Liquidity Modelling and Optimal Liquidation in Bond Markets

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Abstract

The focus of this paper is on the modelling of defaultable bonds and the optimal liquidation of portfolios of such bonds. By developing an existing credit intensity model, we suggest a framework for valuing defaultable bonds that allows for capturing, not only the default risk, but also the liquidity risk. We investigate empirically defaultable bonds and extract their liquidity risk within this model. We observe a regime shift at the beginning of the credit crisis in the liquidity risk of a range of defaultable bonds. The third part of the paper is on optimal liquidation of portfolios of defaultable bonds. The portfolio liquidation problem for bonds can be formulated as a multiple optimal stopping problem, where stopping corresponds to the sale of a certain quantity of the asset. We allow the depth of the market to depend on the liquidity yield of each bond and for sales to put a downward pressure on the price depending on the size of the trade. We then show the relative effect of the change in liquidity over the credit crisis in the amount that a bond portfolio can be liquidated for under this model.

1 Introduction

The liquidation of portfolios of defaultable assets, whether to meet certain liabilities, to increase liquidity or to cut a losing position has always been an important issue. Recently, in the light of the financial crisis, however it has become crucial to the performance and sometimes even the survival of financial firms. This paper focuses on the modelling of defaultable bonds and the optimal liquidation of portfolios of such bonds.

The liquidity risk of an asset is the risk that arises from the inability to sell the asset or the requirement to have to sell it at a low price, because there is no one in the marketplace prepared to buy the asset for a higher price. Simple measures of liquidity risk are the bid-ask spread, or the total trading volume. The recent crisis in the credit markets, which started in July 2007 was a good example of why, along with credit risk, liquidity risk is very important and should not be underestimated. The crisis originated in the US sub-prime mortgage market, but then quickly spread across all credit areas. It affected credit businesses that have hardly anything to do with sub-prime mortgages and represented the general unwillingness of investors to take on credit risk, putting downward pressure on any type of debt. This naturally created low liquidity in the credit market and the newspapers were alternatively using "credit crisis" and "liquidity crisis" to describe the events.

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The two main factors that are believed to explain risky bond spreads over the risk free term structure are the credit risk and the liquidity risk. While much has been done in the direction of modelling the credit risk, there is not much work in the literature on how liquidity risk affects bond prices. Only recently a few papers Ericsson and Renault (2006) and Longstaff, Mithal and Neis (2006), have been published which discuss the liquidity risk. In Ericsson and Renault (2006) a structural model is developed for liquidity and default risk where the illiquidity is modelled as a random shock which causes the bond price to sell at a random discount to the perfectly liquid bond. Longstaff, Mithal and Neis (2006) use credit default swap and bond data to obtain the default and non-default components in US corporate spreads. They use a framework similar to Duffie and Singleton (1999), where the corporate bond cash flows are discounted at an adjusted rate that includes a liquidity yield process. A stochastic liquidity convenience yield is considered in the literature for credit risk free bonds in papers by Kempf and Uhrig-Homburg (2000) and Grinblatt (1995), where they suggest a two factor liquidity model for risk free bonds. Bond prices are functions of the short term interest rate and another exogenous process which accounts for the liquidity. In this work, together with liquidity, we will consider default risk in the pricing of bonds.

We will build on these papers to develop a three factor model, where the different factors are short term interest rate, default intensity and liquidity yield, to allow us to value defaultable bonds. The parameters for the interest rate process can be estimated from the interest rate market and the parameters for the intensity process can be estimated from the CDS market. We can then extract the liquidity process using quotes for different bond issues.

The reduced form version of our framework is similar to the one introduced by Longstaff, Mithal and Neis (2006). Our model differs in the following important assumptions:

- Liquidity risk is individual to a particular bond and cannot be hedged.
- An investor in illiquid bonds expects a higher rate of return than an investor in liquid ones (to compensate for the additional liquidity risk that cannot be hedged).
- We define l as the continuous rate of return, which an investor in the illiquid bond will receive for holding the liquidity risk.
- We assume recovery of treasury. At default, the recovery of the defaulted debt is proportional to the value of a default risk-free asset, which has the same promised cashflows as the defaulted asset. A treasury bond with the same coupon structure is usually used.
- We are modelling the liquidity risk of a bond as a risk of not being able to sell the bond or access cash. For this reason, the recovery also depends on the liquidity yield. Usually the recovery is not received immediately after a default, because there may be some long legal procedures. Valuing the assets of the defaulted entity and their liquidation could also sometimes take a considerable amount of time. In these cases if a bondholder needs to liquidate the position, the bond has to be sold in the distressed/defaulted debt market, which is also affected by liquidity[1].

Our assumptions allow us to obtain a closed form solution for the bond prices in the case where the different driving factors are modelled as independent mean reverting square root (CIR) processes. This compares to prices that are given by a one-dimensional integral in Longstaff, Mithal and Neis (2006). In our model the liquidity premium appears explicitly as a discount on the price of an illiquid bond.

[1] The size of the distressed/defaulted debt market has increased several times over the past few years and especially recently. In the light of the financial crisis, many opportunities have emerged for liquid investors to take on underpriced illiquid problematic debt.

There has been some empirical work on liquidity including that by de Jong and Driessen (2006) and Chacko (2006). The results of de Jong and Driessen (2006) show that the liquidity premia varies across the different quality debt and is positively correlated with it. For the empirical part of our work we use CDS quotes provided by Markit for the period 2001 - 2007. We also use corporate bond data from TRACE and treasury rates data from the WRDS database. We present several cases: AIG, Citigroup, General Electric, Goldman Sachs and Merrill Lynch. We specifically chose financial firms to see how their subprime related writedowns in 2007 have affected the liquidity of their debt products. It is widely agreed that the banking crisis started when problems in the credit markets caused the liquidity to dry up. We will see some empirical evidence for that. Also at the time of the first occurence of the symptoms of the crisis we can see a regime change in the liquidity factor. In 2008, especially after the acquisition of Bear Stearns by JPMorgan, the CDS markets also became illiquid, so it is less straightforward to apply our approach for 2008. During 2008 and especially towards the end of the year the CDS spreads were also very dependent on the counterparty risk as investors became more aware that the large investment banks, the usual counterparties in CDS trades, were also not immune to default.

In the third part of the paper we use the risky bond model to formulate an optimal liquidation problem for bonds and suggest an efficient numerical method to solve it. Due to the small number of trades per day in the bond market it is useful to model time as discrete. The liquidation problem for a portfolio of bonds is then naturally formulated as a discrete time multiple optimal stopping problem, where a stopping time corresponds to a selling time for a fixed unit of the asset. In order to model some of the features of the risky bond market we assume that the amount that can be sold in the market at each time (the depth of the market) depends on the liquidity yield of each bond. We also assume that selling puts a short term downward pressure on the price depending on the size of the trade. Using Aleksandrov and Hambly (2008) we can formulate and solve efficiently this liquidation problem. We then compare the actual return from liquidating a portfolio of 20 Goldman-Sachs bonds before and after the credit crisis. The lack of liquidity results in a 5.5% reduction in the return.

The outline of the paper is as follows. We begin in Section 2 by developing the model, firstly in generality and then reducing it to an analytic expression after making modelling assumptions about the different processes involved. Once the model is in place it can be used to calculate the liquidity risk in some traded corporate bonds and the results of the empirical analysis are given in Section 3. Finally in Section 4 we formulate the liquidation problem and discuss its numerical implementation before giving some results.

2 Intensity and Liquidity - Model Formulation

We will develop a three factor reduced form model for zero coupon bonds where the driving factors are short term interest rates, intensity of default and liquidity yield. We will first consider a general setting where the driving factors are general stochastic processes before specializing to CIR type processes which will allow an explicit solution for bond prices.

In an intensity or reduced form model the default arrival time is modelled as the first arrival time of a Poisson process with some intensity or, if the intensity is stochastic, of a Cox process. For more details on these processes see Cont and Tankov (2004). There is no assumption about the firm's assets or liabilities as the jump time is an exogenous process and thus is independent of the market state variables. The intensity based approach was introduced by Jarrow and Turnbull (1995) and further developed by Duffie and Singleton (1999) and Lando (1998). Intensity models have become popular among practitioners, because they are relatively easy to calibrate to CDS or defaultable bond data.

The default time of an obligor, the first jump time of the Poisson process, is assumed to have an intensity process that is adapted to the market filtration $\mathcal{F}_t$. Corresponding to real markets, the information we have for the arrival time should contain, not only whether there is a default at the current time, but also whether there has been a default at some time in the past. We will discuss the general case, where the interest rate r, jump intensity λ and liquidity yield l are stochastic processes.

The model:

1. The dynamics of the short term interest rate r follow

$$dr = \mu_1(t, r)dt + \sigma_1(t, r)dW_1.$$

2. The default time τ is the first event of a Poisson process with intensity λ satisfying

$$d\lambda = \mu_2(t, \lambda)dt + \sigma_2(t, \lambda)dW_2.$$

3. The liquidity convenience yield l satisfies

$$dl = \mu_3(t, l)dt + \sigma_3(t, l)dW_3.$$

We also assume the driving Brownian motions are correlated in that

$$dW_i dW_j = \rho_{ij}dt, \quad 1 \leq j < i \leq 3.$$

One way to guarantee the existence of strong solutions is to assume global Lischitz condition on the coefficients. For $i = 1, 2, 3$ there exist constants K_i such that

$$||\mu_i(t, x) - \mu_i(t, y)|| + ||\sigma_i(t, x) - \sigma_i(t, y)|| \leq K_i||x - y||, \quad \forall x, y \in \mathbb{R}.$$

The model that we will use, mean-reverting square-root model, does not however satisfy the Lipschitz condition at 0. Despite that, it is well known that the SDE for this process has a strong solution. We also assume the linear growth condition in that for $i = 1, 2, 3$ there are constants K_i such that

$$||\mu_i(t, x)||^2 + ||\sigma_i(t, x)||^2 \leq K_i^2(1 + ||x||^2).$$

to insure that the processes r, λ and l are square integrable.

