A dual approach to some multiple exercise option problems

N. Aleksandrov  ·  B.M. Hambly

Received: date  /  Accepted: date

Abstract This paper considers the pricing of multiple exercise options in discrete time. This type of option can be exercised up to a fixed finite number of times over the lifetime of the contract. We allow multiple exercise of the option at each time point up to a constraint, a feature relevant for pricing swing options in energy markets. It is shown that, in the case where an option can be exercised an equal number of times at each time point, the problem can be reduced to the case of a single exercise possibility at each time. In the general case there is not a solution of this type. By developing a dual representation for the problem, algorithms are suggested for calculating both lower and upper bounds for the prices of such multiple exercise options.

Keywords Multiple optimal stopping  ·  Dual approach  ·  Multiple exercise options  ·  Swing options

Mathematics Subject Classification (2000) 60G40  ·  91B28  ·  90C39  ·  90C46

1 Introduction

There are many different types of financial derivative with early exercise features actively traded in financial markets. The simplest being the American
option in which the holder of the option can exercise and receive the payoff at any time up until maturity. Even in this simple case determining the price accurately can be challenging if the underlying model for the asset price has several driving factors. Our aim here is to consider approaches to developing Monte Carlo algorithms for the pricing of more complex early exercise options.

For most option pricing problems three main types of numerical method are considered in the literature: lattices, finite difference schemes and Monte Carlo methods. The first two approaches usually work particularly well for simple options on a single underlying. With an increase in the dimension of the problem, however, the performance of the lattice and finite difference schemes is typically poor, as the computational effort grows exponentially with the number of underlyings. Thus there is a need to develop efficient Monte Carlo methods as these techniques only see a small growth in computational effort with dimension.

The value of an American option is the return achieved by the holder of the option exercising optimally. Thus it is the solution of an optimal stopping problem and as such is usually tackled with backward dynamic programming. This type of backward recursion is not straightforward for a simulation based approach.

The first attempt to apply Monte Carlo techniques to American option pricing, using a bundling approach was by Tilley in [14]. The estimates obtained there, however, do not converge to the true value. Broadie and Glasserman [5] developed the first algorithm in which the suggested low- and high-biased estimates are proved to converge to the true value. Their approach can also deal with high-dimensional American options. However the disadvantage of their algorithm is that the computational effort still grows exponentially with the number of possible exercise dates.

The method most favored by practitioners these days is the method suggested by Longstaff and Schwartz [11] (and independently by Tsitsiklis and van Roy [15]). The method relies on approximating the value function by linear regression on a suitable space of basis functions and using this to determine the optimal stopping policy. By construction this optimal stopping policy gives a lower bound for the option price. The method is comparatively easy to implement and, for properly chosen regression functions, gives a good estimate for the price, [7].

In the work of Rogers [13] and Haugh and Kogan [8] (see also [1]), ideas for calculating an upper bound for the option price were developed. Both rely on a duality approach, expressing the problem as a minimization over a space of martingales. This provides a method to compute an upper bound, by constructing a martingale which is a good approximation to the optimal one.

More recently, there have been several articles discussing the more general multiple early exercise problem. Pricing a derivative with several early exercise opportunities is equivalent to solving a multiple optimal stopping problem. In this paper we consider such multiple optimal stopping problems where it is possible to stop more than once at the same time point. In option terminology we can exercise the option a certain number of times at a given time. The
motivation for the study of the type of multiple optimal stopping problem considered here comes from the energy market contracts called swing options.

Swing options are actively traded in gas and electricity markets; here we discuss the electricity market. The most common contracts in electricity markets are electricity forward contracts. They are obligations to buy or sell a fixed amount of electricity at a pre-specified price (the forward price $F$) over a certain time period in the future. For simplicity we consider spot forwards in which the price is fixed for a certain time in the future (the expiration time $T$). If $S_t$ is the electricity spot price at time $t$, then the payoff of the forward contract is $(S_T - F)$. The settlement price $S_T$ is usually calculated based on the average price of electricity over the delivery period at the maturity $T$. Based on the delivery period during a day, electricity forwards can be categorized as forwards on on-peak electricity, off-peak electricity and 24 hour electricity.

