The mathematics of voting, power, and sharing – Part 1

Voting systems

A voting system or a voting scheme is a way for a group of people to select one from among several possibilities.

If there are only two alternatives between which to choose, then it is easy: the alternative that is preferred by the majority wins. (Difficulty only arises if there is a tie.)

If there is only one person doing the choosing, then things are easy again: you know your own mind, and you know you prefer A to B, and B to C, and C to D. And if by any chance alternative B were to disappear, then you still prefer A to C and C to D.

It is when several people have to choose among more than two alternatives that things become trickier.

The following is a simple example that illustrates one of the oldest voting paradoxes. Suppose a sizable group of people (say 60) will meet for a celebration in a restaurant, and the restaurant manager wants them to pick one menu for the whole group. For the main courses, she initially offers the choice between salmon and chicken. The organizers consult their group, and find that a majority prefers salmon. But when they call up the restaurant owner, she apologizes: her fish supplier has become less reliable, and she is now offering a choice between chicken and beef instead. After the group is consulted, it turns out that a majority now prefers the chicken choice. In summary, the group

- prefers salmon over chicken;
- prefers chicken over beef.

A day later, the restaurant manager calls back; she has switched to another supplier, who has a superb fish assortment, and she can again offer salmon. On the other hand, the Department of Agriculture recently destroyed large quantities of chicken because they had microbial contamination, so the choice is now between salmon and beef. The organizers feel sure, in view of the ranking above, that their group will largely prefer salmon, but when they ask, they find a clear majority for beef: the group

- prefers beef over salmon.

"Oh well," they think, "people are fickle, and some of them must have changed their minds." Yet, this was not the case: every single person polled had a clear ranking of the three possibilities and stuck to that ranking in a consistent way. Nonetheless, even though every single individual is entirely consistent, the group is not. Let's look at a numerical example: suppose that

- 25 people rank
  1. salmon
  2. chicken
  3. beef
- 20 people rank
  1. chicken
  2. beef
  3. salmon
- 15 people rank
  1. beef
  2. salmon
  3. chicken

So, all in all, 40 people rank salmon higher than chicken, another group of 45 ranks chicken higher than beef, and yet a different grouping of 35 ranks beef higher than salmon: the paradoxical behavior of the group is explained.

This kind of paradox in fact happens all the time, and for things far more serious than menus for celebration dinners, such as presidential elections or votes in congress.

In the case where this type of paradox doesn't happen, that is, when there is one alternative that is always preferred by a majority (although not always the same majority) if it were pitted in a one-on-one race against any one of the others, then we call the winning alternative the "Condorcet" winner [this would be the case for the "chicken" choice in the example above if the third group had changed their ordering to 1) beef, 2) chicken, 3) salmon].

We have just seen that there doesn't always exist a Condorcet winner. But when there exists one, it seems fair that that should be the winning choice for the whole group. Or does it?

There are many different systems in existence to select the "winner". Because the Condorcet method doesn't always yield a winner, it is not used a lot. Other methods are:

- **Plurality**: The candidate who is ranked in first place most often, wins. This is the way in which senators or members of congress are elected in the U.S. in every state.

- **Plurality with run-off**: The two candidates with the most first places are retained, and then a second round run-off election is held between them. This is the system used in the election of the president of France.
- **Sequential run-off or the Hare system**: First, the candidate with the fewest first places is removed, then (after his/her votes have been redistributed among the remaining candidates) the next-bottom candidate, and so on .... This system has been used for years in Australia, Ireland, and also in Cambridge, Mass., and in New York City (although not in situations where only one winner has to be selected, but where several seats are available). It was also recently adopted in several local elections in California.
- **Borda count**: If there are N candidates, then every voter gives N points to his/her first, $N-1$ to the second choice, ... The points that all the voters gave are then added, and the candidate with the most points wins. This system is often used in clubs to decide on admission (or not) of new members.

It turns out that different methods can lead to different outcomes ....

Let's illustrate these paradoxical situations with a few more examples.

Example: Paradox with run-off or sequential run-off. For her assignment in *Math Alive*, a student asks 17 of her friends what kind of breakfast they prefer. Here are their answers:

number of people for each ranking	**6**	**5**	**4**	**2**
cereal	1	2	3	2
danish	2	3	1	1
bagel	3	1	2	3

Let's first get rid of the alternative that got fewest first places: bagel (which had 5, while danish had 6, cereal had 6). That leaves cereal and danish.

With only these two alternatives remaining, the preferences are:

	6	**5**	**4**	**2**
cereal	1	1	2	2
danish	2	2	1	1

so that cereal wins, because it has the most first places now.

