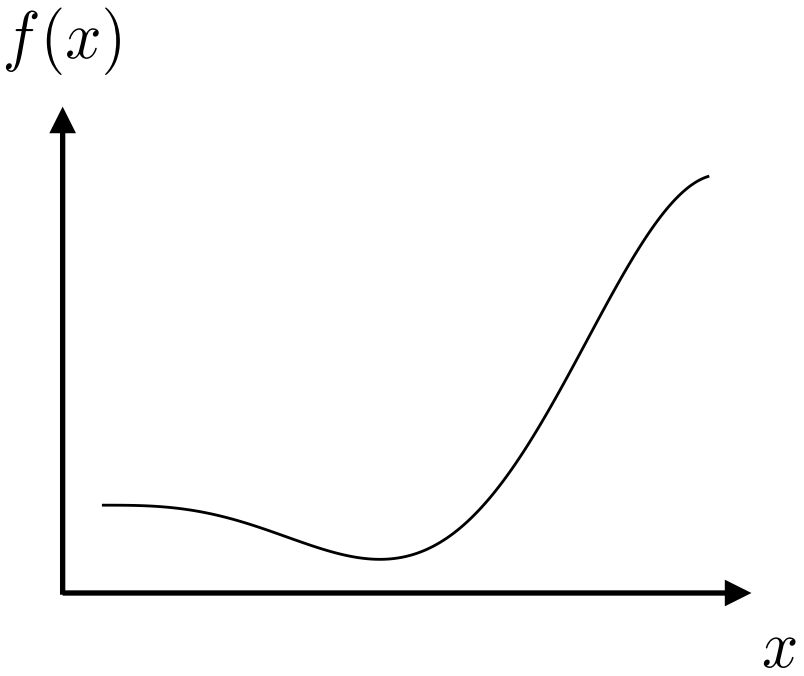

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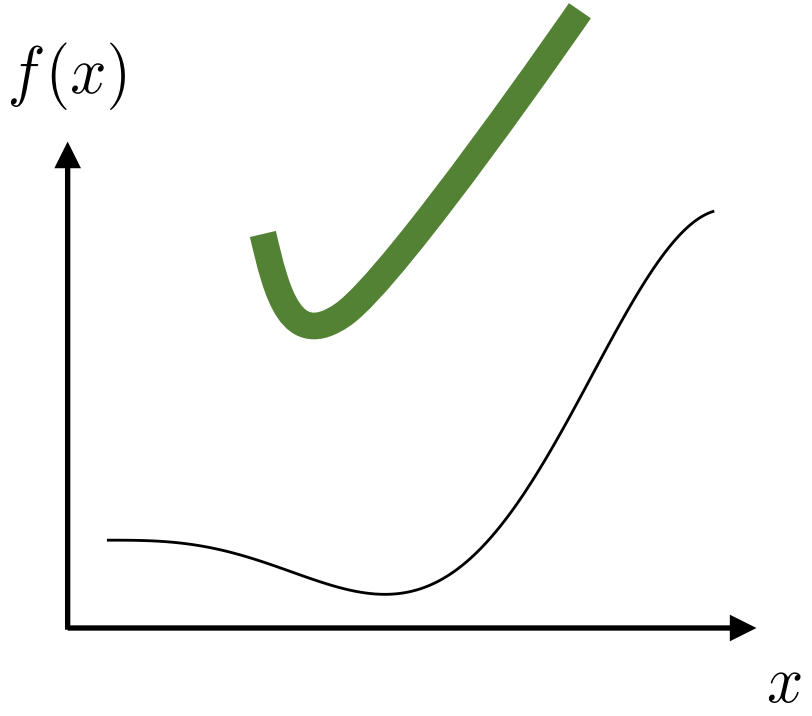

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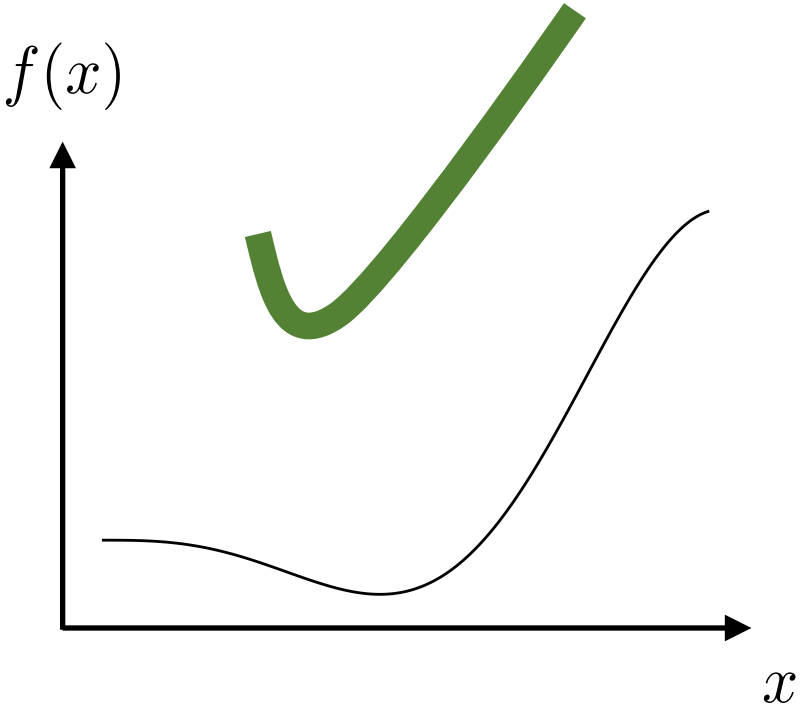

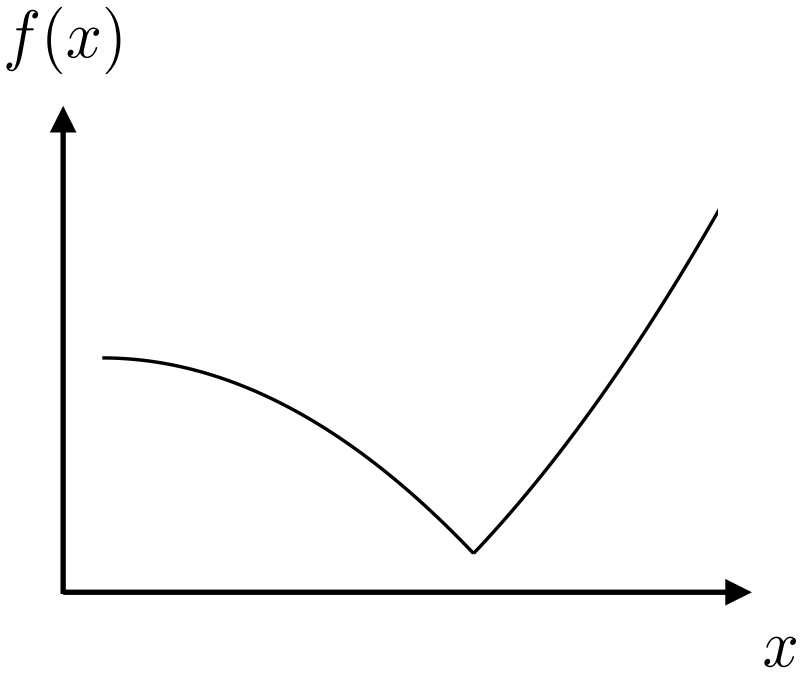

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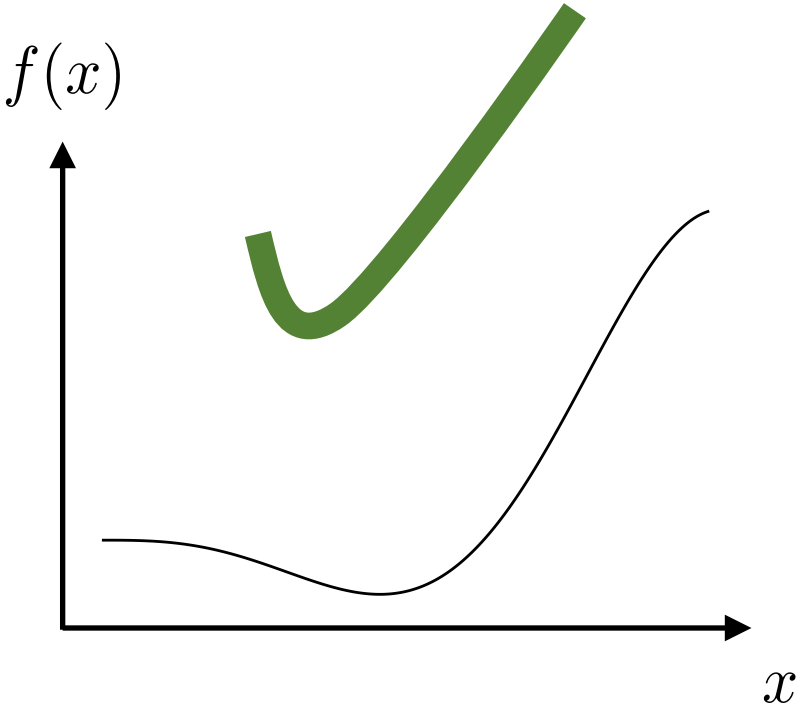

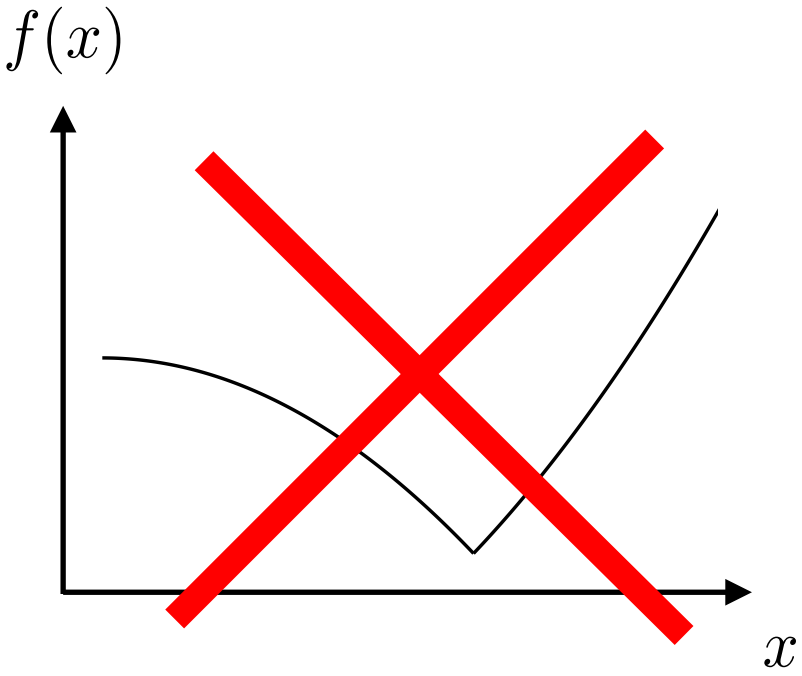

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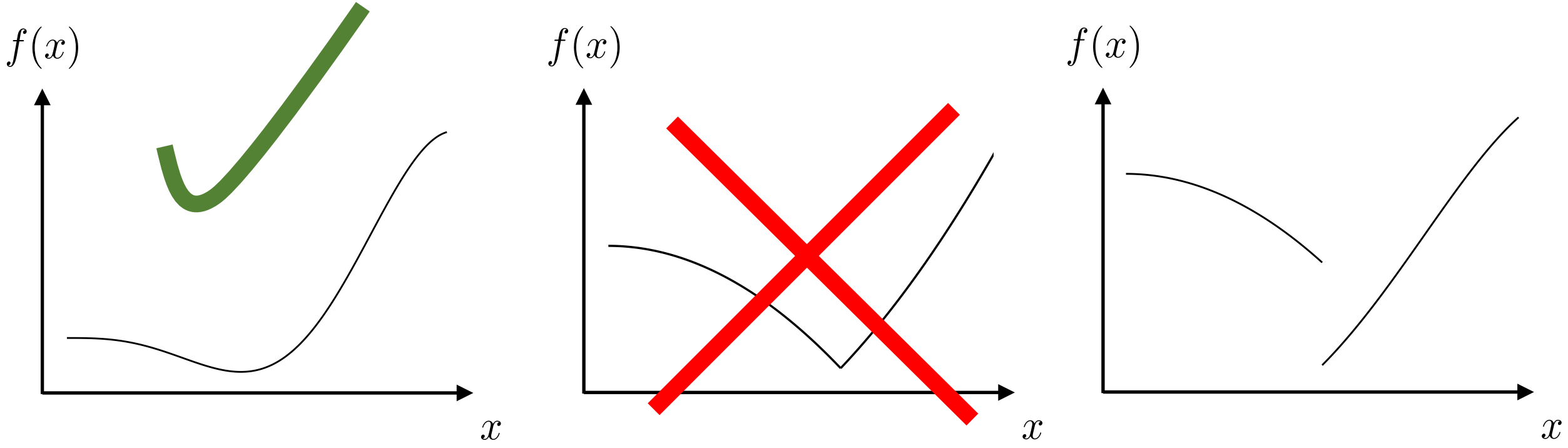

