V Game Theory

V.1 What is Game Theory and an Outline of Two Lectures

Game theory is the study of situations of conflict and cooperation. We define a game as a situation where (i) there are at least two players; (ii) each player has a number of possible strategies that determine the outcome; (iii) associated with each outcome is a set of numerical payoffs to each player.

Of course, games can encompass a huge variety of situations, ranging from traditional games like chess or poker, to social, psychological, biological, or economic games. With only two lectures we are forced to give only a taste of some of the fascinating topics in the mathematical theory of games. Our topics are roughly divided into three topics:

(a) An intriguing game known as "Newcomb's Paradox" that rose to prominence in the 1970s, and that illustrates the paradoxical nature of game theory

(b) Dominated strategies, maximin and minimax

(c) Mixed strategies

There is a somewhat polarized view of game theory in the literature, with some critics that argue that the underlying assumptions of the field are too restrictive, and that the theory is in fact, rather limited in practical use (Martin, 1978). However, as honourary mathematicians in Math Alive, we should view Game Theory from a different light: in a sense, the mathematical analysis of any game will follow the same trend as any other scientific or academic pursuit: the object is to take a real-life situation, reduce it down to its most essential components, and to understand the complexity of the problem beginning from this distilled form.

The point, here, is to learn how very complicated real-life situations can be simplified and studied mathematically and logically.

Most of the material in these two lectures (and many of the examples) is centered upon the very readable book by Straffin (1993), which is suitable for high school students and undergraduates of all disciplines. There are many game theory texts out there, but we can also recommend Binmore (2012) as another lucid and entertaining read.

V.2 Newcomb's problem

For the last ten years, you have been fanatically watching a television gameshow called "*Faith*". The rules of the gameshow are simple: every day, a single contestant is brought onto the stage and stands before two opaque boxes, and Patrick, the enigmatic host.

The left box (Box #1) always contains $1000 in cash. The right box (Box #2) either contains one million dollars or nothing at all. The contestant is offered two choices:

(a) Select both boxes, or

(b) Select Box #2.

There is no physical trickery. Nothing is removed and nothing disappears from the two boxes at this point. The contestant must simply make the choice so as to obtain the best payoff.

The Mentalist. There is a twist. Patrick, the host, is famous for his ability to judge a person's character. You may believe, if you wish, that Patrick possesses psychic abilities or cognitive superpowers. Or more realistically, he might be an extremely perceptive individual and an expert psychologist. The exact mechanism for his abilities is irrelevant. Each day before the game begins, Patrick is handed a dossier that contains all the researched details of this contestant's life: their family, childhood, education, finances, social circles, and so on. By studying the dossier and applying his abilities, Patrick makes a prediction of whether the contestant will choose both boxes or Box #2.

(a) If he believes the contestant will select both boxes, then the right box is kept empty.

(b) If he believes that the contestant has faith, and will select only the right box, then he will ensure there is one million dollars in that box.

Now the gameshow begins, and the contestant is ushered to the stage, and stands before the two boxes.

"I have studied this contestant's past," Patrick announces to the audience, "and I have predicted his actions. Does he have faith in my prediction?"

And then the contestant chooses. Either the contestant takes both boxes (no faith in the host), or takes the one box (faith in the host).

Your observations. For the last ten years, you have been observing this game on the television, and you have been compiling your statistics. You have observed that in a total of 10,000 games, only twice has Patrick been wrong. In other words, in 99.98% of the cases, selecting both boxes would result in only $1,000, whereas selecting Box #2 would award 1 million dollars.

Your turn and the two arguments. Today, you have been given a chance to play the game. You're standing on the stage and it's now your turn to choose. Do you have faith?

Argument 1: Have faith and take Box #2

The host has been observed to possess a 99.98% accuracy rate in predicting the contestants. By choosing Box #2, you will almost certainly get one million dollars. By choosing both boxes, you will almost certainly lose the million dollars. Therefore, take Box #2.

Argument 2: Take both boxes

What does it matter what the host predicted? Either there is one million dollars in the right box, or there isn't. Your decision doesn't make the contents of the box vanish. By taking both boxes, you either win \$1,000 or \$1,001,000. By taking one box, you may leave with nothing. Therefore, you should take both boxes.

What is the correct answer?

To almost everyone it is perfectly clear and obvious what should be done. The difficulty is that these people seem to divide almost evenly on the problem, with large numbers thinking the opposing half is just being silly.

What makes this question so interesting , and why it inspired such debate, is that there is no agreed-upon answer. In the following lecture, we will see the game theoretic principles behind Arguments 1 and 2.