But if the last group of **2** votes changes it mind, and decides to rank cereal above danish instead of the other way around, what happens then? Surely cereal's chances of winning must be better now?! Let's check:

	6	**5**	**4**	**2**
cereal	1	2	3	1
danish	2	3	1	2
bagel	3	1	2	3

The item with fewest first places is now danish (4 versus 8 for cereal, 5 for bagel). Reassigning danish's votes, we get:

	6	**5**	**4**	**2**
cereal	1	2	2	1
bagel	2	1	1	2

so that bagel wins, and cereal loses, even though more voters preferred cereal than before . . . (one could imagine the following newspaper headline after the Irish elections, which use this "Hare" system: O'Grady loses seat, although he would have won if fewer people had voted for him . . .).

Other example: Paradox with Borda scheme: A club of 25 people are planning an outing. They have narrowed down the choices to a trip to the beach, a hike in the mountains, or a day in NYC. Their preference schedule is the following:

	13	**10**	**2**
beach	2	1	3
mountains	3	2	1
NYC	1	3	2

This is in fact a case where there is a Condorcet winner: in the one-on-one contests NYC always wins:

- beach vs. NYC: 15 people prefer NYC; 10 people prefer beach;
- mountains vs. NYC: 23 people prefer NYC; 10 people prefer mountains.

NYC also wins the plurality vote, and is also the winner under the run-off scheme. In a Borda-count, we find the following totals of points:

$$\begin{array}{rrcrcrcrcr} \text{beach:} & 10 & \times 3 & + & 13 & \times 2 & + & 2 & \times 1 = & 58 \\ \text{mountains:} & 2 & \times 3 & + & 10 & \times 2 & + & 13 & \times 1 = & 39 \\ \text{NYC:} & 13 & \times 3 & + & 2 & \times 2 & + & 10 & \times 1 = & 53 \end{array}$$

This does not lead to the same winner, even though NYC won by several other methods . . . . Many of these paradoxes have been known for a very long time. For a long time also, many people have tried to think of schemes that would avoid such paradoxes. Until, in the early 1950's, Kenneth Arrow attacked the problem in a mathematical way.

He started by listing properties that a "fair" voting scheme should have. Imagine that you have two voters, 1 and 2, and three issues, A, B, and C. A voting scheme is a way of distilling out of the two individual preference schemes a ranking for the "group." We shall say that

$$\begin{array}{lll} A \geq_1 B & \text{if} & \text{voter 1 ranks } A \text{ higher than or equal to } B \text{ (we allow ties here);} \\ A =_1 B & \text{if} & \text{voter 1 ranks } A \text{ and } B \text{ at the same level;} \\ A >_1 B & \text{if} & \text{voter 1 prefers } A \text{ to } B \text{ } (i.e., A \geq_1 B \text{ and } \textbf{not } A =_1 B). \end{array}$$

Likewise, for voter 2, we have

$A \geq_2 B$	if	voter 2 ranks A higher than or equal to B (we allow ties here);
$A =_2 B$	if	voter 2 ranks A and B at the same level;
$A >_2 B$	if	voter 2 prefers A to B (i.e., $A \geq_2 B$ and **not** $A =_2 B$).

Now we want to design some procedure for determining from the individual schedules a group preference schedule. For the overall group preference, we use the same symbols, but without indices:

$A \geq B$ (no index)	if	the group outcome for A is higher than or equal to that for B;
$A = B$	if	A and B tie;
$A > B$	if	A is preferred to B,

we do want that procedure to satisfy the following requirements:

1. **(It Shouldn't Hurt if You Go Up in Someone's Opinion)** Suppose that the group rating for x and y (where x and y can be any two choices from A, B, C) is $x \geq y$, starting from preference schedules of the individual voters 1 and 2. If we change the individual schedules so that x either moves up or stays the same in each, and y stays ranked at the same place, then the new social choice should still rank x higher or equal to y. The same statement is true for $x > y$: raising x in individual schedules preserves the higher ranking of x.

2. **(Only Relative Rankings Count)** If the only change in the individual preference schedules concerns the ranking of the third alternative z, but the individual relative rankings of x and y do not change, then the social ranking of x and y should not change.

3. **(Fate Doesn't Stop x From Winning)** For each choice of x, y among A, B, C, there exists some individual preference schedules that lead to $x > y$.

4. **(No Dictators)** For each choice of x, y among A, B, C, there is a combination of individual preference schedules in which $x >_1 y$, yet it does not follow that $x > y$ (otherwise, 1 would be a dictator); the same holds for 2.