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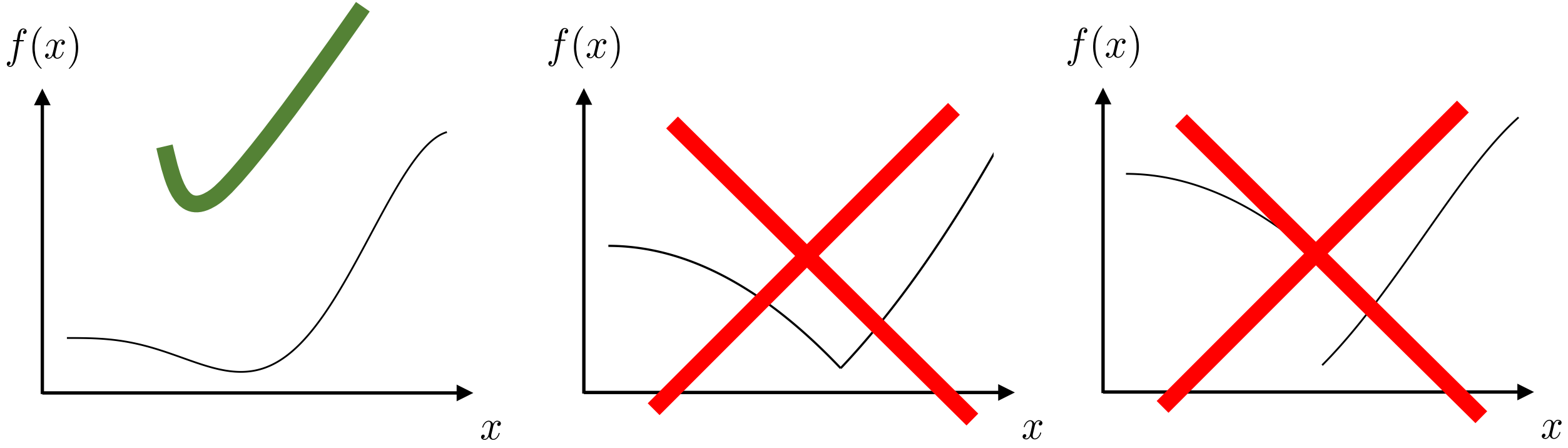

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A Geometric Way of Thinking: Flows on the Line

- Let's take an example: $\dot{x} = f(x) = \sin x$
- The idea is to analyze how $x(t)$ behaves
- If you know calculus, you can solve this system exactly and you get

$$t = -\log|\csc x + \cot x| + \text{constant}$$

- We don't even have $x(t)$, we have $t(x)$ instead, and inverting is impossible!

A Geometric Way of Thinking: Flows on the Line

- Instead of solving exactly, we will plot $f(x)$ against x , i.e. $\dot{x}$ against x

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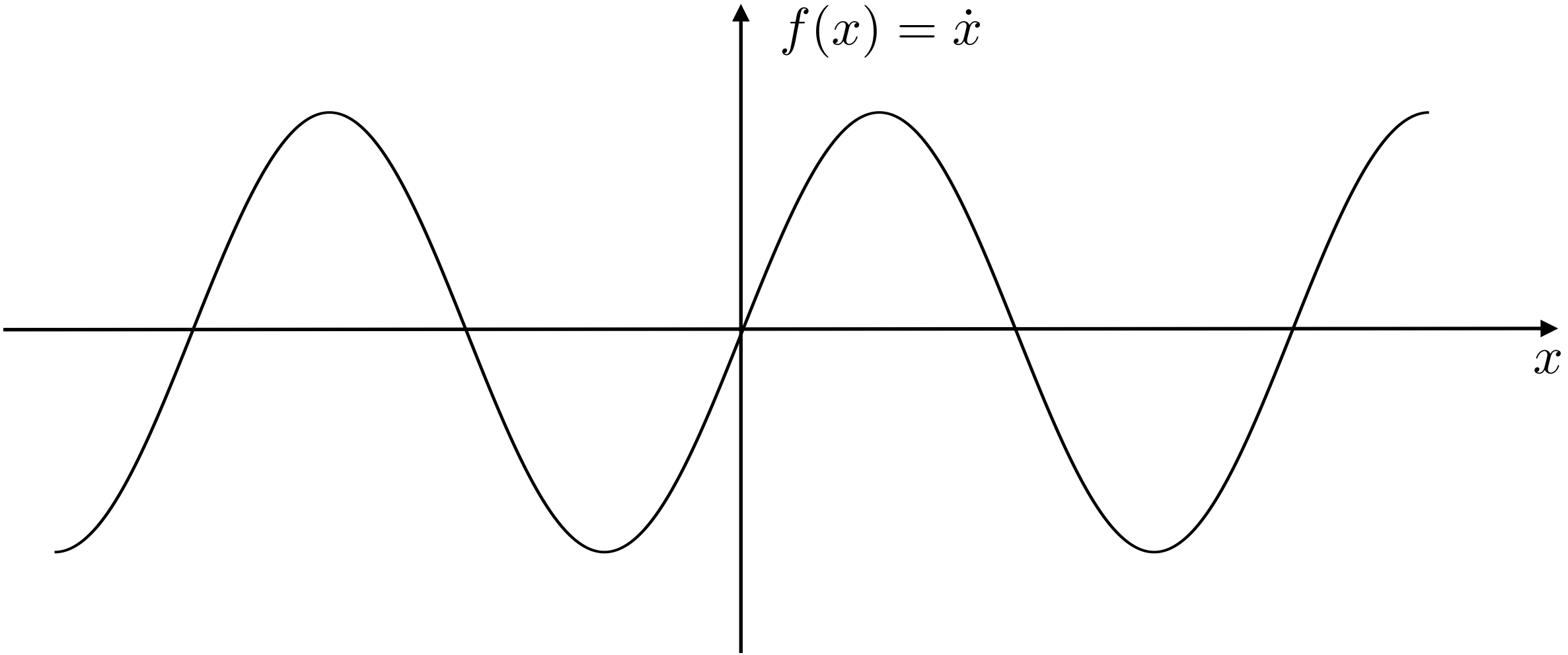

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- Instead of solving exactly, we will plot $f(x)$ against x , i.e. $\dot{x}$ against x
- Then we want to find the **fixed points**, which correspond to points where the system isn't varying with time, i.e. points where

$$\dot{x} = 0$$

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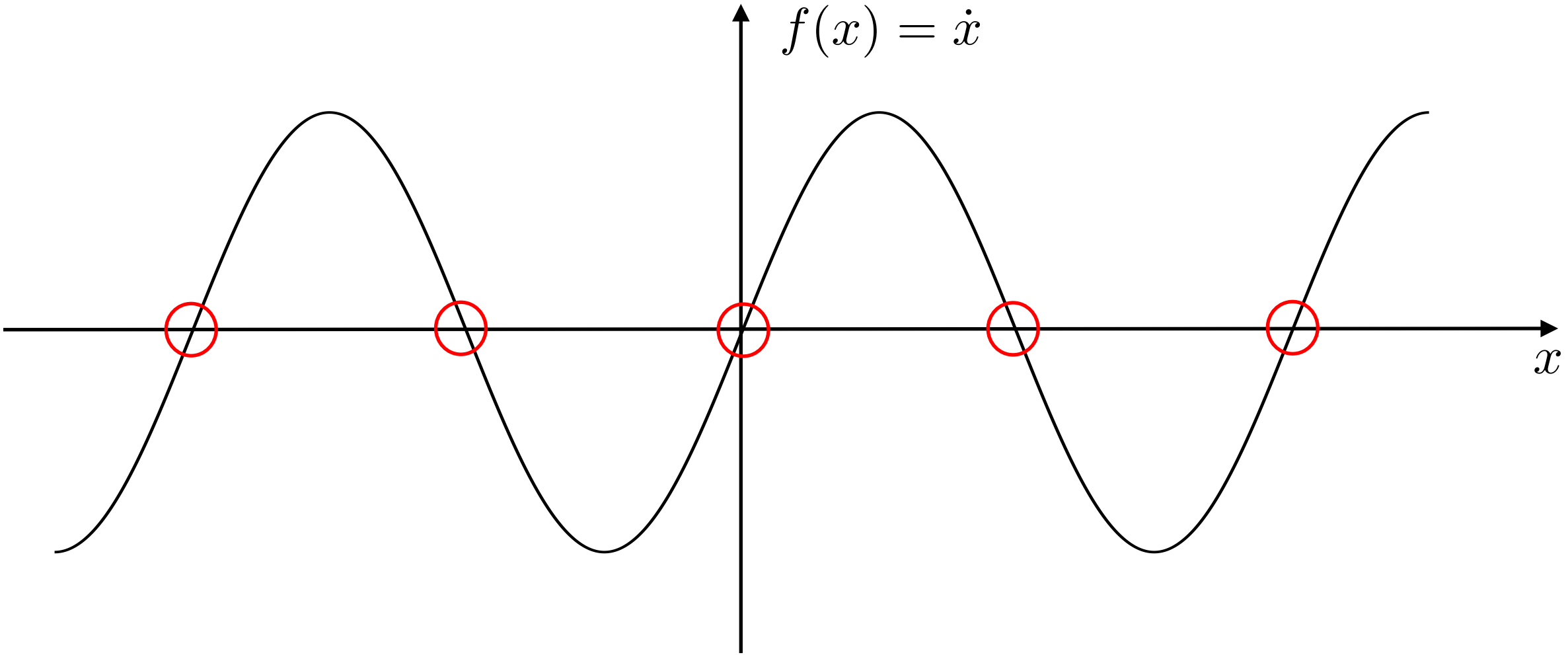

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- There are two kinds of fixed points: **stable** and **unstable**
- The **flow** goes **towards** stable points and **away** from unstable points
- This is determined by the sign of $\dot{x}$

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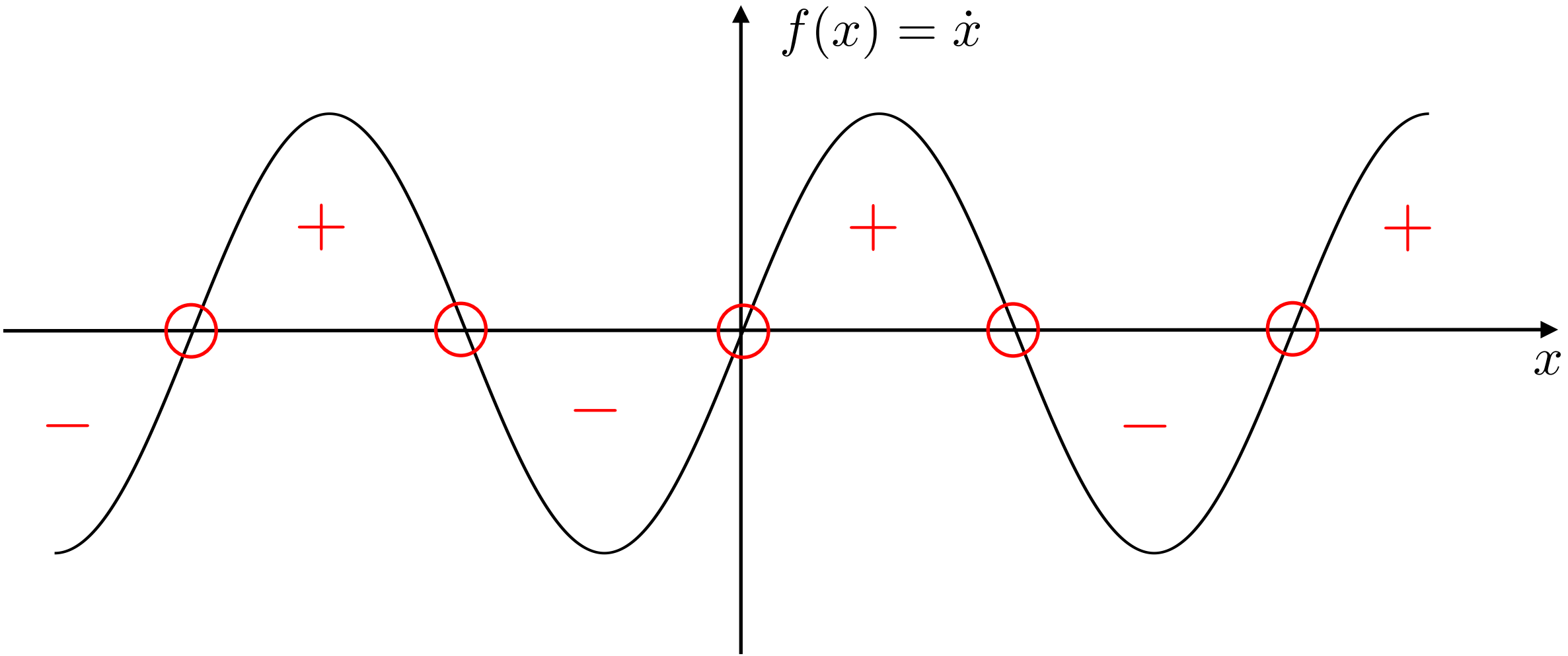