Often the buyer is not sure about the quantity he will want to purchase due to weather changes, particular spot price expectations or other reasons. In that case the forward contracts are coupled with swing options. The simplest swing options give their holder the right to purchase each day (on- or off- peak time) for a specified period electricity at a fixed price $K$ (strike price). Thus the payoff in this case is that of a call $(S_t - K)^+$. When exercising a swing option, the purchased quantity may vary (or swing) between a minimum and a maximum volume. An alternative version is when the swing contract allow the holder either to buy or to sell a certain quantity of electricity on a given day. The total quantity of electricity purchased for the period should be within minimum and maximum volume levels.

There are now several papers ([9], [12], [10], [4], [6], [2], [3]) on pricing swing contracts. In this paper we consider an extension of the ideas developed in [12]. In that paper a stylized swing contract was considered where the number of exercise possibilities was restricted to some number $M < T$, which is an analogue of an American option that can be exercised several times during its lifetime. Our consideration of swing contracts will be slightly more general than this. We will look into options with different volume restrictions each day and a total volume restriction of the contract, though still discrete. This type of restriction is motivated by the fact that the spot price of the electricity together with its demand have some deterministic trends. For example the electricity spot price (and demand) is in general lower on weekends, which could correspond to less need for optionality in the swing contract. In short, the buyer doesn’t have to pay for rights he doesn’t need.

In a number of other papers e.g. [2], [3] the volume constraint is a continuous parameter between specified bounds. We do not consider that case for ease of developing our dual representation. We also avoid any discussion of hedging or other serious financial issues in pricing and risk management of swing contracts and just assume that there is a risk neutral measure and that our aim is to compute the price of a multiple exercise option in this measure.

The paper is organized as follows. In Section 2 we begin by giving the mathematical formulation of the problem for finding the price of the multiple exercise options considered here. We formulate both the optimal stopping
problem and the dynamic programming equations. In Section 3 we prove some properties of the option price process. We prove that the marginal value i.e. the value of an additional exercise opportunity is decreasing as a function of the number of exercise opportunities left. In Section 4 we deal with the case where the option can be exercised up to the same number of times \( k \) for all time levels. This is then the easy case with a solution of bang-bang type in that we prove it is essentially equivalent to \( k \) options in which it is possible to exercise only once at each time point. In Section 5 we obtain a dual representation for the multiple exercise option. In the last section some implementational issues and a numerical example are considered based on the Longstaff-Schwartz regression ideas and the dual representation.

2 Definitions and problem formulation

We consider an economy in discrete time defined up to a finite time horizon \( T \). We assume a complete financial market described by the filtered probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,...,T}, P)\) with \((X_t)_{t=0,1,...,T} \in \mathbb{R}^d\) a discrete \(\mathcal{F}_t\)-adapted Markov chain describing the state of the economy - the price of the underlying assets and any other variables that affect the dynamics of the underlyings. Throughout we will assume that \( P \) is a risk neutral pricing measure and write \( E_t(X) = E(X|\mathcal{F}_t) \) for any random variable \( X \) on our probability space. We will consider multiple exercise options in this market and write \( h_t(X_t) \) for the payoff from one exercise of the option at time \( t \) when the asset price is \( X_t \) (we often suppress the argument of \( h_t \)). We assume that the payoff is non-negative \((h_t(x) \geq 0 \text{ for all } x \in \mathbb{R}^d, t = 0,\ldots,T)\). The holder of such a multiple exercise option has the opportunity of \( k_t \) exercises, that is to receive \( k_t h_t \), at time \( t \), where \( k_t \) will typically vary with time. We will use the notation \( V_{t}^{\pi,m,k} \) for the price (value function) at time \( t \) of an option, which can be exercised \( k = \{k_0, k_1, k_2,\ldots,k_T\} \) times at the corresponding time points and has \( m \) exercise opportunities left. For simplicity of notation we will assume the risk free rate is zero and exclude discounting factors from our expressions.