Let $\mathcal{G}_t$ be the filtration of the state variables $X_t = \{r_t, ...\}$ (usually just the interest rates, but it could have other factors as well) including the jump intensity and the liquidity yield

$$\mathcal{G}_t = \sigma\{X_s, \lambda_s, l_s : 0 \leq s \leq t\}$$

and $\mathcal{H}_t$ be the filtration containing the information on whether or not there has been a default up to time t

$$\mathcal{H}_t = \sigma\{1_{\tau \leq s} : 0 \leq s \leq t\}.$$

The market filtration $\mathcal{F}_t$ i.e. the information about the state variables and whether there has been default or not is then

$$\mathcal{F}_t = \mathcal{G}_t \vee \mathcal{H}_t.$$

Let $I(r(t), \lambda(t), l(t), t, T)$ be the price of the illiquid defaultable bond. In the event of a default with recovery I^d, the increment in I will be $I^d - I$.

We write $1_{\tau>T}$ for the payoff at T, if no default occurs up to time T and $I^d 1_{\tau<T}$ for the payoff at time τ, if there is a default with recovery I^d. Using Duffie and Singleton (1999) the bond price is given by the formula:

$$I(r,\lambda,l,t,T) = \mathbb{E}[e^{-\int_t^T r_s+l_s ds} 1_{\tau>T}|\mathcal{F}_t] + \mathbb{E}[e^{-\int_t^\tau r_s+l_s ds} I^d 1_{\tau<T}|\mathcal{F}_t]. \tag{1}$$

This is equivalent to discounting the cashflows at rate $r+l$. We assume recovery of treasury and also that the recovery takes into account the liquidity of the bond in the defaulted security market, which means that our recovery at time τ is proportional to $\mathbb{E}[e^{-\int_\tau^T r_s+l_s ds}|\mathcal{F}_\tau]$. We write

$$I^d = \delta \mathbb{E}[e^{-\int_\tau^T r_s+l_s ds}|\mathcal{F}_\tau], \tag{2}$$

where δ is a constant.

Theorem 1 *A representation of equation (1) is given by*

$$I(r,\lambda,l,t,T) = \mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t] + \delta(\mathbb{E}[e^{-\int_t^T r_s+l_s ds}|\mathcal{G}_t] - \mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t]). \tag{3}$$

In order to prove this Theorem we begin with two Lemmas.

Lemma 1

$$\mathbb{E}[1_{\tau\geq T}|\mathcal{G}_T \vee \mathcal{H}_t] = 1_{\tau>t} \exp(-\int_t^T \lambda_s ds).$$

Proof:

$$\begin{aligned}
\mathbb{E}[1_{\tau\geq T}|\mathcal{G}_T \vee \mathcal{H}_t] &= 1_{\tau>t}\mathbb{E}[1_{\tau\geq T}|\mathcal{G}_T \vee \mathcal{H}_t] \\
&= 1_{\tau>t}\frac{\mathbb{P}(\{\tau \geq T\} \bigcap \{\tau > t\}|\mathcal{G}_T)}{\mathbb{P}(\tau > t|\mathcal{G}_T)} \\
&= 1_{\tau>t}\frac{\mathbb{P}(\tau \geq T|\mathcal{G}_T)}{\mathbb{P}(\tau > t|\mathcal{G}_T)} \\
&= 1_{\tau>t} \exp(-\int_t^T \lambda_s ds).
\end{aligned}$$

□

Lemma 2

$$\mathbb{E}[e^{-\int_t^T r_s+l_s ds} 1_{\tau\geq T}|\mathcal{F}_t] = 1_{\tau\geq t}\mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{G}_t].$$

Proof:

$$\begin{aligned}
\mathbb{E}[e^{-\int_t^T r_s+l_s ds} 1_{\tau\geq T}|\mathcal{F}_t] &= \mathbb{E}[\mathbb{E}[e^{-\int_t^T r_s+l_s ds} 1_{\tau\geq T}|\mathcal{G}_T \vee \mathcal{H}_t]|\mathcal{F}_t] \\
&= \mathbb{E}[e^{-\int_t^T r_s+l_s ds}\mathbb{E}[1_{\tau\geq T}|\mathcal{G}_T \vee \mathcal{H}_t]|\mathcal{F}_t] \\
&= 1_{\tau\geq t}\mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{F}_t].
\end{aligned}$$

We are now going to express the last expectation as an expectation conditioned on the state variable filtration $\mathcal{G}_t$, rather than on the market filtration F_t. The exponential random variable E is independent of $\mathcal{G}_T$ so

$$\mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{G}_t \vee \sigma(E)] = \mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{G}_t].$$

At the same time we have:

$$\mathcal{G}_t \subset \mathcal{F}_t \subset \mathcal{G}_t \vee \sigma(E).$$

Thus

$$\mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{G}_t \vee \mathcal{H}_t] = \mathbb{E}[e^{-\int_t^T r_s+l_s+\lambda_s ds}|\mathcal{G}_t]$$

and the proof is completed.

□

Proof: (Theorem 1) By the preceding two Lemmas and the fact that

$$\mathbb{E}[e^{-\int_t^\tau r_s+l_s ds}\delta\mathbb{E}[e^{-\int_\tau^T r_s+l_s ds}|\mathcal{F}_\tau]1_{\tau<T}|\mathcal{F}_t] = \delta\mathbb{E}[e^{-\int_t^T r_s+l_s ds}1_{\tau<T}|\mathcal{F}_t], \tag{4}$$

we have

$$\begin{aligned} I(r,\lambda,l,t,T) &= \mathbb{E}[e^{-\int_t^T r_s+l_s ds}1_{\tau>T} + I^d e^{-\int_t^\tau r_s+l_s ds}1_{\tau<T}|\mathcal{F}_t] \\ &= \mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t] + \delta(\mathbb{E}[e^{-\int_t^T r_s+l_s ds}|\mathcal{G}_t] - \mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t]). \end{aligned}$$

□

As a comparison Longstaff, Mithal and Neis (2006) assume recovery at par and the corresponding expression for the zero-coupon bond price using our notation would be

$$I = \mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t] + I^d\mathbb{E}[\int_t^T \lambda_s e^{-\int_t^s r_u+\lambda_u+l_u du}ds|\mathcal{G}_t]. \tag{5}$$

To prove this we use

Lemma 3

$$\mathbb{E}[e^{-\int_t^\tau r_s+l_s ds}1_{\tau<T}|\mathcal{F}_t] = 1_{\tau\geq t}\mathbb{E}[\int_t^T \lambda_s e^{-\int_t^s r_u+\lambda_u du}ds|\mathcal{G}_t].$$

Proof: The density of τ is

$$\frac{\partial}{\partial s}\mathbb{P}(\tau \leq s|\tau > t, \mathcal{G}_T) = \lambda_s e^{-\int_t^s \lambda_u du}.$$

Thus

$$\begin{aligned} &\mathbb{E}[e^{-\int_t^\tau r_s+l_s ds}1_{\tau<T}|\mathcal{F}_t] \\ &= \mathbb{E}[\mathbb{E}[e^{-\int_t^\tau r_s+l_s ds}1_{\tau<T}|\mathcal{G}_T \vee \mathcal{H}_t]|\mathcal{F}_t] \\ &= 1_{\tau\geq t}\mathbb{E}[\int_t^T \lambda_s e^{-\int_t^s r_u+\lambda_u+l_u du}ds|\mathcal{F}_t] \\ &= 1_{\tau\geq t}\mathbb{E}[\int_t^T \lambda_s e^{-\int_t^s r_u+\lambda_u+l_u du}ds|\mathcal{G}_t]. \end{aligned}$$

The last equality is obtained using the same idea as used in the previous Lemma.

□

To compare our model with Longstaff, Mithal and Neis (2006) numerically we assume 40% recovery of par for the Longstaff, Mithal and Neis framework and then we estimate $\delta = 54\%$ for our model using least square regression. We do this using the following procedure. We assume that the driving processes are mean-reverting square root processes and are independent of each other (see the next section for the corresponding formulas). We then calculate the values of (5) and the expectations in (3) for 60,000 different sets of parameters. We then regress the corresponding values with respect to δ for the range of the model parameters. The standard regression error we calculate is 0.054, which shows that the models take similar values.

2.1 Uncorrelated Brownian Motions

In the previous subsection we introduced the general framework we use for modelling credit and liquidity spreads. Here we will look at the case when the Brownian motions are not correlated.