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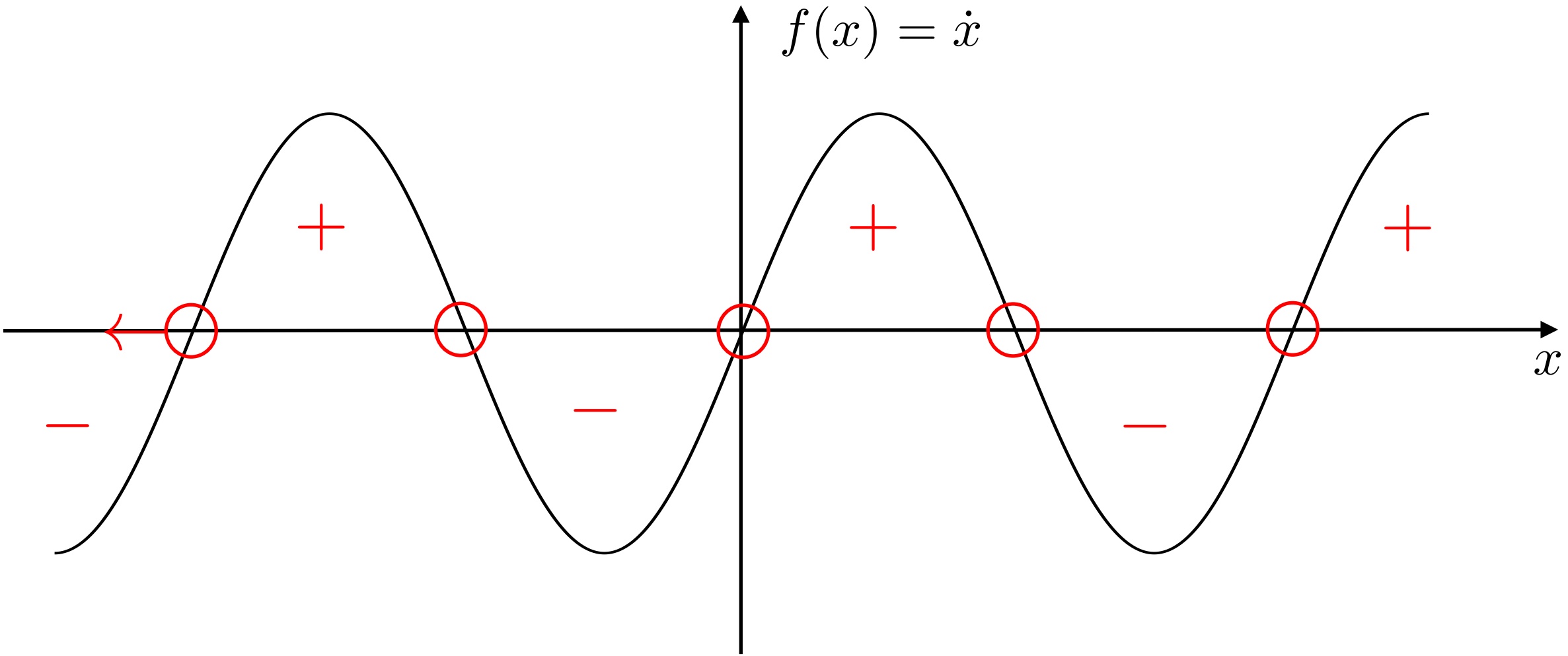

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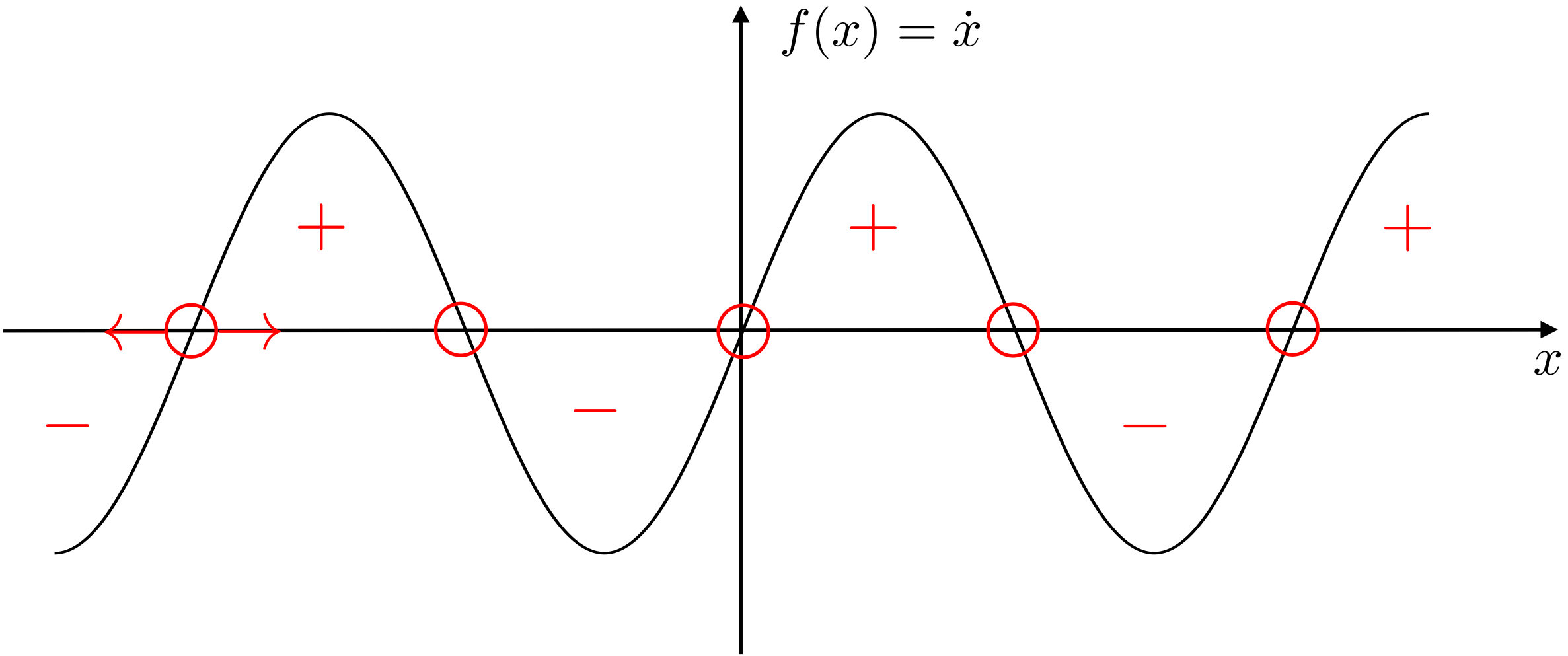

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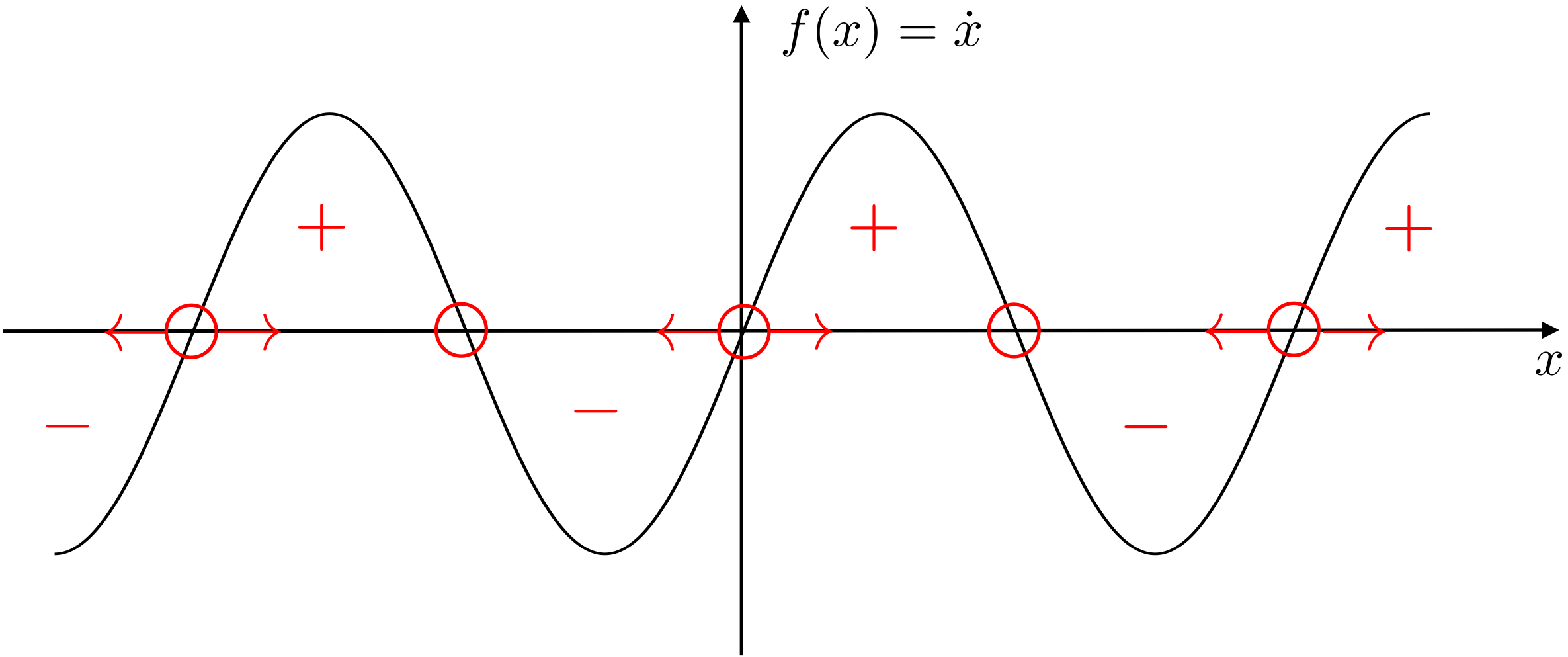