These are all very reasonable assumptions. But then, Arrow showed that:

> There exists **no** possible procedure for deriving $\geq, >, =$ from the $\geq_1, >_1, =_1, \geq_2, >_2, =_2$ that satisfies all these assumptions.

In principle, you could do a computer search among all possibilities, and show that there is something wrong. But you can also argue the case as follows. We are going to derive a few consequences of our conditions $1 \rightarrow 4$ that will contradict each other.

- First of all, if the individual preference schedules are such that $A >_1 B$ and $A >_2 B$, then we must have $A > B$. (Same for other choices of the two alternatives to be ranked.)

  So we have preference schedules in which $A >_1 B$ and $A >_2 B$, and we must show that $A > B$ for the group ranking. We know, from Rule 3, that there are hypothetical schedules for voters 1 and 2 which the result in $A > B$. Let's make hypothetical voters $1'$ and $2'$ whose individual preferences are represented using the symbols $>_1', \geq_1', =_1'$, $>_2', \geq_2', =_2'$, and whose group rankings are denoted using $>', \geq', ='$. Now modify these two hypothetical schedules: raise A to the top on both individual schedules if it isn't there yet. This does not affect the group preference of A over B (by Rule 1). With these modifications, we end up with new hypothetical voters $1''$ and $2''$ and preference schedules (denoted by $>_1'', >_2''$) for which $A >_1'' B$, $A >_2'' B$, and $A >'' B$. Now we can modify the preference schedules $>_1'', >_2''$ further to move C in each of the two preference schedules to the place where it sits in $>_1$ and $>_2$. This last modification turns the schedules $>_1'', >_2''$ exactly into the schedules we were originally given $>_1$ and $>_2$. Since the group ranking of A, B is only affected by the individual rankings of A and B (independently of where the individuals place C) by Rule 2, and since we had $A >'' B$, we must also have $A > B$.

- Next, suppose that whenever $A >_1 B, B >_2 A$, we always find that $A > B$. Then the following argument shows that this would mean that $A >_1 B$ implies $A > B$, regardless of the schedule of 2 (which would be in contradiction with condition 4!).

  We want to show that, if $A >_1 B, B >_2 A$ together always imply $A > B$, then voter 1 is acting as a dictator in choosing A. To prove this dictatorship, take a preference schedule for 1 such that $A >_1' B$ and any preference schedule $>_2'$ for voter 2. We will proceed to show that the group outcome for these is necessarily $A >' B$.

  Modify the schedules as follows: leave schedule 1 alone, but push A to the bottom of schedule 2. Then we have modified schedules $>_1''$ and $>_2''$ wherein $A >_1'' B$ and $B >_2'' A$. By our assumption that $A >_1 B, B >_2 A$ together always imply $A > B$, we must have that the group preference for the modified schedules is $A >'' B$. But now reverse the modification: move A up on the schedule of voter 2 (if any movement is needed) to put it back where it was on schedule $>_2'$. By Rule 1, this means that A cannot fare worse in an election based on $>_1', >_2'$ than in an election based on $>_1'', >_2''$. Since $A >'' B$, we must also have $A >' B$. So A is necessarily given an overall ranking over B when we assume that voter 1 ranks A over B (we assumed $A >_1' B$, but nothing about $>_2'$).

  So this contradiction with the dictator's rule means that it cannot be that $A >_1 B, B >_2 A$ always implies $A > B$ (same for other pairs).

- Next, we show that $A >_1 B$ and $B >_2 A$ **must** imply $A = B$.

  Suppose that we have some schedules $>_1', >_2'$ such that $A >_1' B$, $B >_2' A$, and $A >' B$. Since only relative rankings count (Rule 2), it would follow for **all** schedules that have $A >_1 B$ and $B >_2 A$ that $A > B$. But that (as we just saw) is **not** allowed. So we can't have $A >' B$. Similarly, $B >' A$ is excluded. So it follows that $A =' B$ (same again for other pairs).

- Let us now look at the special preference schedule where

$$\begin{array}{c} A >_1 B >_1 C \\ C >_2 A >_2 B \end{array}$$

(i.e., 1 ranks them A, B, C; 2 ranks them C, A, B). Then $A >_1 B, A >_2 B \Rightarrow A > B$. But also

$$\left.\begin{array}{l} A >_1 C, C >_2 A \Rightarrow A = C \\ B >_1 C, C >_2 B \Rightarrow B = C \end{array}\right\} \Rightarrow A = B \;.$$

But A cannot at the same time outrank B and be equivalent to $B \Rightarrow$ contradiction!