In order to give a formal definition for the value function \( V_{t}^{\pi,m,k} \) we require a little notation. We define an exercise policy \( \pi \) to be a set of stopping times \( \{\tau_i\}_{i=1}^m \) with \( \tau_1 \leq \tau_2 \leq \cdots \leq \tau_m \) and \( \# \{ j : \tau_j = s \} \leq k_s \). Then the value of the policy \( \pi \) at time \( t \) is given by

\[
V_{t}^{\pi,m,k} = E_t(\sum_{i=1}^{m} h_{\tau_i}(X_{\tau_i})).
\]

**Definition 1** The value function is defined to be

\[
V_{t}^{\pi,m,k} = \sup_{\pi} V_{t}^{\pi,m,k} = \sup_{\pi} E_t(\sum_{i=1}^{m} h(X_{\tau_i})).
\]

For our purposes it will be more convenient to work with two alternative formulations. Firstly using dynamic programming it is straightforward to see that the value function can be written as follows.
Lemma 1 (Multiple exercise option price - Dynamic programming formulation). The price $V^*_t,m,k$ at time $t$ of an option with payoff function $\{h_s, t \leq s \leq T\}$ which could be exercised $k_t$ times per single exercise time $s \in \{t, \ldots, T\}$ with $m$ exercise opportunities in total for $m > k_t$ is given by

\[
V^*_t,m,k = k_T h_T,
\]

\[
V^*_t,m,k = \max \{ k_t h_t + E_t[V^*_{t+1}^{m-k_t,k_t}], (k_t - 1) h_t + E_t[V^*_{t+1}^{m-(k_t-1),k_t}], \ldots, h_t + E_t[V^*_{t+1}^{m-1,k_t}], E_t[V^*_{t+1}^{*,m,k_t}] \}.
\]

For $m \leq k_t$ we have

\[
V^*_t,m,k = mh_T,
\]

\[
V^*_t,m,k = \max \{ mh_t, (m-1) h_t + E_t[V^*_{t+1}^{*,m,1,k_t}], \ldots, E_t[V^*_{t+1}^{*,m,k_t}] \}.
\]

Note that for $0 \leq i \leq k_t$ the quantity

\[
(k_t - i) h_t + E_t[V^*_t,m-k_t+i,k_t]
\]

is the payoff under the exercise of the $m$-th, $m-1$-th, ..., $m-k_t+i+1$-th exercise opportunities at time $t$ plus the expected future payoff under the remaining $m-k_t+i$ exercise opportunities. There is also an optimal stopping problem formulation which is again straightforward to derive.

Lemma 2 (Multiple exercise option price - Optimal stopping problem formulation). The price $V^*_t,m,k$ of an option, which could be exercised $k_t$ times per single exercise time $t$ with $m$ exercise opportunities in total is given by

\[
V^*_t,m,k = \max \limits_{t \leq \tau \leq T} E_\tau \left[ \max \{ k_\tau h_\tau + E_\tau[V^*_\tau,m-k_\tau,k_\tau], (k_\tau - 1) h_\tau + E_\tau[V^*_\tau,m-(k_\tau-1),k_\tau], \ldots, h_\tau + E_\tau[V^*_\tau,m-1,k_\tau]\} \right]
\]

(In the max bracket only those terms, which exist are taken)

An useful quantity in the pricing of American options in general is the continuation value $C^*_t$, which is the expected cashflow at the next time-step if the option has not been exercised. In our context we define the continuation value in a similar way.

Definition 2 The continuation value $C^*_t,m,k$ at time $t$ of an option, which could be exercised $k_t$ times per single exercise time $t$ with $m$ exercise opportunities in total is given by

\[
C^*_t,m,k = E_t[V^*_{t+1}^{*,m,k_t}]
\]

i.e. this the expected value of the option given that it is not exercised at the current time-step.
Now the dynamic programming equations can be written in terms of the continuation value.

\[ C^*_T, m, k = 0, \]

\[ C^*_t, m, k = E_t \left[ \max \{ k_{t+1} h_{t+1} + C^*_{t+1}, (k_{t+1} - 1) h_{t+1} + C^*_{t+1}, \ldots, h_{t+1} + C^*_{t+1}, m, k \} \right], \quad m > k_{t+1}, \]

\[ C^*_t, m, k = E_t \left[ \max \{ m h_{t+1}, (m - 1) h_{t+1} + C^*_{t+1}, \ldots, h_{t+1} + C^*_{t+1}, m, k \} \right], \quad m \leq k_{t+1}. \]

When valuing multiple exercise options an important quantity is the value of an additional exercise opportunity.