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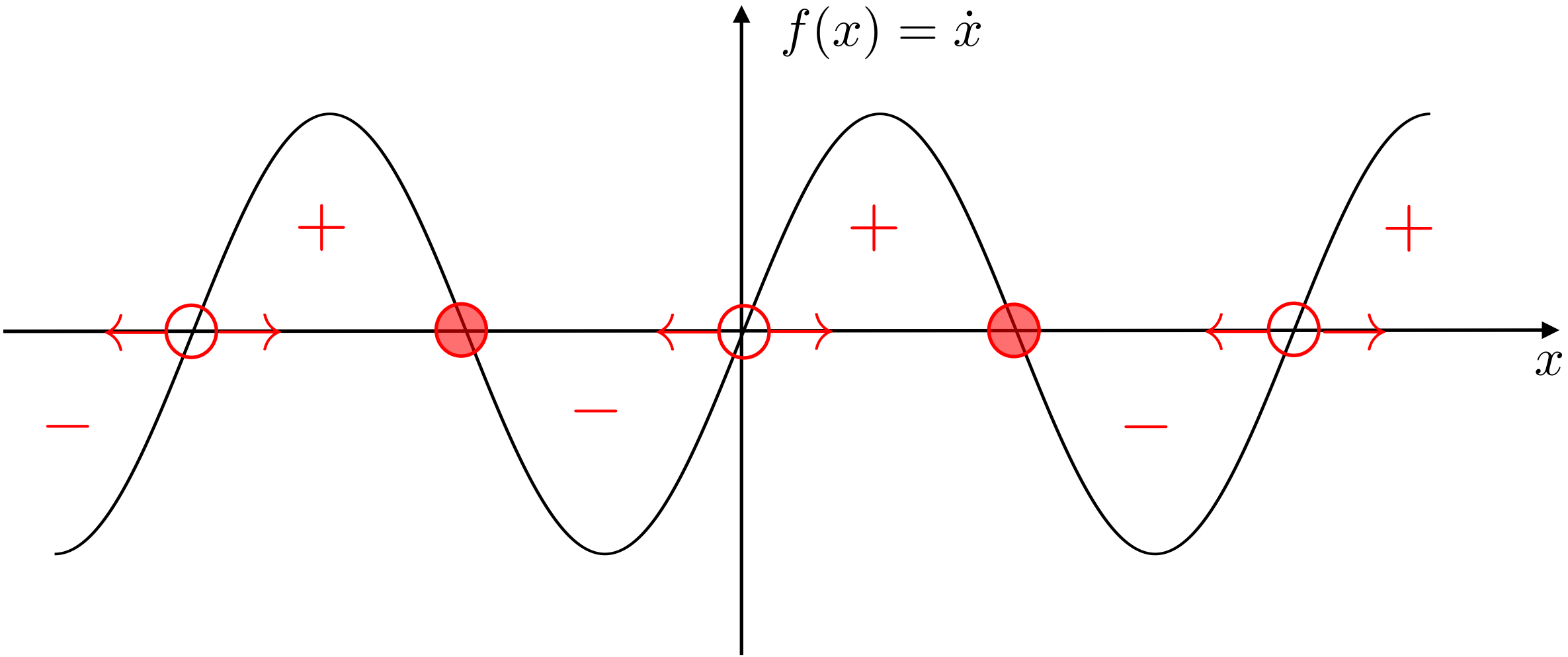

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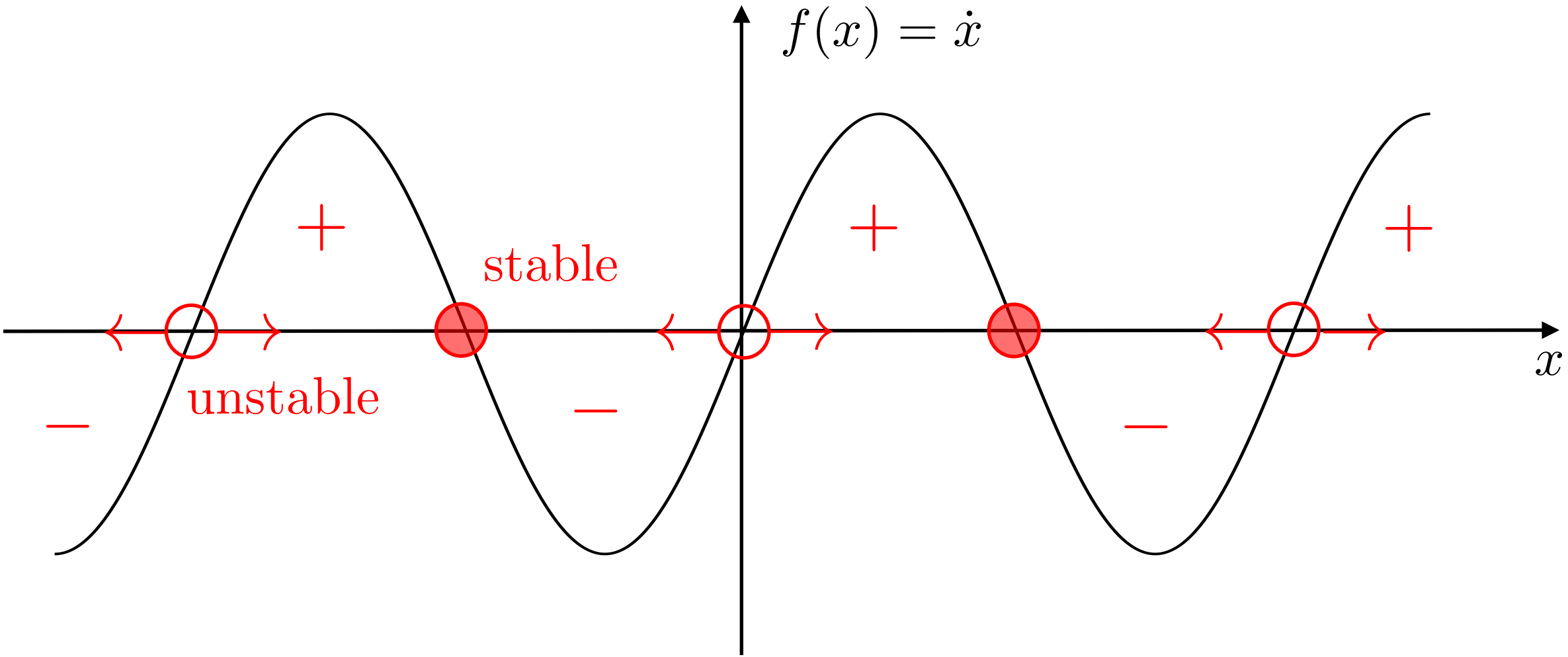
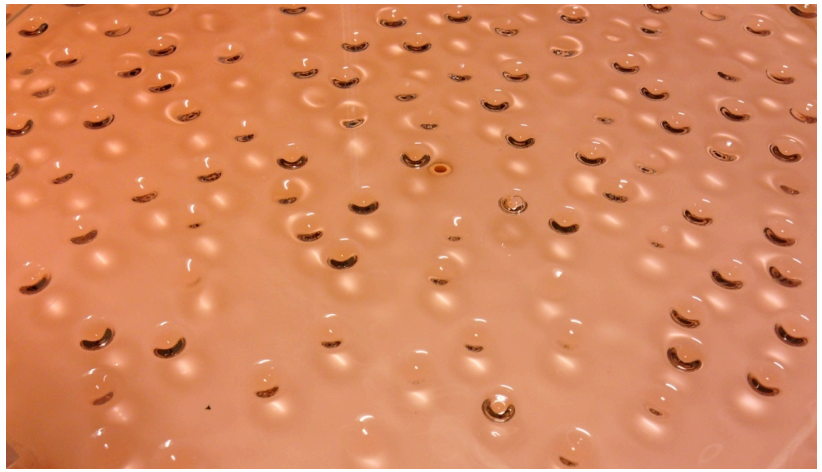


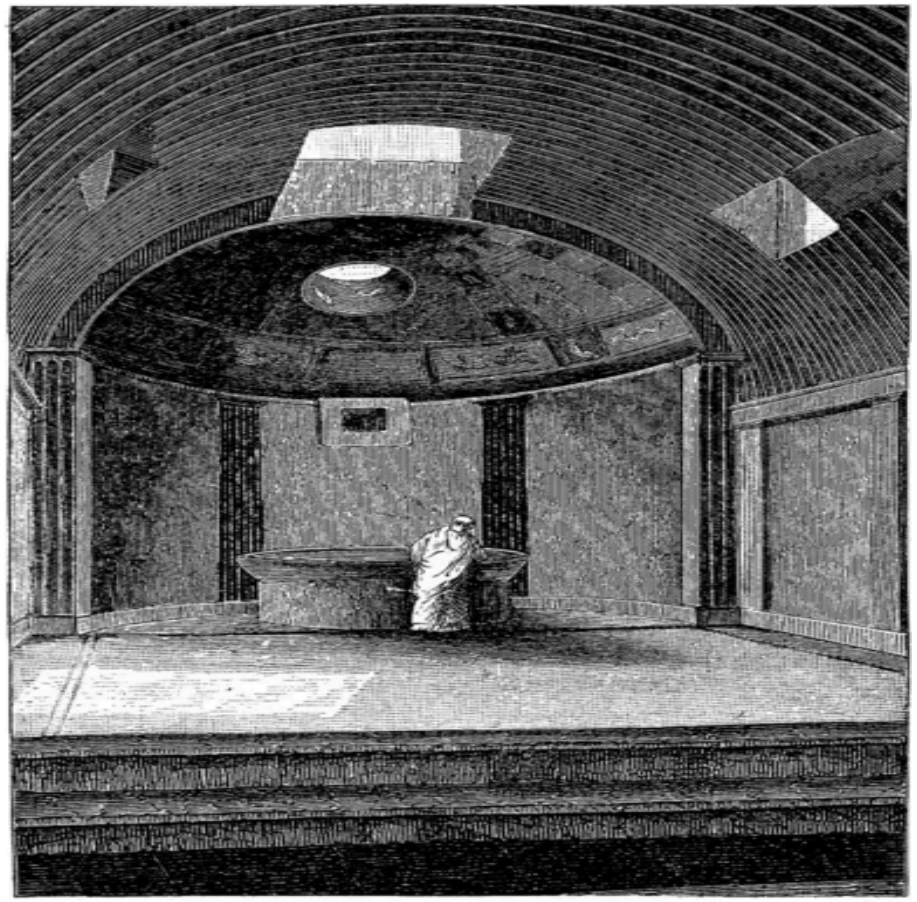
Stable vs. Unstable

Romans and their hot baths!

flat ceiling
will always drip!



curved ceiling may
protect your back!



Simple Example

- Let's start with a simple example

$$\dot{x} = x^2 - 1$$

- First, we must plot this system

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plot(x^2 - 1, {x,-2,2})

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plot(x^2 - 1, {x,-2,2}) - Wolfra

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WolframAlpha computational knowledge engine.

plot(x^2 - 1, {x,-2,2})

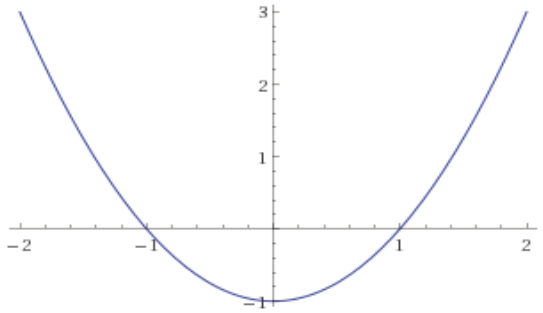
Web Apps Examples Random

Input interpretation:

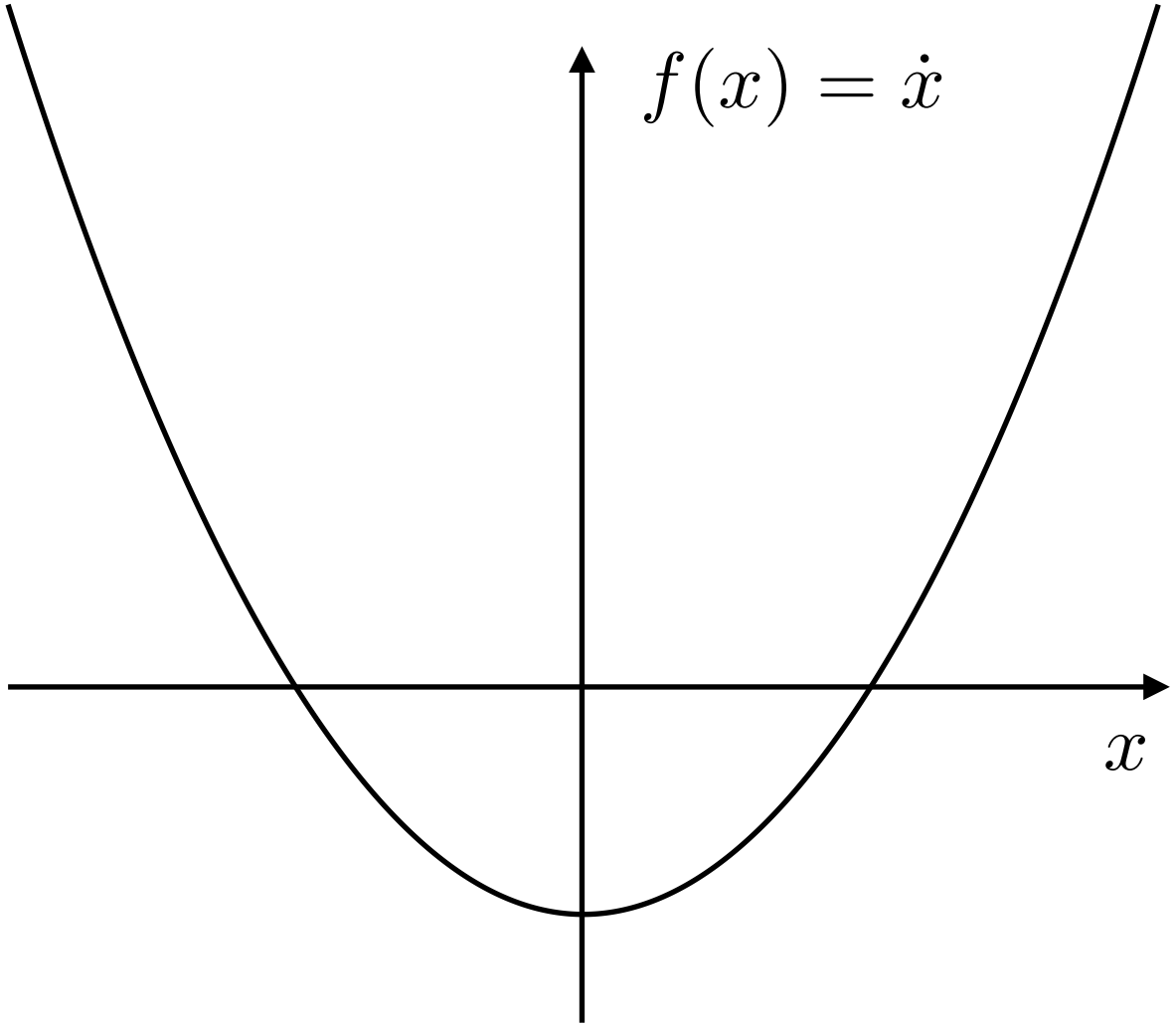
plot	$x^2 - 1$	$x = -2$ to 2

Open code

Plot:



Simple Example



$$\dot{x} = x^2 - 1$$

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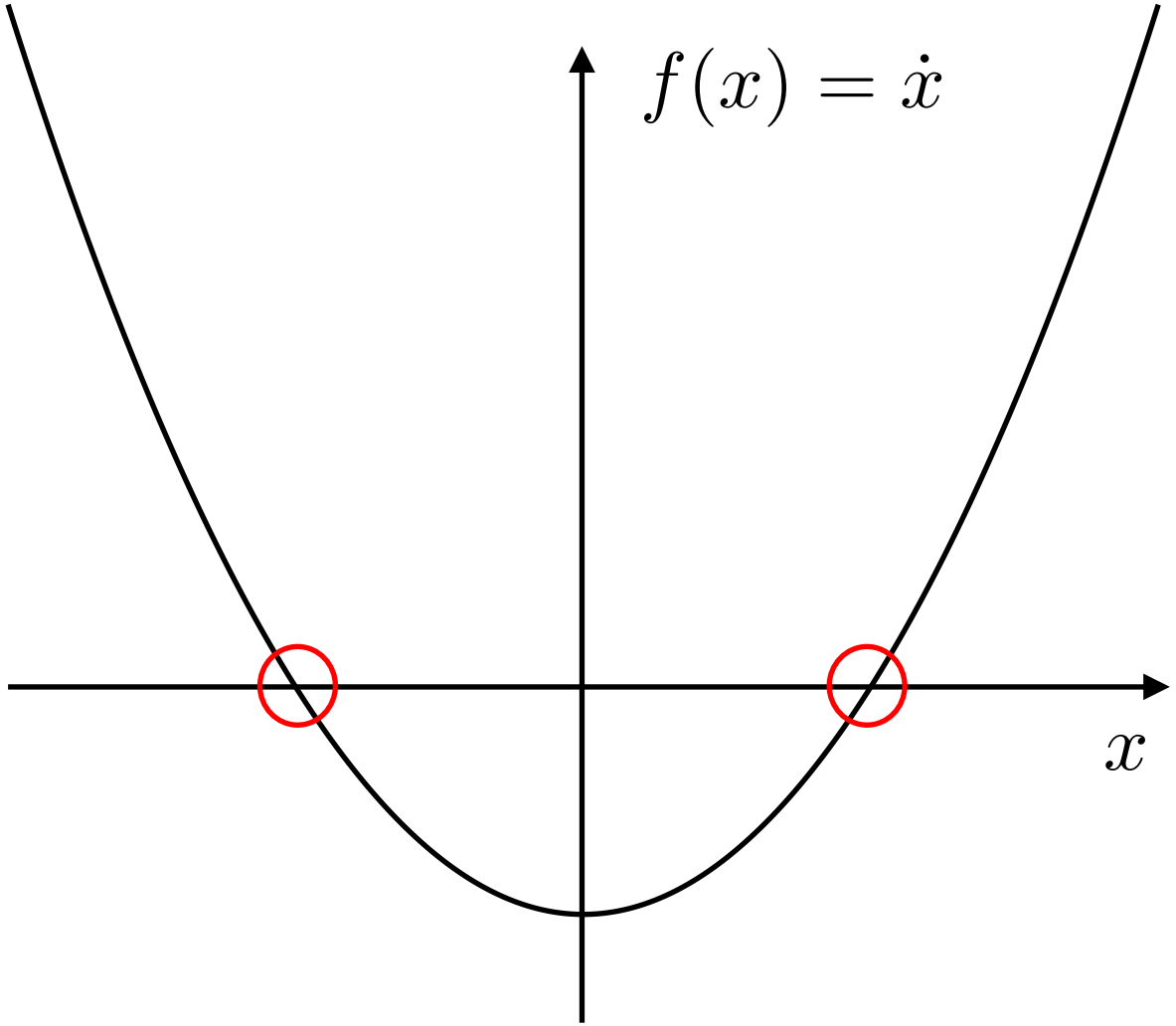

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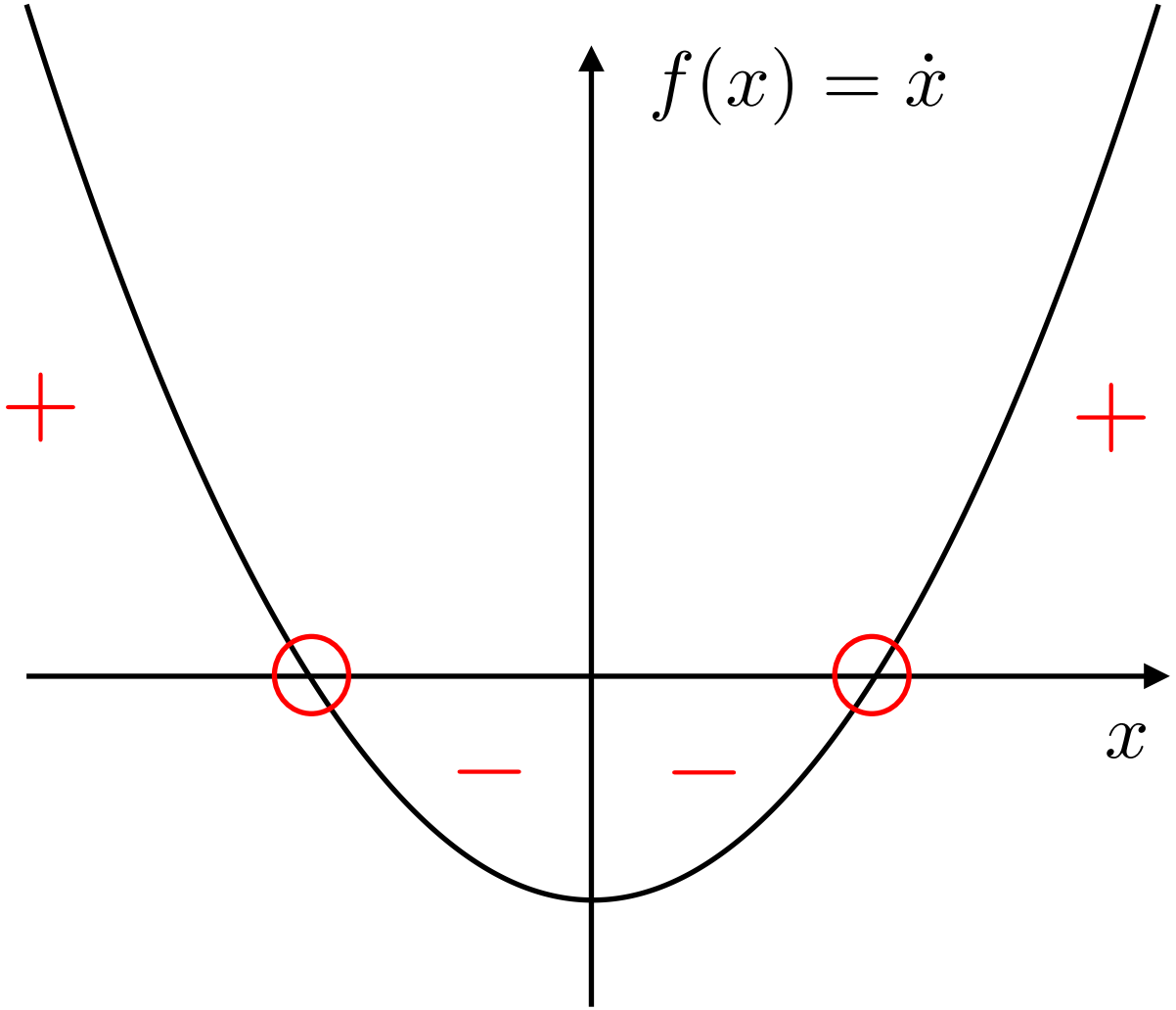

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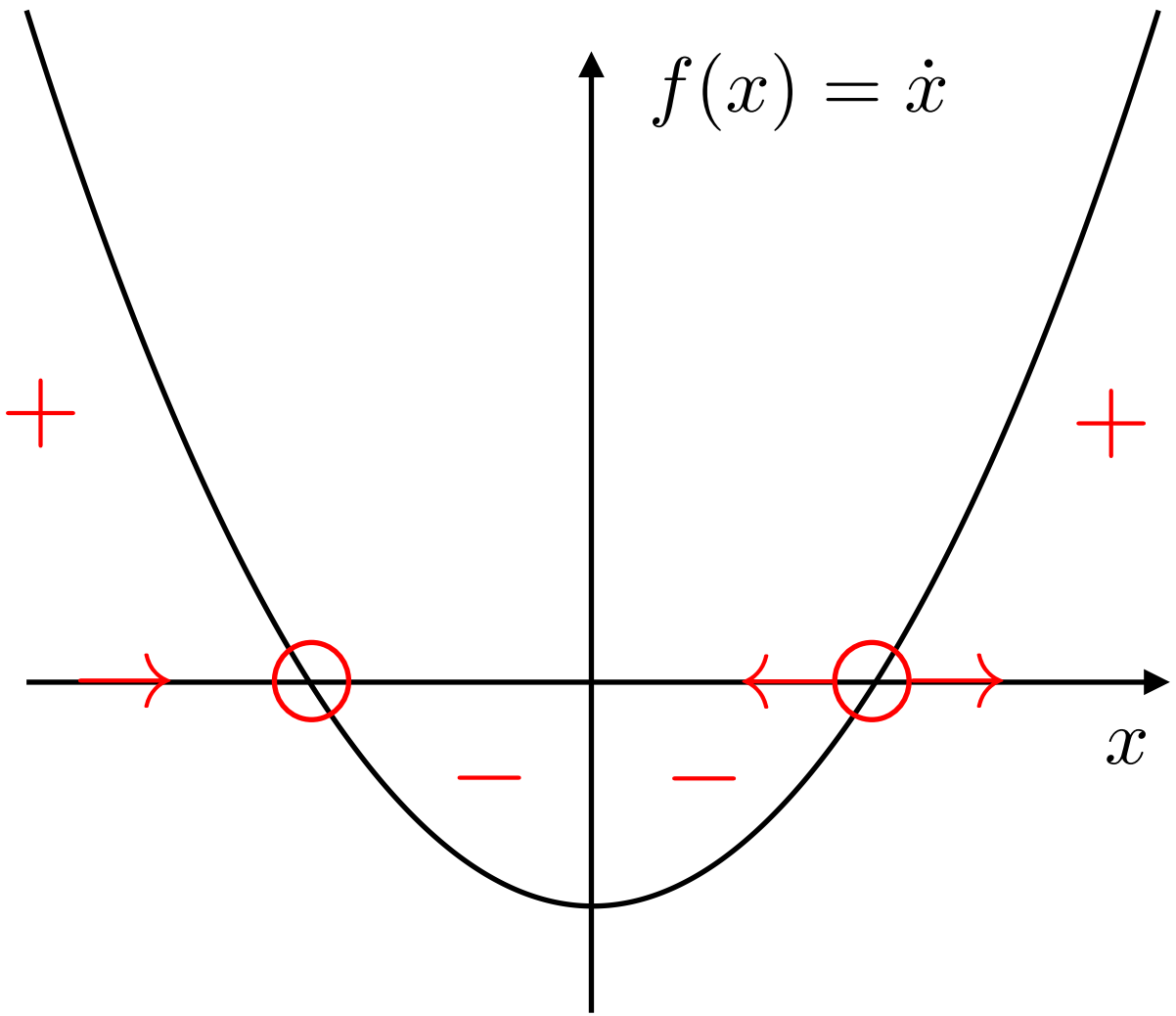