We looked only at this special case (two voters, three alternatives), but the same argument holds for N voters, K alternatives, under similar conditions on the would-be "ideal" voting scheme.

(You don't even have to assume that all possible permutations of the K alternatives are "admissible" preferences for all voters. For instance, if you asked rankings of 20 or so menus by 100 people, then you would expect that the vegetarians among the group never rank meat dishes near the top, thus excluding some individual preference schedules. As long as there is one group of three alternatives where people can rank in any order, the argument will hold.)

For this surprising and unexpected result, Kenneth Arrow was awarded the Nobel Prize in Economics.

Arrow's theorem shows that there is no ideal voting system, that is, no system that will translate **all** possible individual preference schemes into a group preference scheme that is fair (in the sense of Arrow's conditions).

But maybe only a few of all possible schemes lead to a problem? When there is a Condorcet winner, it is certainly true that the first two conditions are satisfied:

- If, in all individual schemes, y stays put and x moves up or stays put, then the relative group ordering of x versus y cannot deteriorate.
- Whether x or y wins in a pairwise comparison for the group, depends only on the relative positions of x and y in the individual rankings.

So trouble can arise only when there is no Condorcet winner. How often can this happen?

Let's look at the case of three voters, three alternatives.

Each voter has six possibilities for his/her ranking:

$$\begin{array}{cccccc} A & B & C & B & A & C \\ B & C & A & A & C & B \\ C & A & B & C & B & A \end{array}$$

The total number of possibilities is $6 \times 6 \times 6 = 216$.

How many of all these combinations lead to a voting paradox, *i.e.*, to the absence of a Condorcet winner?

Take any of the six possibilities for voter 1, and let's check what possibilities for 2 and 3 exclude a Condorcet winner. **Example**: Voter 1 picks

$$\begin{array}{c} C \\ A \\ B \end{array}$$

If voter 2 picks C first, then C will win. In order to avoid a Condorcet winner, 2 must therefore have ranked A or B first. Let us explore these two possibilities:

1. Voter 2 picks B first. If this voter's second choice is A then we have

   $$\begin{array}{cc} C & B \\ A & A \\ B & C \end{array}$$

   Then the third voter must have ranked A first, otherwise there would be a Condorcet winner. That means that voter 3 has two possible rankings only:

   $$\begin{array}{ccc} A & \text{or} & A \\ B & & C \\ C & & B \end{array}$$

   But in both cases there is a Condorcet winner, namely A. Our assumption that voter 2 picked A in second place always leads to Condorcet winner, so let's examine what happens when voter 2's second choice is C:

   $$\begin{array}{cc} \mathbf{1} & \mathbf{2} \\ C & B \\ A & C \\ B & A \end{array}$$

   It now follows that for 3 we have no choice if we want to avoid having a Condorcet winner:

First pick	A	
Second pick	B	(if 3 picked C in 2nd place,
Third pick	C	then C would be Condorcet winner)

2. In the second possibility, voter 2 picks A first.

Then we have again two possibilities for the second choice of 2: B or C

First possibility		**Second possibility**	
1	**2**	**1**	**2**
C	A	C	A
A	B	A	C
B	C	B	B

both A and C have already outranked B twice. A has outranked C once, C has outranked A once. If 3 ranks A ahead of C, A is the Condorcet winner. If 3 ranks C ahead of A, C is the Condorcet winner. In this case there is therefore always a Condorcet winner.

$\Rightarrow$ only first possibility remains.

Now 3 must rank B first (otherwise there is a Condorcet winner) and C second (otherwise we have $C\ A\ B \Rightarrow$ Condorcet winner). $\begin{matrix} A & B & A \\ B & C & C \end{matrix}$

There is thus only one possibility for 3 if we know that there is no Condorcet winner.

So, for each of voter 1's six possibilities, there are two for voter 2 and then no more choices for voter 3, that can lead to the absence of a Condorcet winner; in total we have thus twelve cases.

Only twelve out of 216 possibilities give a paradox, or $\frac{12}{216} = \frac{2}{36} = \frac{1}{18} \simeq .056 \rightarrow$ only 5.6% of cases. Maybe the "voting paradoxes" are much ado about almost nothing then?