The first expectation from (1) can be written as

$$\begin{aligned}\mathbb{E}[e^{-\int_t^T r_s+\lambda_s+l_s ds}|\mathcal{G}_t] &= \mathbb{E}[e^{-\int_t^T r_s ds}e^{-\int_t^T \lambda_s ds}e^{-\int_t^T l_s ds}|\mathcal{G}_t] \\ &= \mathbb{E}[e^{-\int_t^T r_s ds}|\mathcal{G}_t]\mathbb{E}[e^{-\int_t^T \lambda_s ds}|\mathcal{G}_t]\mathbb{E}[e^{-\int_t^T l_s ds}|\mathcal{G}_t],\end{aligned}$$

where

$$P(r,t,T) = \mathbb{E}[e^{-\int_t^T r_s ds}|\mathcal{G}_t]$$

is the default risk-free zero coupon bond price,

$$\Lambda(\lambda,t,T) = \mathbb{E}[e^{-\int_t^T \lambda_s ds}|\mathcal{G}_t]$$

is the survival probability and we can think of

$$D(l,t,T) = \mathbb{E}[e^{-\int_t^T l_s ds}|\mathcal{G}_t]$$

as a liquidity discount factor.

We can write the illiquid bond price as the liquid bond price multiplied by a discount factor

$$I(r,\lambda,l,t,T) = D(l,t,T)L(r,\lambda,t,T) \tag{6}$$

where

$$L(r,\lambda,t,T) = P(r,t,T)\Lambda(\lambda,t,T) + \delta P(r,t,T)(1-\Lambda(\lambda,t,T)). \tag{7}$$

In this uncorrelated case the credit and liquidity spreads γ_c and γ_l can be separated as

$$\begin{aligned}\gamma_c &= -\frac{\ln(L(r,\lambda,t,T)/P(r,t,T))}{T-t} = -\frac{\ln(\Lambda(\lambda,t,T)+\delta(1-\Lambda(\lambda,t,T))}{T-t}, \\ \gamma_l &= -\frac{\ln(I(r,\lambda,l,t,T)/L(r,\lambda,t,T))}{T-t} = -\frac{\ln(D(l,t,T))}{T-t},\end{aligned}$$

and the total spread is

$$\gamma = \gamma_c + \gamma_l.$$

2.2 Mean reverting square root process model

We now consider a specific model for the different driving factors. For the short term interest rate, we consider the CIR model as originally suggested in Cox, Ingersoll and Ross (1985) and we also assume the same model (with different parameters) for the other factors.

The model:

1. The dynamics of the short term interest rate r satisfy

$$dr = k(\theta - r)dt + \sigma_r \sqrt{r} dW_1.$$

   where k, θ and σ_r are constant parameters.

2. The intensity of default λ satisfies

$$d\lambda = \rho(\nu - \lambda)dt + \sigma_\lambda \sqrt{\lambda} dW_2,$$

   where ρ, ν and σ_λ are constant parameters

3. The liquidity convenience yield l satisfies

$$dl = \beta(\mu - l)dt + \sigma_l \sqrt{l} dW_3,$$

   where β, μ and σ_l are constant parameters.

We also assume that the driving Brownian motions are not correlated.

For all the mean-reverting square-root processes in this part we impose the positivity condition i.e. $k\theta > \sigma^2/2$, which guarantees that the process does not reach zero. The main properties of the CIR process are that it is always positive and that it is mean reverting. The drift is positive if $r < \theta$ and negative if $r > \theta$ and thus r is pulled to the level θ with speed k. As in Cox, Ingersoll and Ross (1985) it is easy to show that bond prices P, without credit or liquidity risk, satisfies the partial differential equation

$$\frac{\sigma_r^2}{2} r P_{rr} + k(\theta - r)P_r + P_t = rP$$

with terminal condition

$$P(r, T, T) = 1.$$

The closed form solution is

$$P(r, t, T) = A(t, T)e^{-B(t,T)r},$$

where

$$\begin{aligned} A(t, T) &= \left[\frac{2\omega e^{k+\omega}(T-t)/2}{(\omega + k)(e^{\omega(T-t)} - 1) + 2\omega} \right]^{\frac{2k\theta}{\sigma_r^2}}, \\ B(t, T) &= \frac{2(e^{\omega(T-t)} - 1)}{(\omega + k)(e^{\omega(T-t)} - 1) + 2\omega}, \\ \omega &= (k^2 + 2\sigma_r^2)^{1/2}. \end{aligned}$$

We will calculate $\mathbb{E}[e^{-\int_t^T r_s ds}|\mathcal{G}_t]$, $\mathbb{E}[e^{-\int_t^T \lambda_s ds}|\mathcal{G}_t]$ and $\mathbb{E}[e^{-\int_t^T l_s ds}|\mathcal{G}_t]$. The first expectation above is the value of the risk free zero coupon bond. From Feynman-Kac theorem the two other expectations are solutions to the analogous partial differential equations

$$\frac{\sigma_\lambda^2}{2}\lambda\Lambda_{\lambda\lambda} + \rho(\nu - \lambda)\Lambda_\lambda + \Lambda_t = \lambda\Lambda, \Lambda(\lambda, T, T) = 1$$

and

$$\frac{\sigma_l^2}{2} l D_{ll} + \beta(\mu - l)D_l + D_t = lD, D(l, T, T) = 1.$$

As before, these can be explicitly solved as

$$\Lambda(\lambda, t, T) = C(t, T)e^{-E(t,T)\lambda},$$

where

$$C(t,T) = \left[\frac{2\omega e^{(\rho+\omega)(T-t)/2}}{(\omega+\rho)(e^{\omega(T-t)}-1)+2\omega}\right]^{\frac{2\rho\nu}{\sigma_\lambda^2}},$$
$$E(t,T) = \frac{2(e^{\omega(T-t)}-1)}{(\omega+\rho)(e^{\omega(T-t)}-1)+2\omega},$$
$$\omega = (\rho^2 + 2\sigma_\lambda^2)^{1/2}.$$

and

$$D(l, t, T) = G(t, T)e^{-H(t,T)l},$$

where

$$G(t,T) = \left[\frac{2\eta e^{(\beta+\eta)(T-t)/2}}{(\beta+\eta)(e^{\eta(T-t)}-1)+2\eta}\right]^{\frac{2\beta\mu}{\sigma_y^2}},$$
$$H(t,T) = \frac{2(e^{\eta(T-t)}-1)}{(\eta+\beta)(e^{\eta(T-t)}-1)+2\eta},$$
$$\eta = (\beta^2 + 2\sigma_y^2)^{1/2}.$$

Thus from (6) and (7) we have an analytic expression for the bond price given by

$$I(r, \lambda, l, t, T) = D(l, t, T)\left(P(r, t, T)\Lambda(\lambda, t, T) + \delta P(r, t, T)(1 - \Lambda(\lambda, t, T))\right),$$

with P, Λ, D as given above.

3 Empirical Work

In this section we will give some empirical observations about the liquidity premia of several U.S. corporate bonds. We will discuss the change in the liquidity environment over the summer of 2007 and also estimate the parameters for our liquidity model. The data we use is the following:

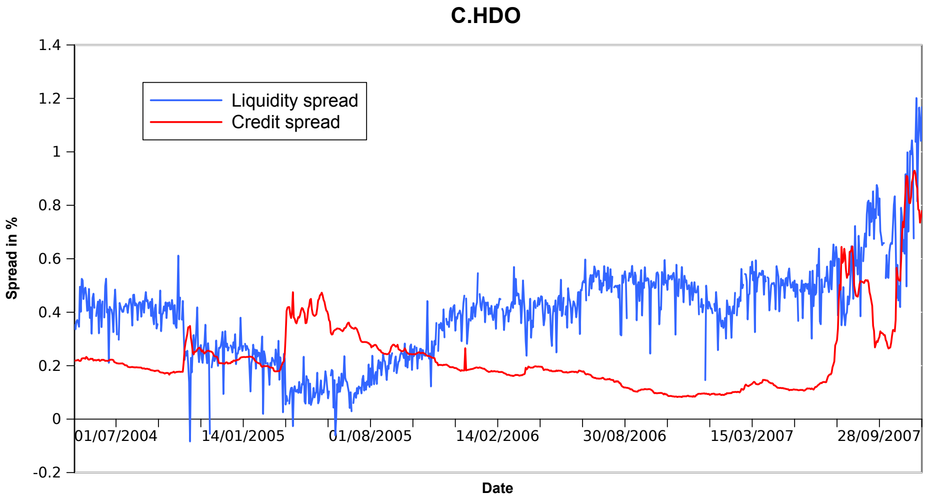

Figure 1: Credit and liquidity spreads for the Citigroup bond - C.HDO issued on 22/07/2004 with maturity 29/07/2009 and issue size USD1bn.

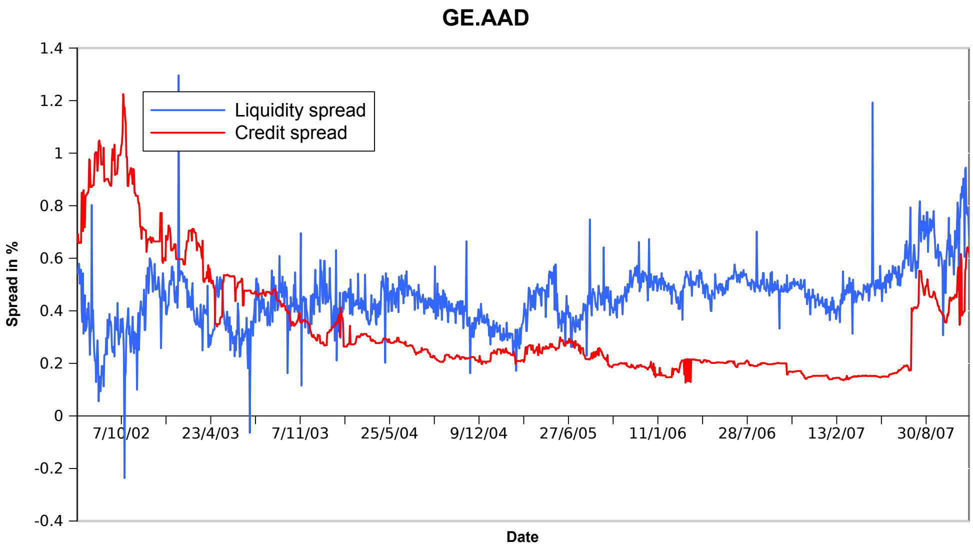

Figure 2: Credit and liquidity spreads for the General Electric bond - GE.AAD issued on 31/05/2002 with maturity 15/06/2012 and issue size USD3.75bn.