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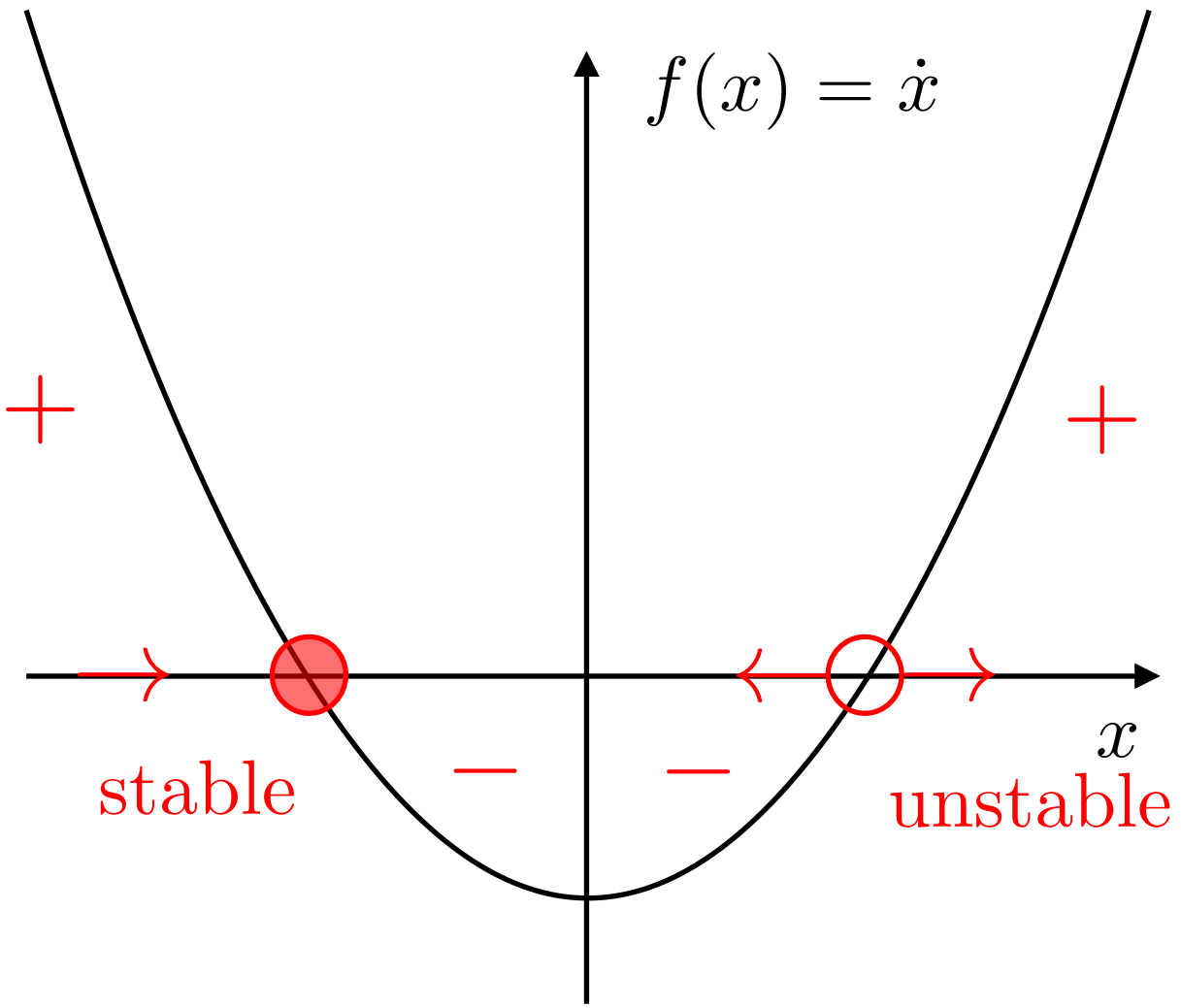

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Population Growth

- Suppose you have a species and you're interested in how it will grow in time or if it could possibly die out
- The species has population $N(t)$ at time t
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- In this simple model, the population will grow indefinitely

Population Growth

In the model $\dot{N} = rN$

- We did not specify what may cause the population to die
- We did not consider how the presence of a **carrying capacity** may hinder growth
- We also did not consider what happens if a disease breaks out

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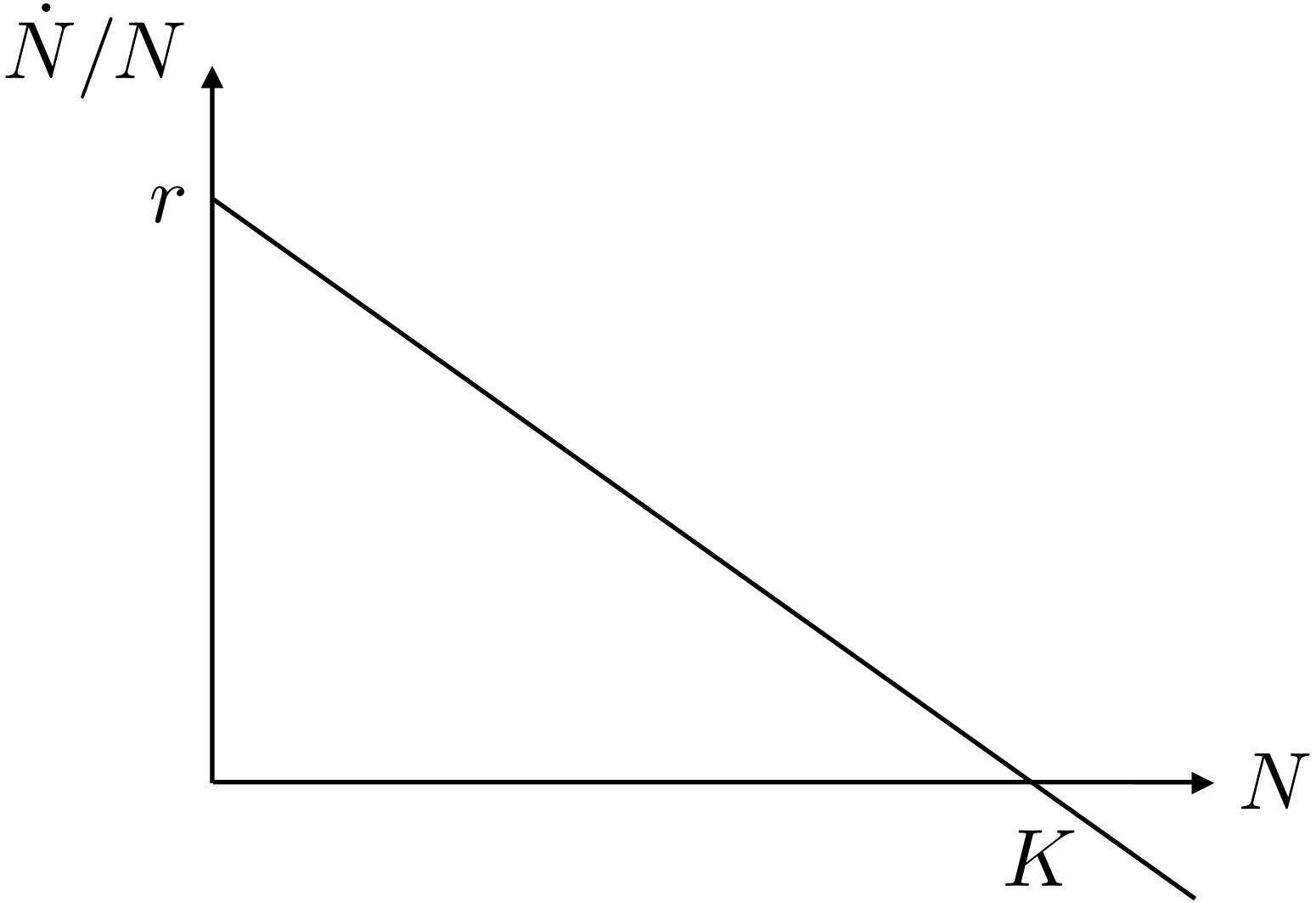
To model the effects of **overcrowding**, demographers usually assume that the per-capita growth-rate $\dot{N}/N$ decreases as N becomes sufficiently large

When N is small, growth rate is r; when N is larger than a carrying capacity K, growth rate is negative (death rate higher than birthrate)