Let us look at what happens for more general cases, for higher numbers of voters and of alternatives among which to choose. Here is a table of the percentage of preference schedules that give rise to a paradox for general numbers of voters and of alternatives:

Number of Alternatives	Number of Individuals 3	5	7	9	11	...	Limit
3	0.056	0.069	0.075	0.078	0.080	...	0.080
4	0.111	0.139	0.150	0.156	0.160	...	0.176
5	0.160	0.200	0.215	0.230	0.251	...	0.251
6	0.202	0.255	0.258	0.284	0.294	...	0.315
7	0.239	0.299	0.305	0.342	0.343	...	0.369
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Limit	1.000	1.000	1.000	1.000	1.000	...	1.000

For three alternatives and an arbitrary number of voters, it is never worse than .080, which is less than 1/12. But more alternatives rapidly increase the likelihood of a paradox ....

This is one reason why Congress in the U.S. (as well as in many other countries) uses voting procedures that are binary (i.e. only two alternatives are pitted against each other at any time;

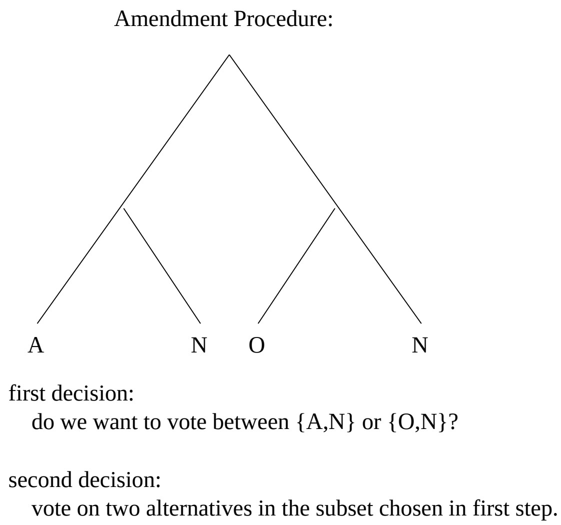
the winning alternative then leads another choice, ...). But that can lead to other paradoxes, where the final outcome is not the one that would have received a majority vote.

In the U.S. Congress, before a bill is voted upon, there is the process of amendment, which can change the bill finally presented for approval or disapproval. If we have a bill, and a proposed amendment, there are then three alternatives for what might eventually end up happening:

O, the original bill is passed
A, the amended bill is passed
N, no bill is passed

However, the Congress does not vote on these three options simultaneously, but rather follows a *voting tree* which offers a binary choice (*i.e.*, a choice between two things) at each of two stages.

The first stage is the amendment stage, to see whether the bill is to be amended or not. The second stage is the final vote to see if the bill (now either amended or left in its original state) is to be passed into law. So if the first vote determines to amend the bill, then the second vote is between the options $\{A, N\}$, while if the first vote determines to leave the bill unamended, then the second vote is between the options $\{O, N\}$. The voting tree illustrates this:



The voting tree shows that there are two ways to defeat a bill: to defeat it as it is and to defeat an amended version of it. Now imagine that it's 2097, and the Silly Party is in power. They have drafted the bill HR123 which, if passed, would give ten free Metropolitan Opera tickets to every US citizen. Since the Silly Party is in the majority (60%) in 2097, it seems like this is the wave of the future.

The Sensible Party (only 40% of the representatives) really doesn't want this to happen, but it looks like there is little they can do, since they are a minority in the House, and the Senate is solidly Silly, as is the President. However, the opportunity comes when the more extreme radical wing of the Silly Party decides that they want to introduce an ammendment to the bill that gives a double helping of free Met tickets to residents of Oklahoma (their reasoning, if you can make anything of it, is that if you live in a state named after a Broadway musical, you need more Met tickets). This sharply divides the Silly Party into those who favor the ammendment (the Very Silly, consisting of 15% of the total representatives) versus the Slightly Silly (45% of the total representatives). The Slightly Silly representatives would rather have no bill than the ammended version. However, if the ammendment fails, the Very Silly wing would still vote for the unammended bill.

So the preferences for Very Silly representatives are $A > O > N$. For Slightly Silly representatives, the preferences are $O > N > A$.

How about the Sensible Party? They definitely prefer N to either O or A. And they can get their top preference. If the bill is left unamended, the combined Silly Party has the strength to pass it. So the Sensible Party votes for the amendment, joining their 40% to the Very Silly wing's 15%. Thus the bill is amended and proposed for final approval. Then, when the time to vote comes, the Sensible Party, along with the Slightly Silly representatives, for a total of 85% of the votes, vote the bill down. So the final option taken is N, no bill. Although 60% of the voters favored the passage of the original bill O, it did not pass because of the order of the voting in the voting tree.