- Fixed maturities U.S. Treasury rates (taken from the WRDS database of Wharton Business School). We use Treasury rates as opposed to Libor rates, because the Treasury rates are the closest rates to default risk-free (at least in theory). There is a significant amount of credit risk in the Libor curve. In 2008 we saw significant increase in the TED spread (the difference between the interest rates for three-month U.S. Treasuries contracts and the three-month Eurodollars contract as represented by the LIBOR), which illustrated that point. The industry usually compares the CDS spread to the Asset Swap Spread[2] of a name, because of financing issues.

[2]The Asset Swap Spread is the spread over LIBOR in the floating leg of a swap where the fixed leg is the bond's coupon payments

The difference of the two is called the basis. We will however only look at the treasury curve for the reason already stated. The data we have taken from WRDS is obtained in the following way: Yields on Treasury nominal securities at "constant maturity" are interpolated by the U.S. Treasury from the daily yield curve for non-inflation-indexed Treasury securities. This curve, which relates the yield on a security to its time to maturity, is based on the closing market bid yields on actively traded Treasury securities in the over-the-counter market. These market yields are calculated from composites of quotations obtained by the Federal Reserve Bank of New York. The constant maturity yield values are read from the yield curve at fixed maturities, currently 1, 3 and 6 months and 1, 2, 3, 5, 7, 10 and 20 years. Given these yield values at constant maturities, we use cubic splines to interpolate the yield curve. We then read the needed yield off that yield curve.

- Credit default swap data (provided by Markit). The CDS quotes were for 6 months and 1, 2, 3, 4, 5, 7, 10, 15, 20 and 30 years. Again cubic splines were used to build the whole credit curve.
- Bond yield data (taken from Trade Reporting and Compliance Engine (TRACE)). TRACE is a vehicle that facilitates the mandatory reporting of over the counter secondary market transactions in eligible fixed income securities. All brokers/dealers who are Financial Industry Regulatory Authority member firms have an obligation to report transactions in corporate bonds to TRACE under an SEC approved set of rules. As TRACE reports all trades in a particular bond, we calculate the bond yields of the considered names by averaging the daily yields of all transactions weighted by their sizes.

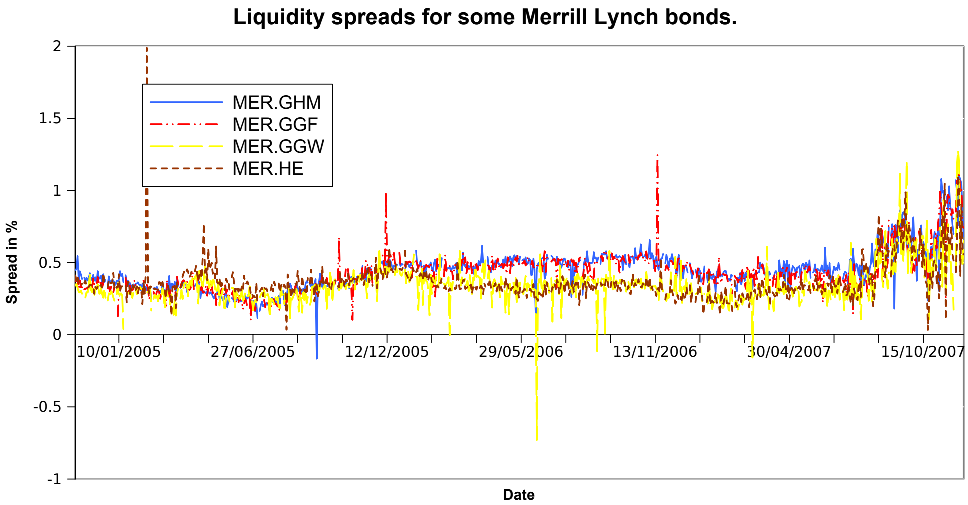

Figure 3: Liquidity spreads for the Merrill Lynch bonds - MER.GHM, MER.GGF, MER.GGW, and MER.HE

Once we have build the interest rate and credit curves we then calculate the liquidity spread as the difference between the bond yield and the sum of the interest rate and the default spread for a particular maturity. In this way we have liquidity yield to maturity. Some visual examples of liquidity yields are given on figures 1-5. On figures 1 and 2 we can see a comparison of the credit and the liquidity spreads of two bond issues of Citigroup and General Electric. We can see that the liquidity spread is very significant and in some cases larger than the credit spread.

On figure 6 we have plotted the average liquidity yield of 40 different bonds issued by 10 large US corporations. We can clearly see that there is some sort of change in the behaviour of the average yield

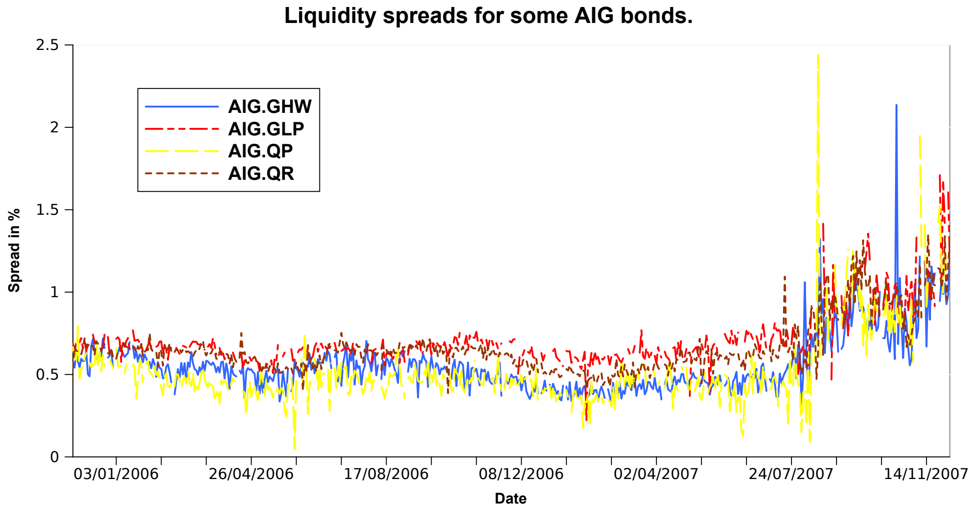

Figure 4: Liquidity spreads for the AIG bonds - AIG.GHW, AIG.GLP, AIG.QP, and AIG.QR

at the beginning of the credit crisis in July-August 2007. The square on figure 6 marks the 1 August 2007, the day after the Bear Stearns High-Grade Structured Credit Fund and Bear Stearns High-Grade Structured Credit Enhanced Leveraged Fund filed for Chapter 15 bankruptcy. The collapse of these two hedge funds with capital of over 1.6 billion USD started the panic in the credit markets. On figures 3-5 which represent liquidity yields for several bonds issued by Merrill Lynch, AIG and Goldman Sachs we can see that they all have behaviour similar to the average.

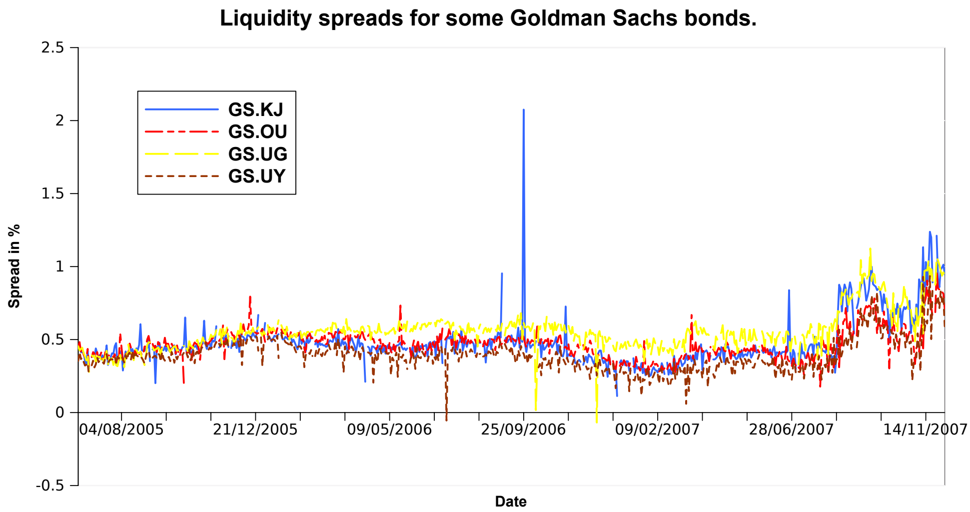

Figure 5: Liquidity spreads for the Goldman Sachs bonds - GS.KJ, GS.OU, GS.UG, and GS.UY