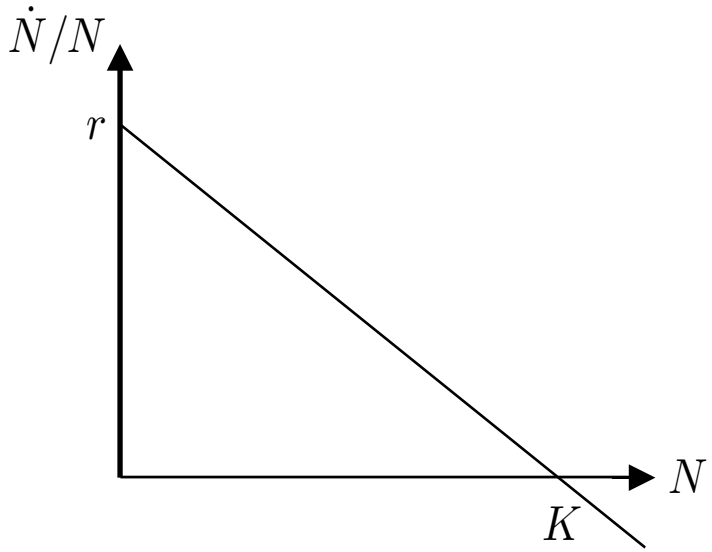
Population Growth



Population Growth: the logistic equation

This leads to the **logistic model** for population growth

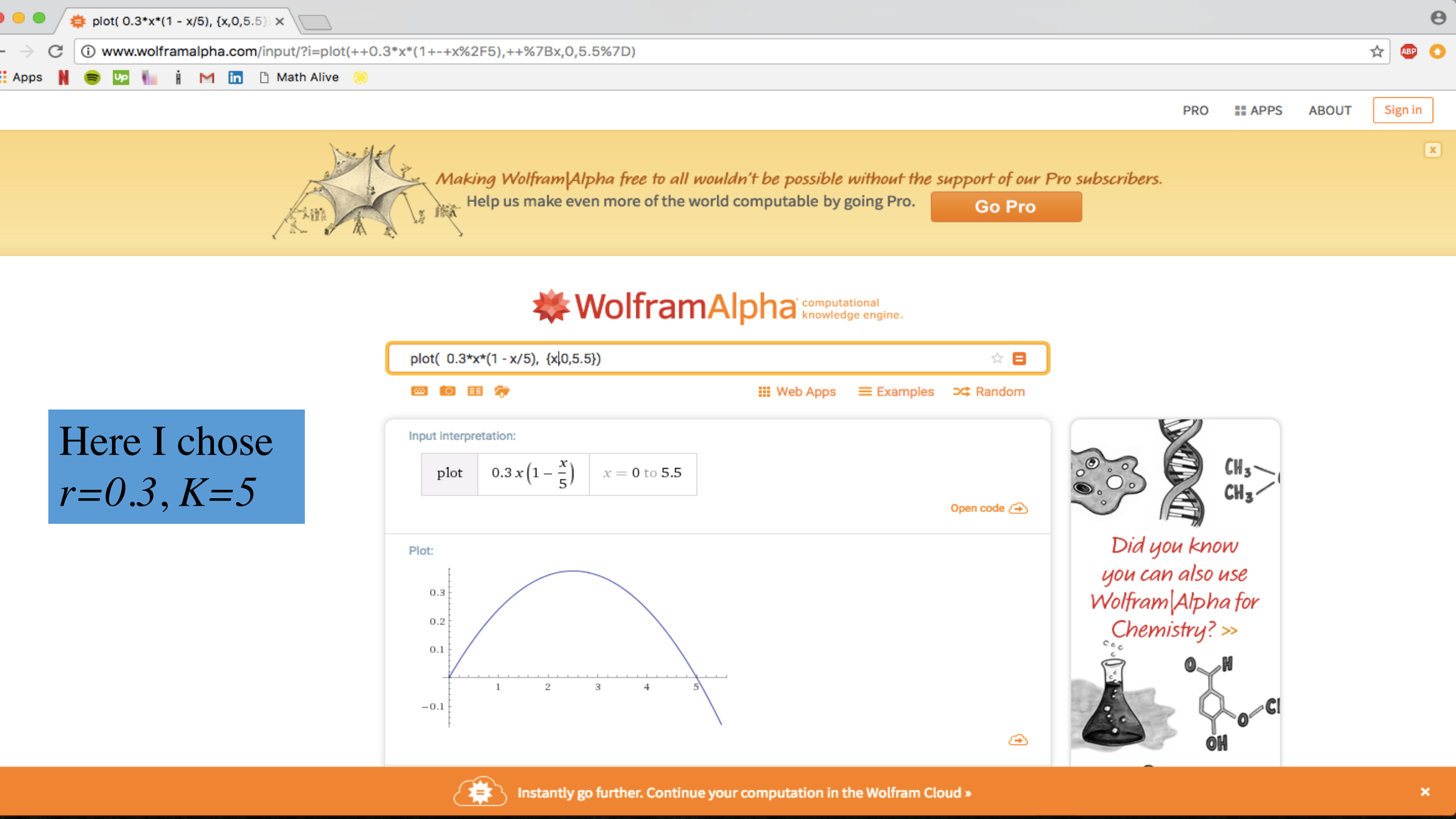
$$\dot{N} = rN\left(1 - \frac{N}{K}\right)$$



How to solve this?

1. $N \geq 0$: a negative population makes no sense
2. We plot the function to find the fixed points
3. We find the values of these fixed points and analyze what they mean

Here I chose
$r=0.3$, $K=5$



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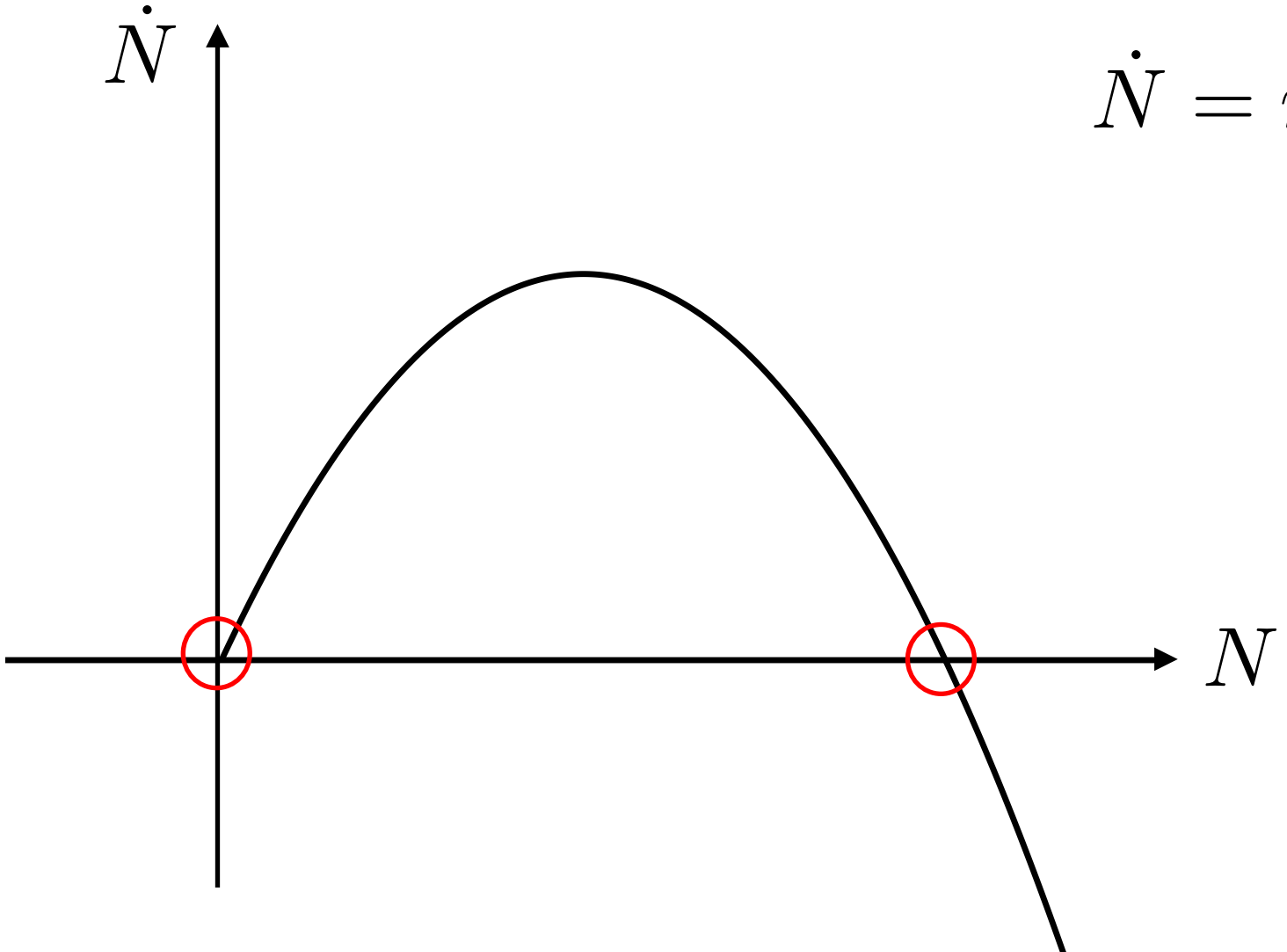

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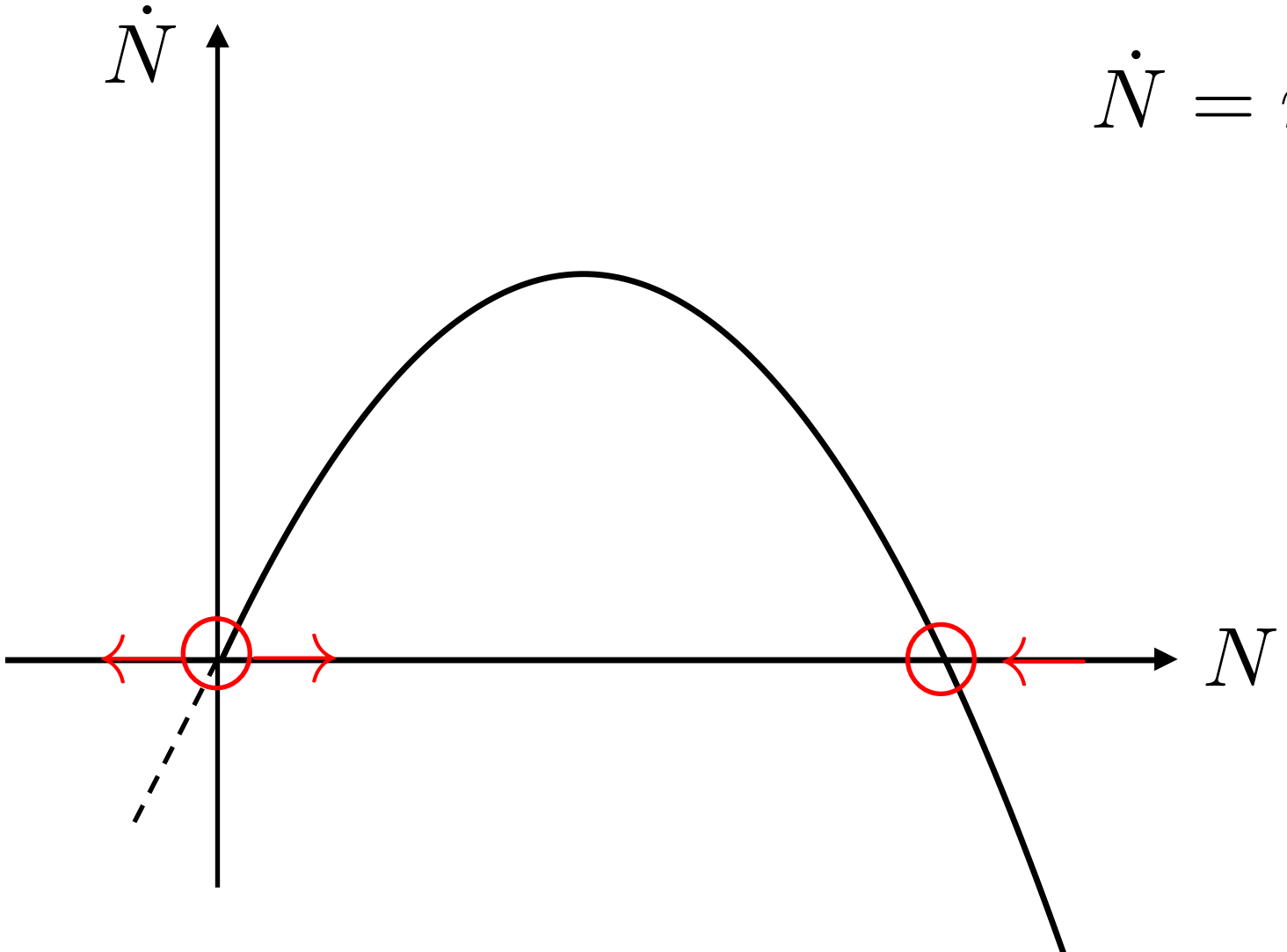

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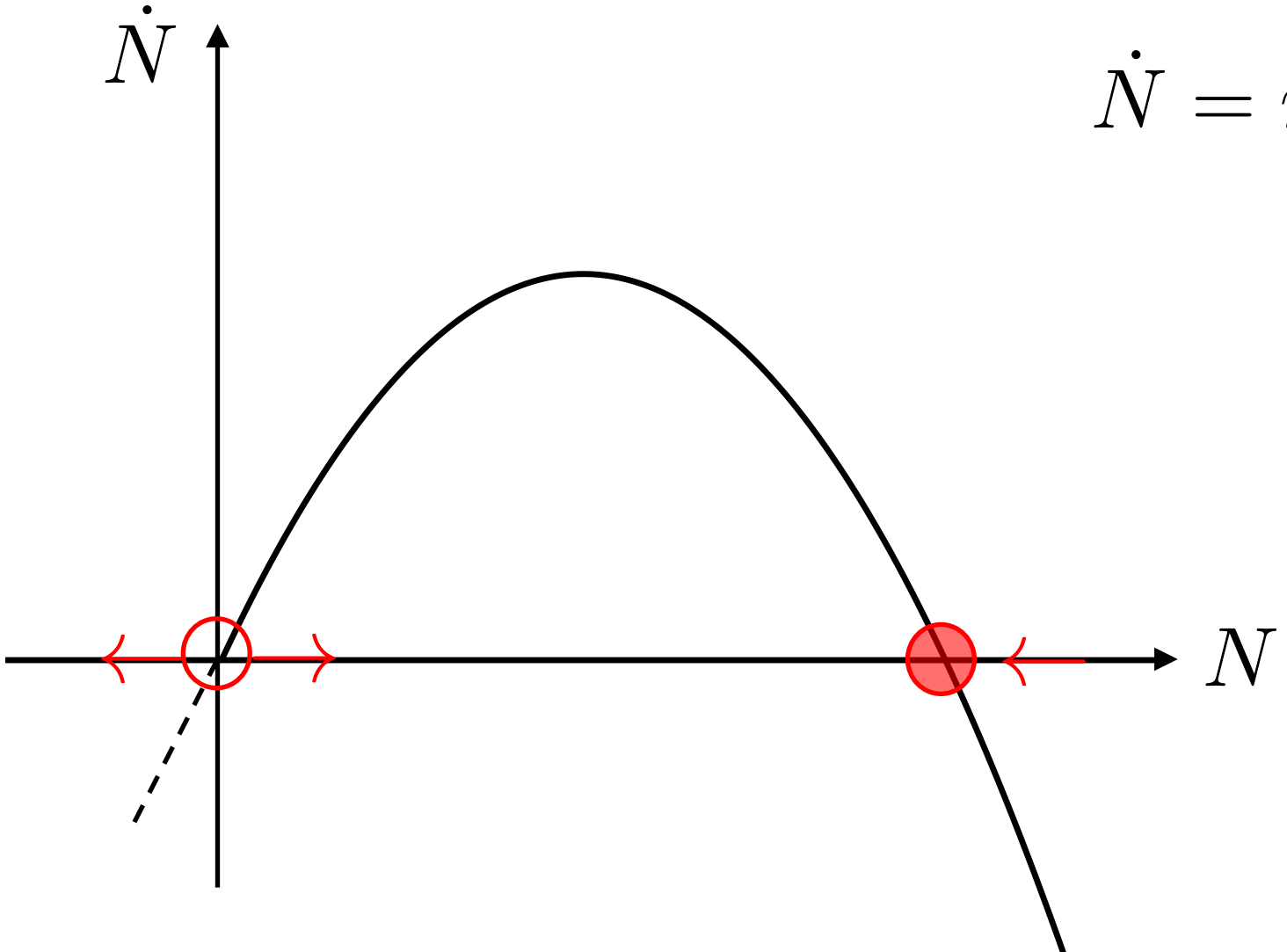