V.3 Matrix Games

We begin with the simplest versions of a game played between two players, Rose and Colin. Rose can choose amongst three strategies, whereas Colin chooses amongst two. Their payoffs are represented in table form (a "matrix"):

		Colin	
		A	B
Rose	A	2, -2	-3, 3
	B	0, 0	2, -2
	C	-5, 5	10, -10

To play the game, both players secretly write down their chosen strategy on a piece of paper. On cue, then they turn over their papers and determine their payoffs. If, for example, Rose plays A and Colin plays A, then Rose gets 2 and Colin loses 2. Notice that in the above game, the payoffs for each outcome sum to zero.

Definition (Zero-sum game). A zero-sum game is one in which the sum of the individual payoffs for each outcome sum to zero.

We focus on zero sum games primarily because their mathematical theory is simpler. Since in such games, Colin will simply lose what Rose has gained (and vice versa), then it is sufficient for us to only write down the payoffs for Rose.

Suppose that Rose plays C hoping Colin will play B so she gets 10. Knowing this, Colin realizes that it is better to play A, so that he earns 5. Foreseeing this, Rose decides it would be better to play A. Round and round it goes. In this case, we see that this type of reasoning does not leave to a resolution that is appealing to either player.

V.4 Dominance, saddle points

Consider the following zero-sum game:

		Colin A	Colin B	Colin C	Colin D
Rose	A	12, -12	-1, 1	1, -1	0, 0
Rose	B	5, -5	1, -1	7, -7	-20, 20
Rose	C	3, -3	2, -2	4, -4	3, -3
Rose	D	-16, 16	0, 0	0, 0	16, -16

As we mentioned earlier, it is sufficient for us to only write down the payoffs for Rose. Therefore, we can write the above game matrix as,

		Colin A	Colin B	Colin C	Colin D
Rose	A	12	-1	1	0
Rose	B	5	1	7	-20
Rose	C	3	2	4	3
Rose	D	-16	0	0	16

Notice that the game is biased towards Rose (see Rose C). Moreover, Colin C is a particularly bad strategy since, regardless of what Rose plays, Colin B will always be preferable to Colin C. We say that B dominates C. This leads us to formulate the dominance principle.

Proposition (Dominance Principle). *A rational player should never play a dominated strategy.*

Moving on, we see that the CB entry (Rose C, Colin B) is special. It is the most cautious strategy: by playing C, Rose is assured to win at least 2 regardless of what Colin does; similarly, by playing Colin B, Colin is assured to lose no more than 2. We call this an *equilibrium* outcome or *saddle point*. If Colin knows that Rose will play C, he will play B. If Rose knows that Colin will play B, she will play C. Neither player has any apparent incentive to change their strategy. This leads us to formulate the following definition

Definition (Saddle point). *In a zero-sum matrix game, an outcome is a saddle point if the outcome is a minimum in its row and maximum in its column.*

The argument that players will prefer not to diverge from the saddle point leads us to offer the following principle of game theory:

Proposition (Saddle Point Principle). *If a matrix game has a saddle point, both players should play it.*

Finding the saddle point. There is a convenient algorithm to finding the saddle point of any zero-sum two-player game. This is illustrated below.

		Colin				
		A	B	C	D	row min
	A	12	-1	1	0	-1
Rose	B	5	1	7	-20	-20
	C	3	2	4	3	$\boxed{2}$
	D	-16	0	0	16	-16
column max		12	$\boxed{2}$	7	16	

Rose's strategy is to select her maximin (the maximum of the row minima), while Colin's strategy is to select his minimax (the minimum of his column maxima). By doing so, each player has chosen the most cautious strategy, for which in Rose?s case, she can guarantee a payoff of at least her maximin value, and Colin can guarantee a loss of at most his minimax value. Assuming that

$$\text{minimax(columns)} = \text{maximin(rows)} \tag{1}$$

then there exists a saddle point. Notice that if a saddle point exists, it may not necessarily be unique. However, if multiple saddle points exist, then they must be equal in value.

V.5 Mixed Strategies

However, not all games have saddle-point solutions. For example, consider the following game based on matching pennies: both players hold a penny in their hands, heads up (H) or tails up (T). Upon cue, they show the other player their choice. Rose wins if the two sides match, and loses otherwise. Rose's payoffs are shown in the game matrix below:

		Colin	
		H	T
Rose	H	1	-1
	T	-1	1

For this game, there is no saddle point, and indeed, no optimal pure strategy. After all, if Rose did have an optimal strategy she should play, Colin would simply play the opposite in order to counter. There is no pure solution to this game, but we find a resolution by playing a mixed strategy, whereby players are allowed to use a random device to decide which strategy to play. For example, Colin may instead flip a coin into the air in order to help decide whether to play H or T. This would give him a mixed strategy of $\frac{1}{2}$H and $\frac{1}{2}$T. Rose will do the same, and this guarantees that in the long run, both players neither win nor lose.