As we have separated the liquidity yield and we are primarily interested in it we will estimate our model parameters for the liquidity process only. We gave the explicit calculation in the case when we assume that the liquidity process is a CIR process. In this part we will also consider the Vasicek model for the liquidity process. The parameter estimation will be done using the Kalman filter. We will use the methodology from Bolder (2001) and Chen and Scott (2003). We will give a short description of

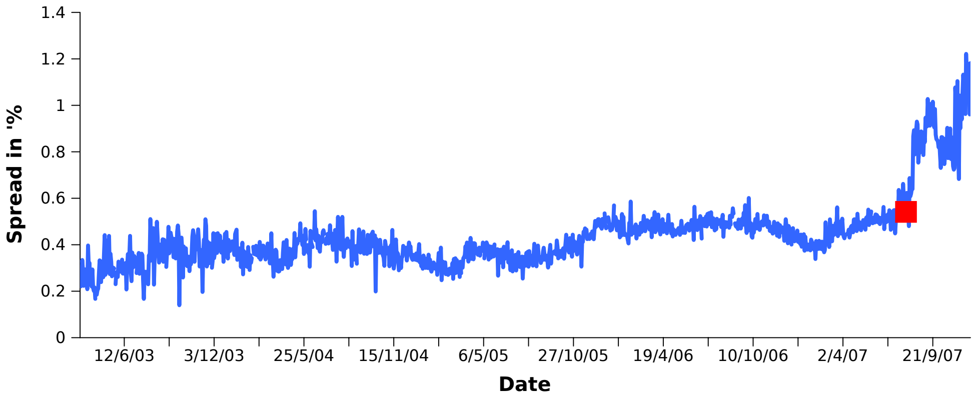

Figure 6: Average liquidity spread calculated over 40 bond issues by 10 obligors.

the procedure in Appendix A, but for more details see the two papers.

Because of the difference in the behaviour of the liquidity yield before the beginning of the crisis and during the crisis we have estimated the model parameters for the liquidity process for two time intervals - before 1 August 2007 and after. The results are given in Table 1. We can see that the parameters that maximise the likelihood function in the Kalman filter are different before and after. After 1 August 2007 in most cases we have higher means, variances and speed of mean reversion for both the mean reverting process and CIR process.

4 Portfolio Liquidation Problem

The liquidation of defaultable assets has become very important during the current financial crisis. Examples where this may be required in the financial industry include:

- A bank liquidating toxic assets to cut losses, to obtain liquidity or to improve its leverage ratio[3]. From the beginning of the crisis and especially after the bankruptcy of Lehman Brothers, banks were very aggressive to bring the leverage ratios down and contract their balance sheets. Morgan Stanley, for example shrunk its balance sheet from USD 1,131,649 mil in Mar 2008 to USD 626,023 mil in Mar 2009, bringing the leverage ratio from 27.7x to 11.2x for the same period. See figure 7.
- A prime broker liquidating clients assets after a default or unmet margin calls. In the event of a hedge fund default its prime broker would in many cases have to liquidate the collateral.
- A buy-side firm (hedge fund, pension fund or insurance company) selling assets to meet redemptions or cut losses. According to data compiled by Hedge Fund Research Inc. Hedge fund assets shrank to $1.2 trillion at the end of 2008 from the June peak of $1.9 trillion due to market losses and investor withdrawals.

It has very often been the case that when liquidating a portfolio, the actual resulting cash is very different from the book value. This is because most quotes in the marketplace are only valid for limited amounts. This is a different complication to the fact that some financial products are not

[3]Leverage ratio is the ratio between the core capital of the firm i.e. shareholders' equity and the total consolidated assets. It is a measure of the riskiness of the firm.

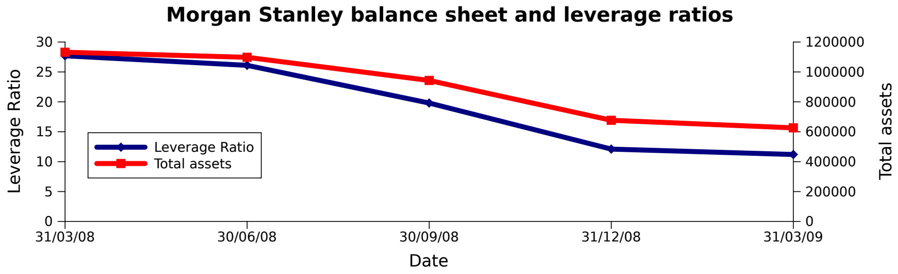

Figure 7: Source: Morgan Stanley Reports.

actively traded and the price is marked-to-model as opposed to mark-to-market. Most financial firms who mark-to-market use the quoted market price to value their assets. This however has proven to be misleading. Here we show how this occurs for the defaultable illiquid securities we have considered. Thus not only an asset price model, but also a liquidation model and a projection for the resulting cash from a liquidation should be used when valuing a position.

We consider defaultable illiquid securities - corporate and government bonds and will suggest a framework for designing a liquidation strategy for such assets. The problem of liquidation of equities or foreign exchange positions in a limit order book setting has been widely considered in the literature Smith et al. (2003), Almgren and Chriss (2000) and others. Most of these approaches rely on modelling the limit order book and bid-ask spread explicitly. Unfortunately this is only realistic in fairly liquid markets, which equities and foreign exchange are, and unrealistic in the more illiquid bond markets. The liquid stocks and currency pairs trade every second, even more often, very liquid U.S. corporate bond issues trade several times a day. It has been calculated in Bessembinder et al. (2006) that the average bond trades in the OTC market are of size USD 2.7 mil. This more discrete and sizable nature of trading in defaultable bonds requires a different modelling approach for the liquidation problem. On figure 8 we can observe the typical trading behaviour of a bond issue. After issuance the bond trades heavily for some time. Very often buy-and-hold institutional investors buy into it and the amount of trading in the issue stabilises at a lower level.

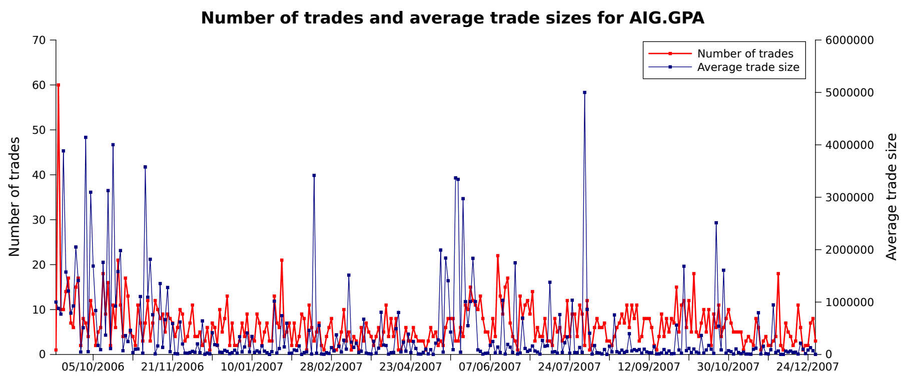

Figure 8: Average trade size and number of trades for the American International Group - AIG.GPA issue.

We will state here the main economic assumptions for our liquidation problem:

1. The liquidation of the position occurs in discrete time, on a daily basis. Given the infrequency of the trading in the corporate bond markets, this is not an unrealistic assumption.

2. The financial institution liquidating the position is a price taker and its influence on the bond price only temporary. It affects the price on the day, depending on the amount of bonds it sells that day, by putting a downward pressure on the price. On the next day, however, the bond price is determined by the economic fundamentals - interest rates, probability of default and the liquidity of the bond on that day.

3. The depth of the market i.e. the maximum amount the firm can sell on a given day is determined by the liquidity of the bond. High liquidity of the bond (low liquidity yield) corresponds to a large amount that could be sold and vice versa.

The liquidation is done over a finite time horizon $\{1, 2, ..., T\}$. The amount that could be sold in the market, which depends on the depth or the liquidity of the market, is given by k_t ($t = \{1, 2, ..., T\}$). k_t is a decreasing function of the liquidity yield l_t and has a discrete distribution. So the seller can sell up to k_t units of the asset at time t. The price of the defaultable bond that depends on the factors r_t, λ_t and l_t is only for limited amounts of the security. This corresponds to the fact that usually the price the market-makers announce or the price on the screen is for limited amounts. For larger amounts there is a price correction. Thus the cashflow from selling one unit at time t would be

$$h_t^1 = I(r, \lambda, l, t, T).$$

However for more units of the security the price and thus the cashflows from the sale will adjust to

$$h_t^1 \geq h_t^2 \geq h_t^2 ... \geq h_t^{k_t},$$

with cumulative cashflow from the sale of p units of the asset

$$H_t^p = \sum_{i=1}^{p} h_t^i, \qquad p = 1, 2, ..., k_t.$$

The difference between $h_t^1, h_t^2, ..., h_t^{k_t}$ represents the price impact of the sale. In this way the liquidity of a specific security affects its price on the screen and the total amount the market could take at any one time. On the other hand the amount to be sold introduces an additional effect on the total cashflow. We assume that the asset price is affected by the sale only temporarily, i.e. after a sale the price goes back to the price level given by r_t, λ_t and l_t before the next time step. This is a reasonable assumption if the seller's position is not a large portion of the market in that security and the other market participants are not aware of the liquidation plans. Otherwise the sale may have a larger impact on the price lasting longer.

Here we will make some more assumptions about the microstructure of the bond market.

1. The quote $I(r, \lambda, l, t, T)$ is good for a trade that is of size equal to the average bond trade in the issue. The price is then affected by every consecutive block of that size. So if twice the amount of an average trade is to be sold, the first block could be sold for the announced price $I(r, \lambda, l, t, T)$ and the second block will be sold at a discount.