**Definition 3** The marginal value of one additional exercise opportunity is denoted by \( \Delta V^*_t, m, k \) for \( m \geq 1: \)

\[ \Delta V^*_t, m, k = V^*_t, m, k - V^*_t, m-1, k. \]

The marginal value for \( m = 1 \) is just the option value for one exercise opportunity

\[ \Delta V^*_t, 1, k = V^*_t, 1, k. \]

From the dynamic programming equations we can see that

\[ E_t[V^*_{t+1}, m, k] \leq V^*_t, m, k. \]

Thus the process \( V^*_t, m, k \) is a supermartingale and has a Doob decomposition

\[ V^*_t, m, k = V^*_0, m, k + M^*_t, m, k - D^*_t, m, k, \]

where \( M^*_t, m, k \) is a martingale vanishing at \( t = 0 \) and \( D^*_t, m, k \) is a previsible increasing process also vanishing at \( t = 0 \). The increments of the process \( D^*_t, m, k \) are given by the previsible parts of the option price process

\[ D^*_t, m, k - D^*_t-1, m, k = V^*_t, m, k - E_t[V^*_{t+1}, m, k]. \]

The martingale difference is

\[ M^*_t, m, k - M^*_t-1, m, k = V^*_t, m, k - E_t[V^*_{t+1}, m, k]. \]

Corresponding to the marginal values we can further introduce \( \Delta M^*_t, m, k \) and \( \Delta D^*_t, m, k \) as

\[ \Delta M^*_t, m, k = M^*_t, m, k - M^*_t-1, m, k, \]

\[ \Delta D^*_t, m, k = D^*_t, m, k - D^*_t-1, m, k. \]
Here $\Delta M_{t}^{*,m,k}$, being a difference of martingales is again a martingale and $\Delta D_{t}^{*,m,k}$ is a previsible process. Both vanishing at $t = 0$. Of course the marginal continuation values can be given by

$$\Delta C_{t}^{*,m,k} = C_{t}^{*,m,k} - C_{t}^{*,m-1,k}.$$ 

For simplicity of notation we introduce

$$A_{t}^{*,m,l} = [h_{t} - E_{t}[\Delta V_{t+1}^{*,m,k}]] + [h_{t} - E_{t}[\Delta V_{t+1}^{*,m-1,k}]] + \ldots + [h_{t} - E_{t}[\Delta V_{t+1}^{*,m-l+1,k}]] + \ldots + ]_{+},$$

where we have embedded in a sum the positive parts of $l$ terms of the kind $h_{t} - E_{t}[\Delta V_{t+1}^{*,m-k}]$. 

### 3 Properties of the marginal values

Here we will state and prove several properties of the marginal values, which we will use later.