In order to better illustrate, we use a slightly more non-trivial game. Suppose we have the below matrix game

		Colin	
		A	B
Rose	A	2	-3
	B	0	3

Recall the expected value: given events with probabilities $p_1, p_2, \ldots, p_n$, and pay-offs $a_1, a_2, \ldots, a_n$, the expected value is

$$\text{Expected value} = E = p_1 a_1 + p_2 a_2 + \cdots + p_n a_n. \tag{2}$$

Suppose that Rose knows that Colin will play a mixed strategy of $\frac{1}{2}$A and $\frac{1}{2}$B. If she plays 100% of A, then she can expect

$$E(A, \tfrac{1}{2}A + \tfrac{1}{2}B) = \left(\tfrac{1}{2} \times 2\right) + \left(\tfrac{1}{2} \times (-3)\right) = -\tfrac{1}{2}, \tag{3}$$

so if this game were played many times, Rose would expect to win on average -1/2 per game. On the other hand, if she plays 100% of B, she would earn

$$E(B, \tfrac{1}{2}A + \tfrac{1}{2}B) = \left(\tfrac{1}{2} \times 0\right) + \left(\tfrac{1}{2} \times 3\right) = \tfrac{3}{2}. \tag{4}$$

Clearly, if Rose knows or guesses that Colin is playing the mixed strategy of $\frac{1}{2}$A and $\frac{1}{2}$B, then by playing B, she is ensured a better payoff. This leads us to formulate another principle of game theory:

Proposition (Expected Value Principle). *If you know your opponent is playing a mixed strategy and will continue to play it, you should use a strategy that maximizes your expected value.*

Now, however, Colin wishes to maximize his own payoffs given that Rose is either playing A or B. If we assume that Colin will play a mixed strategy with probabilities p for A and $(1-p)$ for B (where p lies between 0 and 1), then his expected values are

$$E_C(A, pA + (1-p)B) = (p)(-2) + (1-p)(3), \tag{5a}$$
$$E_C(B, pA + (1-p)B) = (p)(0) + (1-p)(-3). \tag{5b}$$

Now, if one of these payoffs is higher than the other, then this means Rose possesses a strategy that can take advantage of Colin's strategy. Then, he should perhaps choose p so that both these values are equal

$$-2p + 3(1-p) = -3(1-p) \implies p = 3/4. \tag{6}$$

Thus, if Colin plays $\frac{3}{4}$A and $\frac{1}{4}$B, he will ensure that Rose wins no more than $\frac{3}{4}$ per game. Now by the same logic, Rose needs a mixed strategy according to probabilities q for A and $(1-q)$ for B. Her expected values are

$$E_R(qA + (1-q)B, A) = (q)(2) + (1-q)(0), \tag{7a}$$
$$E_R(qA + (1-q)B, B) = (q)(-3) + (1-q)(3). \tag{7b}$$

Again, if Rose chooses q so that both these values are equal, then there is no choice Colin can take that will take advantage of Rose's mixed strategy. Thus

$$2q = -3q + 3(1-q) \implies q = 3/8. \tag{8}$$

If Rose plays $\frac{3}{8}$A + $\frac{5}{8}$B, she will be guaranteed to win at least on average 3/4 per game. In other words, if

- Colin plays $\frac{3}{4}$A + $\frac{1}{4}$B,
- Rose plays $\frac{3}{8}$A + $\frac{5}{8}$B,

then Rose is assured to win at least $v = 3/4$ (Rose's value), and Colin is assured to lose no more than $v = 3/4$ (Colin's value). Neither player has much (apparent) incentive to diverge from this strategy. The value of the game is $v = 3/4$, and we consider this solution to be optimal.

In fact, we can say much more than this. Given any $m \times n$ zero-sum matrix game, it must always possess an optimal solution. This was proven by John von Neumann in 1928 (see von Neumann (1959) for a translated version in 1959).

Theorem (Minimax Theorem). *Every $m \times n$ matrix game has a solution. That is, there is a unique number, v, called the value of the game, and there are optimal (pure or mixed) strategies for Rose and Colin such that*

(1) If Rose plays her optimal strategy, Rose's expected payoff is $\geq v$, no matter what Colin does.

(2) If Colin plays his optimal strategy, Rose's expected payoff is $\leq v$, no matter what Rose does.