2. The maximum that could be sold in the market at any one day, in a perfectly liquid market is equal to twice the average daily volume in the issue. This means at each time t the maximum amount that can be sold in the market k_t (which is a random variable) takes values in the set $(1, 2, ..., M)$, where

$$M = \text{integer part}\left[\frac{2 * \text{average daily volume}}{\text{average daily trade}}\right]. \tag{8}$$

For the numerical example we will consider a distribution for k_t that is derived from l_t in the following way.

$$k_t = \begin{cases} 6, & \text{if } l_t < \mu - 2\hat{\sigma} \\ 5, & \text{if } \mu - 2\hat{\sigma} \leq l_t < \mu - \hat{\sigma} \\ 4, & \text{if } \mu - \hat{\sigma} \leq l_t < \mu \\ 3, & \text{if } \mu \leq l_t < \mu + \hat{\sigma} \\ 2, & \text{if } \mu + \hat{\sigma} \leq l_t < \mu + 2\hat{\sigma} \\ 1, & \text{if } \mu + 2\hat{\sigma} \geq l_t \end{cases}$$

where $\hat{\sigma}^2 = \frac{\sigma_l^2 \mu}{2\beta}$. μ is the stationary mean of the process l_t. $\hat{\sigma}^2$ is the stationary variance of the mean reverting process with the same speed and level of mean reversion as l and constant variance σ_l^2.

We also assume that the sale of the asset impacts the price through a discount factor, that is

$$h^i = \frac{I}{(1+\alpha)^{i-1}}. \tag{9}$$

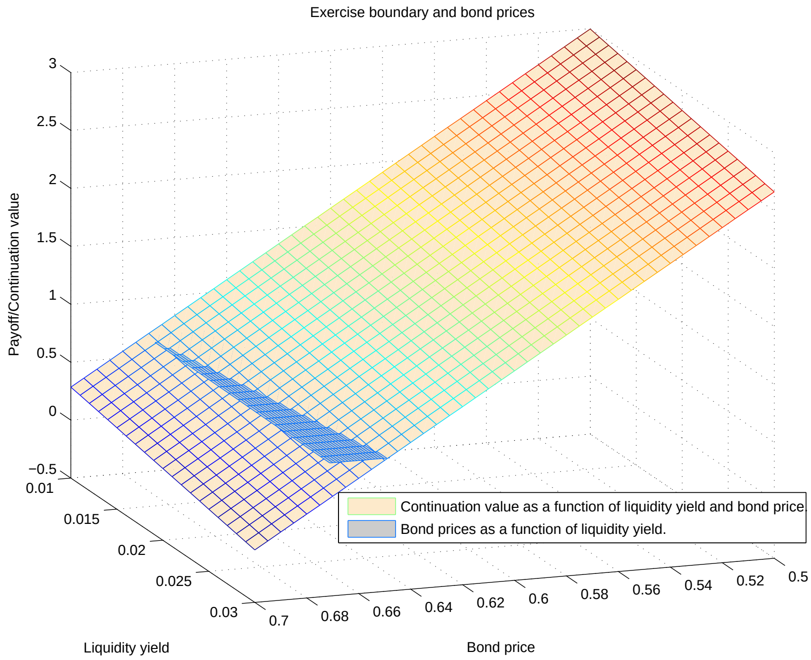

Figure 9: The intersection between the continuation value as a function of the liquidity yield and bond price and bond price as a function of the liquidity yield is the exercise boundary.

We will use the notation $V_t^{*,m}$ for the optimal value for which the position can be liquidated. We define a liquidation policy $\pi_{\mathbf{k}}$ to be a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_m \leq \tau_{m-1} \leq \cdots \leq \tau_1$ and $\#\{j : \tau_j = s\} \leq k_s$. We will often drop the subscript and just write π for ease of notation. Then the value of the policy $\pi_{\mathbf{k}}$ at time t is given by

$$V_t^{\pi_{\mathbf{k}},m} = \mathbb{E}_t\Big(\sum_{s=t}^{T} H_s^{\#\{j:\tau_j=s\}}(X_s)\Big).$$

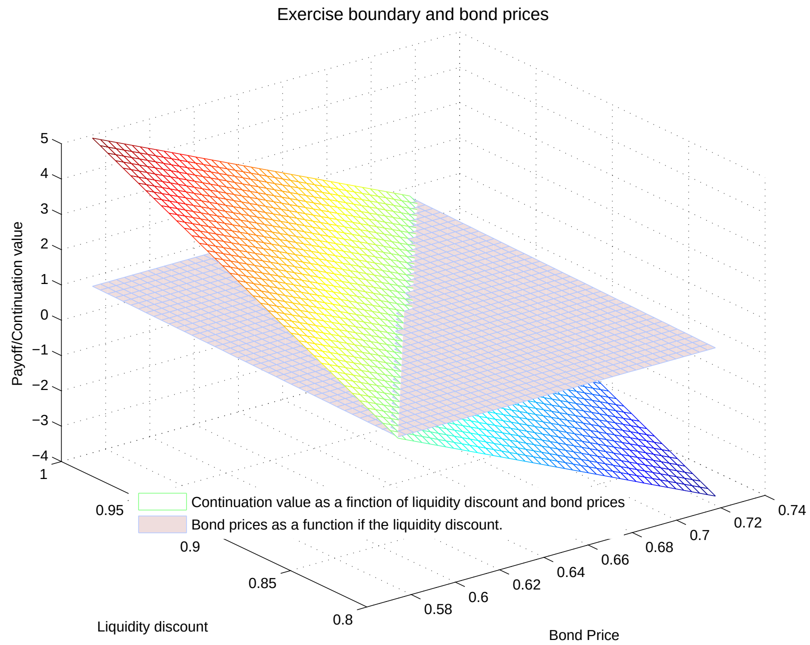

Figure 10: The intersection between the continuation value as a function of the liquidity discount and bond price and bond price as a function of the liquidity discount is the exercise boundary.

Definition 1 *The value function is defined to be*

$$V_t^{*,m} = \sup_{\pi_{\mathbf{k}}} V_t^{\pi_{\mathbf{k}},m} = \sup_{\pi_{\mathbf{k}}} \mathbb{E}_t(\sum_{s=t}^{T} H_s^{\#\{j:\tau_j=s\}}(X_s)).$$

We denote the corresponding optimal policy $\pi^* = \{\tau_m^*, \tau_{m-1}^*, ..., \tau_1^*\}$.

For our purposes it will be more convenient to work with an alternative formulation. Using dynamic programming it is straightforward to see that the value function can be written as follows.

Lemma 4 *(Dynamic programming formulation). The value function* $V_t^{*,m}$ *at time* t *is given by*

$$\begin{aligned} V_T^{*,m} =& H_T^{\min\{k_T,m\}}, \\ V_t^{*,m} =& \max\{H_t^{\min\{k_t,m\}} + \mathbb{E}_t[V_{t+1}^{*,m-\min\{k_t,m\}}], \\ & H_t^{\min\{k_t,m-1\}} + \mathbb{E}_t[V_{t+1}^{*,m-(\min\{k_t,m-1\})}], \\ & ..., H_t^1 + \mathbb{E}_t[V_{t+1}^{*,m-1}], \mathbb{E}_t[V_{t+1}^{*,m}]\}. \end{aligned}$$

Note that for $0 \le i \le k_t$ the quantity

$$H_t^{\min\{k_t,m-i\}} + \mathbb{E}_t[V_{t+1}^{*,m-\min\{k_t,m-i\}}]$$

is the cashflow from the sale of the m-th, $m-1$-th, ..., $m - \min\{k_t, m-i\} + 1$-th units at time t plus the expected future payoff from the remaining $m - \min\{k_t, m-i\}$ units.

We will give definitions for few other useful quantities.

Definition 2 *The continuation value* $C_t^{*,m}$ *at time* t*,is given by*

$$C_t^{*,m} = \mathbb{E}_t[V_{t+1}^{*,m}].$$

Now the dynamic programming equations can be written in terms of the continuation value.

$$\begin{aligned} C_T^{*,m} &= 0, \\ C_t^{*,m} &= \mathbb{E}_t\big[\max\{H_{t+1}^{\min\{k_{t+1},m\}} + C_{t+1}^{*,m-\min\{k_{t+1},m\}}, H_{t+1}^{\min\{k_{t+1},m-1\}} + C_{t+1}^{*,m-(\min\{k_{t+1},m-1\})}, \\ &\quad ..., H_{t+1}^1 + C_{t+1}^{*,m-1}, C_{t+1}^{*,m}\}\big] \end{aligned}$$

An important quantity is the value of an additional unit of the asset in the position.

Definition 3 *The marginal value of one additional unit is denoted by* $\Delta V_t^{*,m}$ *for* $m \geq 1$*:*

$$\Delta V_t^{*,m} = V_t^{*,m} - V_t^{*,m-1}.$$

The marginal value for $m = 1$ *is just the value for one unit*

$$\Delta V_t^{*,1} = V_t^{*,1}.$$

Of course the marginal continuation values can be given by

$$\Delta C_t^{*,m} = C_t^{*,m} - C_t^{*,m-1}.$$

4.1 Numerical Solution

Here we will introduce a method to compute an approximation to the optimal liquidation policy. It can be calculated by a generalization of the Monte Carlo regression method introduced by Longstaff and Schwartz (1998). A more complete discussion about the method can be found in Aleksandrov and Hambly (2008).