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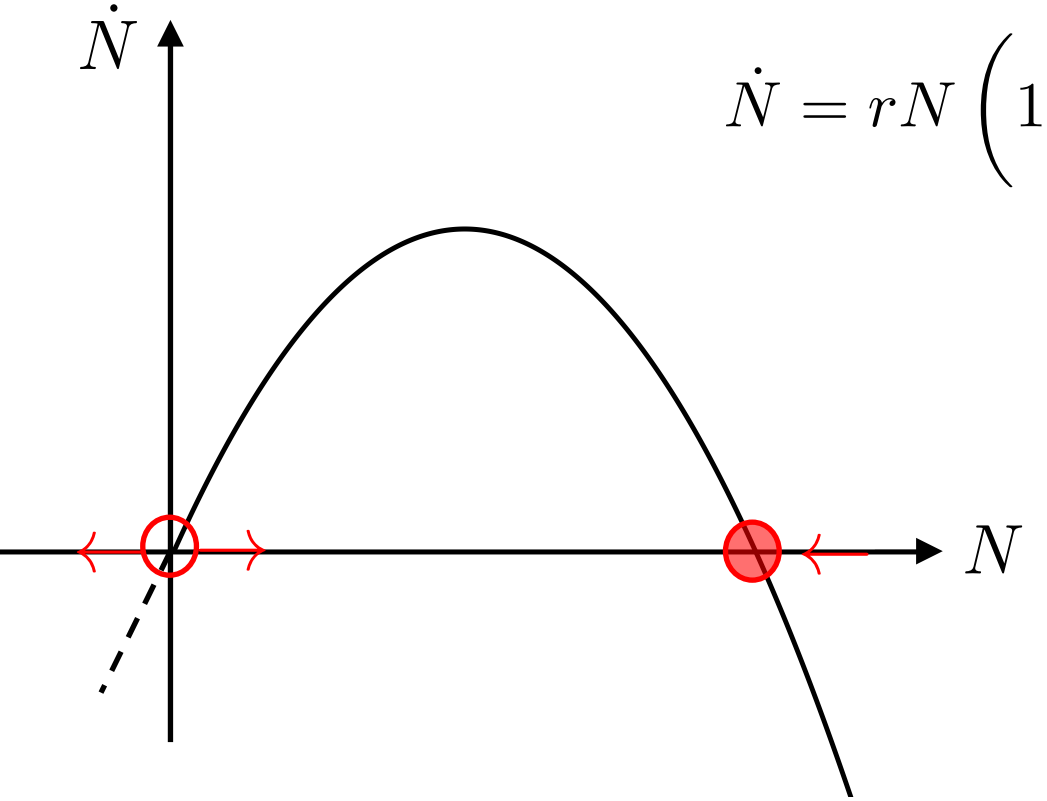

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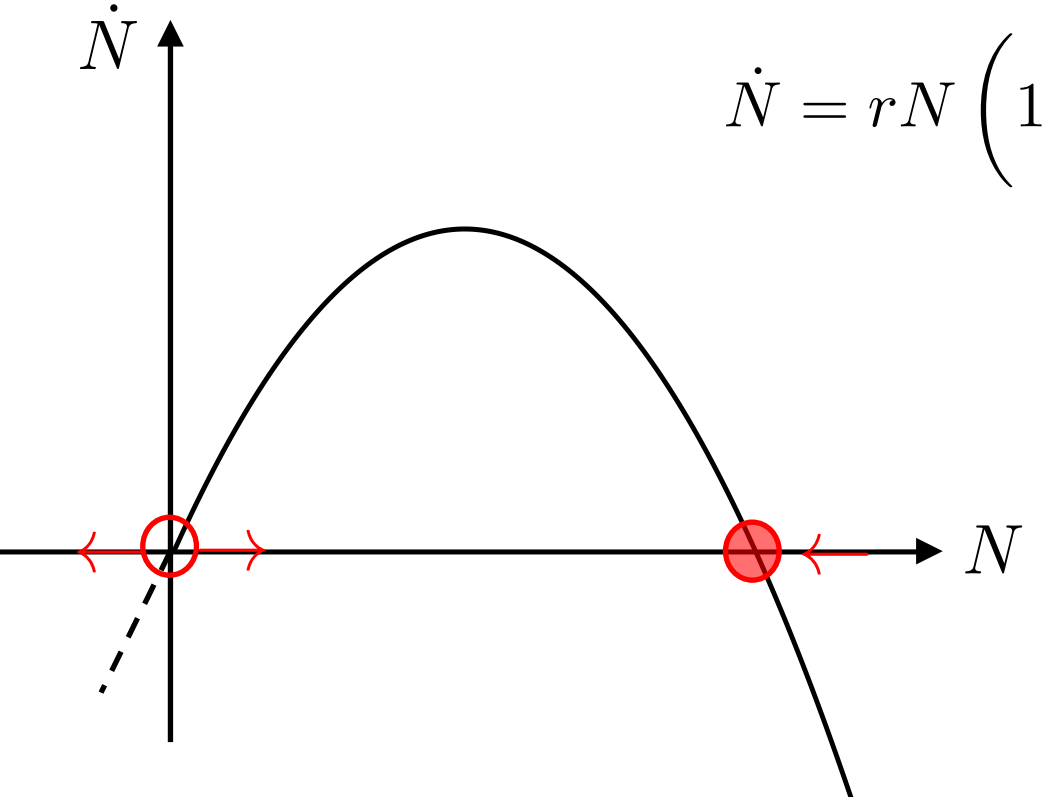

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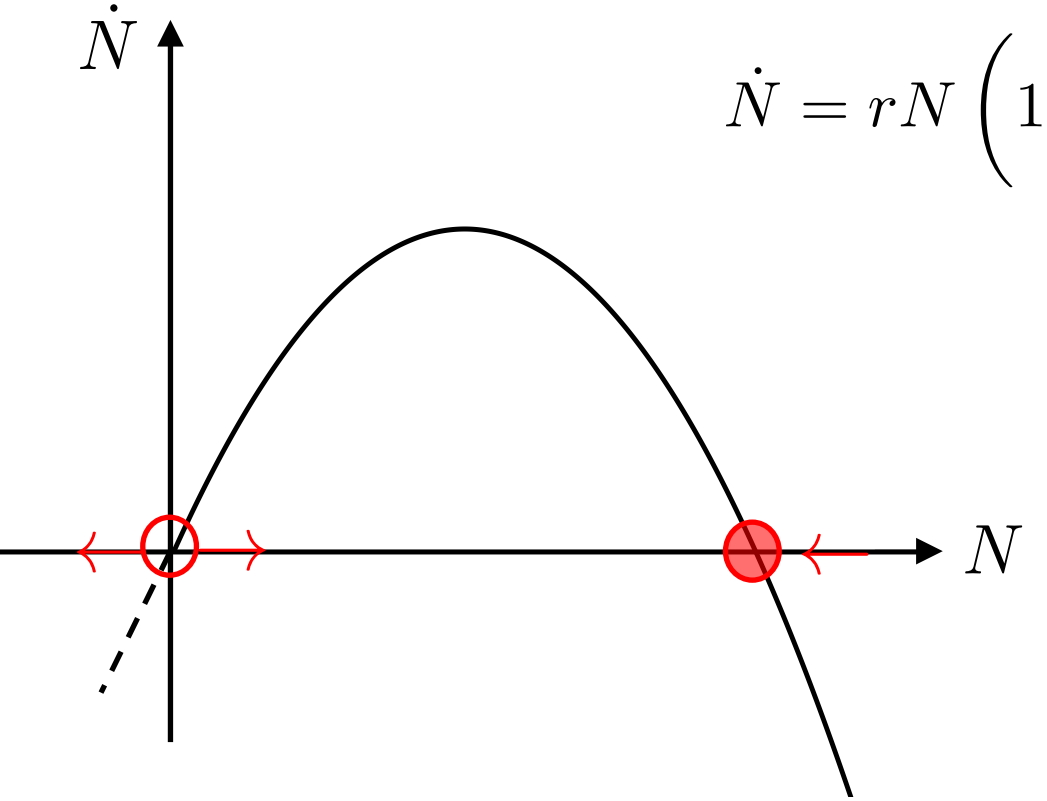

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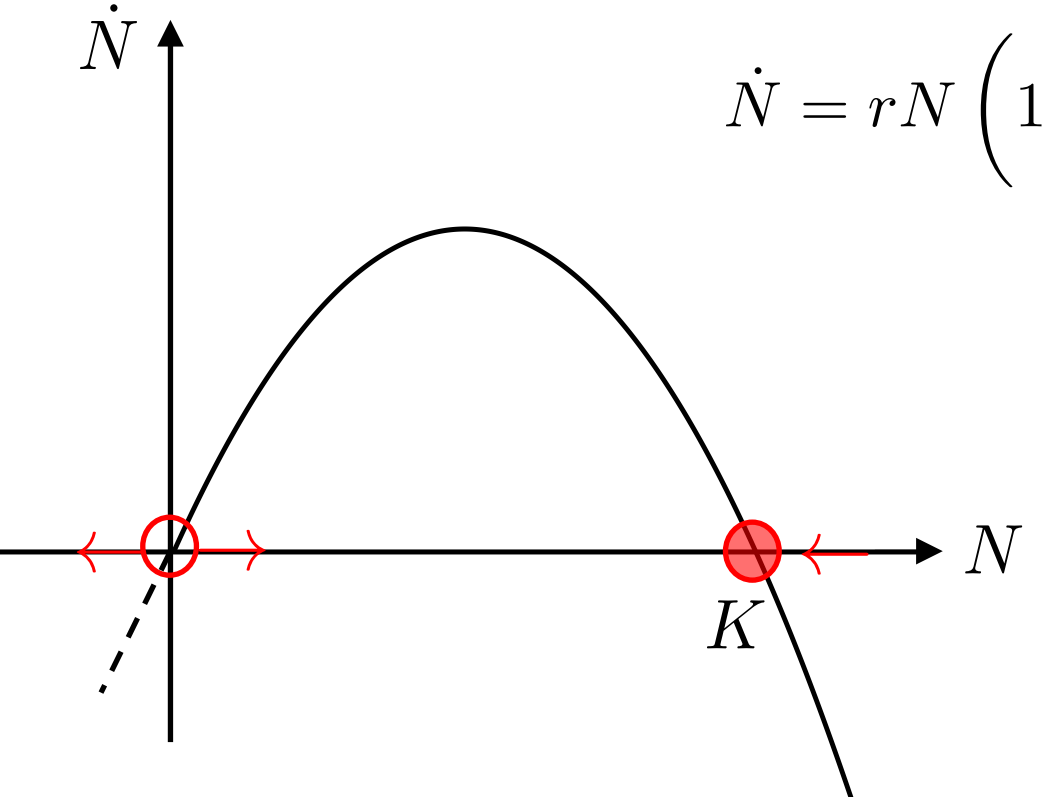

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