With the ordering of the continuation values (the proof of this can be found in Aleksandrov and Hambly (2008))

$$\mathbb{E}_t[\Delta V_{t+1}^{*,m}] \leq \mathbb{E}_t[\Delta V_{t+1}^{*,m-1}], \qquad \forall m, \forall t$$

we can determine the optimal liquidation policy at time level t as

if $i = \max(j : h_t^j \geq \mathbb{E}_t[\Delta V_{t+1}^{*,m-j+1}] | 1 \leq j \leq \min\{m, k_t\})$ - sell i units.

if $\mathbb{E}_t[\Delta V_{t+1}^{*,m}] > h_t^1$ - do not sell.

In terms of the marginal continuation values we can write the strategy as

if $i = \max(j : h_t^j \geq \Delta C_t^{*,m-j+1} | 1 \leq j \leq \min\{m, k_t\})$ - sell i units.

if $\Delta C_t^{*,m} > h_t^1$ - do not sell.

The idea here is to work backwards in time, approximating the marginal continuation value with a linear combination of basis functions. In this way an approximation of the optimal liquidation strategy is found and consequently a lower bound for the value function.

Algorithm

Let $\psi_i : \mathbb{R}^d \to \mathbb{R}$ for $i = 1, \ldots, l$ be the basis functions used for the regression.

1. For all times $t \in \{0, 1, 2, ..., T\}$ and at each point $x \in \mathbb{R}^d$ we define $\Delta \hat{C}_t^m(x)$, an approximation to the $m-$th marginal continuation value $\Delta C_t^{*,m}$, by

$$\Delta \hat{C}_t^m(x) = \sum_{i=1}^{l} c_{t,i}^m \psi_i(x).$$

Of course the optimal continuation values are not known. Thus we use non-optimal continuation values C_t^m defined as follows.

2. Suppose that, working backwards in time and forward from one unit, approximations $\Delta \hat{C}_{t+1}^m, \Delta \hat{C}_{t+1}^{m-1}, ..., \Delta \hat{C}_{t+1}^{m-\min\{m-1,k_{t+1}\}}$ to the $m-$th, $m-1$,...,$m-\min\{m-1,k_{t+1}\}$ marginal continuation value functions have been obtained. Then for path j define the approximate continuation value $C_t^{m,(j)}$ to be

$$C_t^{m,(j)} = \begin{cases} H_{t+1}^{\min\{k_{t+1}^{(j)},m\}}(X_{t+1}^{(j)}) + C_{t+1}^{m-\min\{k_{t+1}^{(j)},m\},(j)}, \\ \qquad \text{if } h_{t+1}^{\min\{k_{t+1}^{(j)},m\}}(X_{t+1}^{(j)}) \geq \Delta \hat{C}_{t+1}^{m-\min\{k_{t+1}^{(j)},m-1\}}(X_{t+1}^{(j)}) \\ H_{t+1}^{\min\{k_{t+1}^{(j)},m-1\}}(X_{t+1}^{(j)}) + C_{t+1}^{m-\min\{k_{t+1}^{(j)},m-1\},(j)}, \\ \qquad \text{if } \min\{k_{t+1}^{(j)}, m-1\} = \max(i : h_{t+1}^i(X_{t+1}^{(j)}) \geq \Delta \hat{C}_{t+1}^{m-i+1}(X_{t+1}^{(j)}) | 1 \leq i \leq \min\{m, k_{t+1}\}) \\ \qquad \vdots \\ C_{t+1}^{m,(j)}, \\ \qquad \text{if } \Delta \hat{C}_{t+1}^m(X_{t+1}^{(j)}) > h_{t+1}(X_{t+1}^{(j)}) \end{cases}$$

The non-optimal $m-$th marginal continuation values are also defined by

$$\Delta C_t^{m,(j)} = C_t^{m,(j)} - C_t^{m-1,(j)}.$$

3. Let $\psi = (\psi_1, \psi_2, ..., \psi_l)$ and $\vec{c}_t^m = (c_{t,1}^m, c_{t,2}^m, ..., c_{t,l}^m)$. If n paths are simulated, an estimate for the regression coefficients would be

$$\vec{c}_t^m = \arg\min_{c \in \mathbb{R}^l} \sum_{j=1}^{n} \left(\Delta C_t^{m,(j)} - \sum_{i=1}^{l} c_{t,i}^m \psi_i(X_t^{(j)})\right)^2.$$

The explicit formulas for the coefficients are

$$\begin{aligned} \vec{c}_t^m &= \Psi^{-1} v^m, \\ \Psi_{l,p} &= \frac{1}{n} \sum_{j=1}^{n} \psi_l(X_t^{(j)}) \psi_p(X_t^{(j)}), \\ v_l^m &= \frac{1}{n} \sum_{j=1}^{n} \psi_l(X_t^{(j)}) \Delta C_t^{m,(j)}. \end{aligned}$$

4. Once the coefficients $c_{t,1}^m, c_{t,2}^m, ..., c_{t,l}^m$ are obtained we can approximate the m-th marginal continuation value, and from there the stopping rule, at any point in the state space. We work backwards in time until we reach $t = 0$.

4.2 Numerical Example

Here we can use the regression approach to find the optimal liquidation strategy. In this way we can show a lower bound for the cash flow that can be guaranteed from the liquidation of the position. As basis functions we use $1, D, \Lambda, P$ and I, which are calculated at each time point of the simulated paths. Here the analytical expression for the bond price is crucial for the simulations. In figures 9 and 10 we can see the exercise boundary obtained as the intersection between the payoff and the continuation value. We consider the following example in the intensity setting. We want to liquidate a position of 20 units of a defaultable zero-coupon bond over 20 trading days (roughly one month). Here we assume that one unit of the bond has notional one. Using the parameter values recovery $\delta = 0.53$, $\sigma_r = 0.2, k = 0.5, r_0 = 0.05, \sigma_\lambda = 0.3, \rho = 2, \nu = 0.03, \lambda_0 = 0.03, \sigma_l = 0.2, \beta = 0.5, \mu = 0.02, l_0 = 0.02$ and maturity of the bond 5 years, we calculate that the price of one unit as 0.6671. In this case the book value of our holding will be $13.34 (= 20 \times 0.6671)$. Using our liquidation model we obtain that on average we can liquidate the portfolio for 11.9717 (with a standard deviation of 1.1476). Thus the actual cash flow from the liquidation is 10.26% less than the book value. The book value is actually more than one standard deviation away ($13.34 > 11.9717 + 1.1476 \sim 13.12$) from what could be achieved after a liquidation using our suggested method and model. In figure 11 we can see the dependence of the liquidation value on the short term price impact α as well as the discount of the liquidation value from the book value.

This discount is due to:

- Downward price pressure from the sale.
- Price fluctuations over the liquidation period (in some cases they are positive and the effect is marginal)
- In some cases inability to sell the whole amount over the liquidation period, due to the unknown random liquidity of the market. We impose a hard restriction here and at the end of the period the unsold units are worthless.

To observe the impact of the regime change in the liquidity factor that occurs around August 2007 on the liquidation value we run the calculation with numbers before and after 1 August 2007. We use the average numbers for the liquidity parameters (given in Table 1) for one of the more stable companies in our set - Goldman Sachs. We run the liquidation algorithm for the average liquidity parameters before and after 1 August 2007, everything else we keep the same. The set up of this liquidation problem is the same as the one above - we want to liquidate 20 zero coupon bonds issued by Goldman, over a horizon of 20 trading days. The selling constraints k_t are also given in the same form as above. The calculation shows that before the beginning of the crisis we can liquidate the position, under our model assumptions, on average for 11.13. After the beginning of the crisis however we can liquidate the same position for average of 10.54. That is a 5.5% discount.

5 Summary

Many improvements and generalisations of the single firm models have been introduced since their first appearances in Merton (1974) and Jarrow and Turnbull (1995), for the structural and intensity approaches respectively. The bond spreads of defaultable bonds over the treasuries are in most of those cases treated as purely default driven. There is empirical evidence that this is not true. Particularly recently, because of the credit crisis good quality debt has been largely undervalued, because of lack

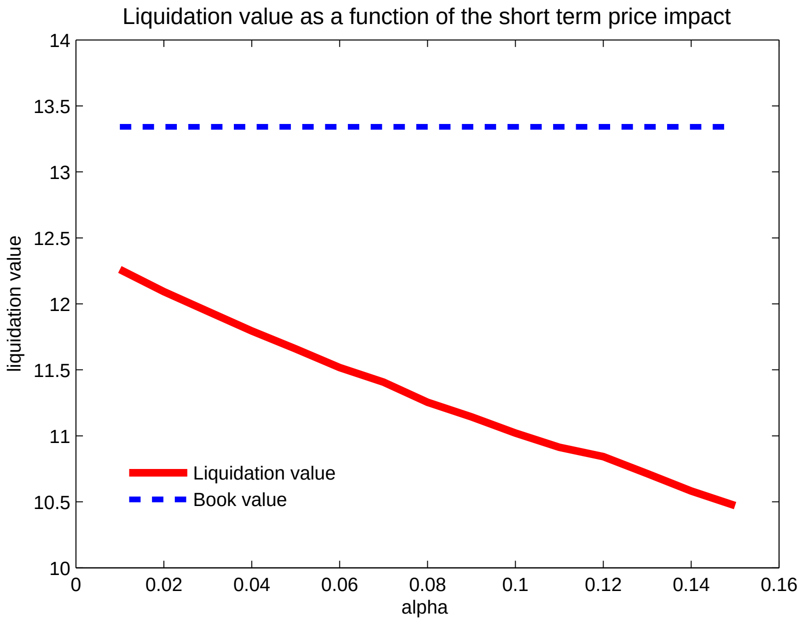

Figure 11: Dependence of the liquidation value on the short term price impact.

of liquidity in the credit market. To cover writedowns and margin calls due to bad debts, funds had to liquidate positions with a better marked-to-market (good quality debt) in a low liquidity environment and thus take big losses. This illustrates another point that the portfolio is only worth what the other market participants are prepared to pay for it and that may be very different from its book value. After these events liquidity risks in the debt market will be watched more carefully and maybe banks and investment funds will also try to adjust their existing default risk models to capture the liquidity risk as well.

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A Kalman Filter for the liquidity process.

We have a one factor model and one observed yield. The formulation of the Kalman filter consists of measurement equations and transition equations. For the liquidity discount factors we have the representation

$$D(l,t,T) = G(t,T)e^{-H(t,T)l},$$

both in the case of mean reverting process and CIR process (G and H are known functions). The liquidity spread can be written as

$$\gamma_l = -\frac{\ln(D(l,t,T))}{T-t} = -\frac{\ln(G(t,T))}{T-t} + \frac{-H(t,T)l}{T-t}.$$

So if we have observation of the liquidity spread on dates $t_1, t_2, ..., t_n$ we can write the measurement equations as

$$\gamma_l(t_i,T) = \alpha(t_i,T) + \xi(t_i,T)l(t_i) + z_i.$$

where z_i is a normal random variable with variance r^2. To discretise the stochastic differential equation we rely on the scheme

$$l(t_i,T) = \mu(1-e^{-\beta\Delta t}) + e^{-\beta\Delta t}l(t_{i-1},T) + \epsilon_i,$$

where

$$\epsilon_i|\mathscr{F}_{i-1} \sim \mathscr{N}(0,Q)$$

Q here is in the case of a mean reverting process

$$Q = \frac{\sigma_l^2}{2\beta}(1-e^{-2\beta\Delta t})$$

and for a CIR process

$$Q = \frac{\mu\sigma_l^2}{2\beta}(1-e^{-\beta\Delta t})^2 + \frac{\sigma_l^2}{\beta}(e^{-\beta\Delta t} - e^{-2\beta\Delta t})l(t_{i-1})$$

For shortness of notation we write:

$$l(t_i,T) = V + Wl(t_{i-1},T) + \epsilon_i,$$

We will give a short description of the procedure from Bolder (2001). The measurement equations define the filtrations $\mathscr{F}_i = \sigma\{\gamma_l(t_1,T), \gamma_l(t_2,T), ..., \gamma_l(t_i,T)\}$.

1 Initializing the state vector. The unconditional mean and variance are used for starting the recursion. The unconditional mean, for both the mean reverting and the CIR models is:

$$\mathbb{E}[l(t_1,T)] = \mathbb{E}[l(t_1,T)|\mathscr{F}_0] = \mu$$

The unconditional variance for the mean reverting model is:

$$\mathrm{var}[l(t_1,T)] = \mathbb{E}[l(t_1,T)|\mathscr{F}_0] = \frac{\sigma_l^2}{2\beta}$$

and the CIR it is

$$\mathrm{var}[l(t_1,T)] = \mathbb{E}[l(t_1,T)|\mathscr{F}_0] = \frac{\sigma_l^2\mu}{2\beta}$$

2 Forecasting the measurement equation. The conditional forecast of the measurement equation has the following form:

$$\mathbb{E}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}] = \alpha + \xi\mathbb{E}[l(t_i,T)|\mathscr{F}_{i-1}]$$

The conditional variance is:

$$\mathrm{var}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}] = \xi^2\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}] + r^2.$$

3 Updating the state vector. We update the conditional equation by:

$$\mathbb{E}[l(t_i,T)|\mathscr{F}_i] = \mathbb{E}[l(t_i,T)|\mathscr{F}_{i-1}] + \xi\frac{\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}]}{\mathrm{var}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}]}(\gamma_l(t_i,T) - \mathbb{E}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}])$$

and variance:

$$\mathrm{var}[l(t_i,T)|\mathscr{F}_i] = \big(1 - \xi^2\frac{\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}]}{\mathrm{var}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}]}\big)\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}]$$

4 Forecasting the state vector. Using the state equation, in this step, a forecast is calculated based on the conditional updated values for the previous period.

$$\mathbb{E}[l(t_{i+1},T)|\mathscr{F}_i] = V + W\mathbb{E}[l(t_i,T)|\mathscr{F}_i]$$

The conditional variance here is:

$$\mathrm{var}[l(t_{i+1},T)|\mathscr{F}_i] = \mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}] - W^2\mathrm{var}[l(t_i,T)|\mathscr{F}_i] + Q$$

5 Maximum likelihood function. The last step is to construct the log-likelihood function Using the first four steps we generate forecasts for the state vector. Under the assumption that the prediction errors of the measurement system are Gaussian, we can calculate the log-likelihood function.

$$f = -\frac{nN\ln(2\pi)}{2} - \frac{1}{2}\sum_{i=1}^{N}[\ln|\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}]| + \frac{(\gamma_l(t_i,T) - \mathbb{E}[\gamma_l(t_i,T)|\mathscr{F}_{i-1}])^2}{\mathrm{var}[l(t_i,T)|\mathscr{F}_{i-1}]}]$$

To find the optimal set of parameters we use an optimization method on f.

In the case of a CIR process the unobservable state variables are distributed conditionally as noncentral χ^2. In the procedure above the distribution has been approximated by a normal with matching first two moments.

Table 1: Parameters estimations for the mean reverting (MR) and CIR processes before and after 1 August 2007.

bond	process	before			after		
		μ	σ	β	μ	σ	β
AIG.GHW	MR	0.0051	0.0029	0.4848	0.0065	0.0042	0.9126
	CIR	0.0075	0.0120	0.8805	0.0070	0.0460	1.2946
AIG.GLP	MR	0.0060	0.0080	1.8014	0.0132	0.0230	4.5542
	CIR	0.0048	0.0124	1.5359	0.0062	0.0267	0.6166
AIG.GPA	MR	0.0057	0.0099	3.8473	0.0106	0.0253	7.6145
	CIR	0.0057	0.0099	0.6286	0.0057	0.0266	0.8944
AIG.QR	MR	0.0056	0.0074	1.3027	0.0096	0.0179	4.2142
	CIR	0.0062	0.0127	0.8781	0.0075	0.0230	1.5203
C.GDS	MR	0.0047	0.0018	0.1520	0.0101	0.0475	10.0751
	CIR	0.0031	0.0273	13.4326	0.0101	0.1163	14.1264
C.HDI	MR	0.0026	0.0056	1.2881	0.0083	0.0183	14.1306
	CIR	0.0030	0.0143	0.9776	0.0053	0.0017	0.0027
C.HDO	MR	0.0031	0.0093	1.6973	0.0101	0.0206	5.7834
	CIR	0.0036	0.0078	0.2472	0.0046	0.0223	1.4203
C.NW	MR	0.0052	0.0046	0.8685	0.082	0.0387	0.0190
	CIR	0.0054	0.0030	0.0132	0.053	0.0038	6.1158
GE.AAD	MR	0.0039	0.0091	1.4348	0.0085	0.0121	3.6040
	CIR	0.0051	0.0148	0.9050	0.0055	0.0176	2.6849
GE.ACE	MR	0.0039	0.0088	1.6064	0.0065	0.0135	3.7464
	CIR	0.0068	0.0087	0.4093	0.0061	0.0242	1.9282
GE.ADF	MR	0.0044	0.0077	1.4919	0.0055	0.094	2.6747
	CIR	0.0051	0.0111	0.6088	0.0054	0.0289	4.0542
GE.HBO	MR	0.0072	0.0105	2.5913	0.0058	0.062	6.7346
	CIR	0.0045	0.0225	3.8041	0.0065	0.0296	4.1422
GS.KJ	MR	0.0045	0.0080	1.2536	0.0073	0.0142	4.9076
	CIR	0.0042	0.0145	0.6730	0.0041	0.0200	0.9629
GS.OU	MR	0.0046	0.0096	1.9975	0.0085	0.0162	5.1010
	CIR	0.0052	0.0134	0.7265	0.0059	0.0294	6.1019
GS.UG	MR	0.0051	0.0090	1.9966	0.0111	0.0179	4.0201
	CIR	0.0043	0.0099	0.5404	0.0059	0.0159	1.1647
GS.UY	MR	0.0040	0.0074	3.0421	0.0066	0.0158	8.5737
	CIR	0.0034	0.0136	1.5917	0.0034	0.0239	1.1541
MER.GGF	MR	0.0037	0.0096	1.9618	0.0095	0.0222	5.9708
	CIR	0.0047	0.0147	0.9194	0.0031	0.0236	0.6763
MER.GGW	MR	0.0031	0.0136	8.6038	0.0055	0.0294	28.7537
	CIR	0.0038	0.0212	2.1055	0.0054	0.0394	17.8579
MER.GHW	MR	0.0039	0.0080	1.5875	0.0081	0.0239	5.2213
	CIR	0.0040	0.0213	3.5702	0.0042	0.0248	1.7148
MER.HE	MR	0.0038	0.0160	7.0717	0.0061	0.0228	18.9605
	CIR	0.0069	0.0102	0.3572	0.0061	0.0378	11.3464