**Proposition 1** The increment of the process $D_{t}^{*,m,k}$ can be expressed as

$$D_{t+1}^{*,m,k} - D_{t}^{*,m,k} = A_{t}^{*,m,\min\{k_{t}, m\}}.$$

**Proof:** Using the dynamic programming formulation and assuming $k_{t} < m$,

\[
\begin{align*}
D_{t+1}^{*,m,k} - D_{t}^{*,m,k} &= V_{t+1}^{*,m,k} - E_{t}[V_{t+1}^{*,m,k}] \\
&= \max\{k_{t}h_{t} + E_{t}[V_{t+1}^{*,m-k_{t},k}], (k_{t} - 1)h_{t} + E_{t}[V_{t+1}^{*,m-(k_{t}-1),k}], \\
&\quad \ldots, h_{t} + E_{t}[V_{t+1}^{*,m-k_{t}+1,k}], E_{t}[V_{t+1}^{*,m,k}] - E_{t}[V_{t+1}^{*,m,k}] \} \\
&= \left[\max\{k_{t}h_{t} + E_{t}[V_{t+1}^{*,m-k_{t},k}], (k_{t} - 1)h_{t} + E_{t}[V_{t+1}^{*,m-(k_{t}-1),k}], \right. \\
&\quad \left. \ldots, h_{t} + E_{t}[V_{t+1}^{*,m-k_{t}+1,k}], E_{t}[V_{t+1}^{*,m,k}] \} - E_{t}[V_{t+1}^{*,m,k}] \right] + \\
&= [h_{t} - E_{t}[\Delta V_{t+1}^{*,m,k}]] + \ldots + [k_{t}h_{t} + E_{t}[V_{t+1}^{*,m-k_{t},k}], (k_{t} - 1)h_{t} + E_{t}[V_{t+1}^{*,m-(k_{t}-1),k}]] + \ldots + [k_{t}h_{t} + E_{t}[V_{t+1}^{*,m-k_{t},k}], (k_{t} - 1)h_{t} + E_{t}[V_{t+1}^{*,m-(k_{t}-1),k}]] + \ldots + [h_{t} - E_{t}[\Delta V_{t+1}^{*,m,k}]] + \ldots + h_{t} - E_{t}[\Delta V_{t+1}^{*,m,k}] + \ldots + ]_{+}.
\end{align*}
\]

Proceeding in this way we have the result. The case where $k_{t} \geq m$ follows the same argument. 

Now we will write the marginal value as an optimal stopping problem.

**Proposition 2** The marginal value $\Delta V_{t}^{*,m,k}$ can be written as

$$\Delta V_{t}^{*,m,k} = \max_{t \leq \tau \leq T} E_{t}[h_{\tau} + A_{\tau}^{*,m-1,\min\{k_{\tau}-1,m-1\}} - D_{\tau+1}^{*,m-1,k}] + D_{t}^{*,m-1,k}.$$
Proof: Using the optimal stopping formulation

\[ \Delta V_t^{*,m,k} = V_t^{*,m,k} - V_t^{*,m-1,k} \]

\[ = \max_{t \leq \tau \leq T} E_{t} \left[ \max\left\{ k\tau \omega + E_{\tau}[V_{\tau+1}^{*,m-1,k}], (k\tau - 1)\omega + E_{\tau}[V_{\tau+1}^{*,m-(k\tau-1),k}], \ldots, h_{\tau} + E_{\tau}[V_{\tau+1}^{*,m-1,k}] \right\} \right] - V_t^{*,m-1,k} \]

\[ = \max_{t \leq \tau \leq T} E_{t} \left[ \max\left\{ k\tau \omega + E_{\tau}[V_{\tau+1}^{*,m-1,k}], (k\tau - 1)\omega + E_{\tau}[V_{\tau+1}^{*,m-(k\tau-1),k}], \ldots, h_{\tau} + E_{\tau}[V_{\tau+1}^{*,m-1,k}] \right\} \right] - V_t^{*,m-1,k} \]

Thus

\[ \Delta V_t^{*,m,k} = \max_{t \leq \tau \leq T} E_{t} \left[ h_{\tau} + A^{*,m-1,\min(k\tau-1,m-1)}_{\tau+1} - (D^{*,m-1,k}_{\tau+1} - D^{*,m-1,k}_{\tau}) \right] + V_t^{*,m-1,k} \]

\[ + (M^{*,m-1,k}_{\tau+1} - M^{*,m-1,k}_{\tau}) + \left( M^{*,m-1,k}_{\tau} - M^{*,m-1,k}_{\tau-1} \right) \]

\[ - V_t^{*,m-1,k} \]

The term \( E_{\tau}[V_{\tau+1}^{*,m-1,k}] \) is just the predictable part of the option value process. Also from the Doob decomposition of \( V_{\tau+1}^{*,m-1,k} \) we have

\[ V_{\tau+1}^{*,m-1,k} = V_{\tau+1}^{*,m-1,k} + (M^{*,m-1,k}_{\tau+1} - M^{*,m-1,k}_{\tau}) + (D^{*,m-1,k}_{\tau+1} - D^{*,m-1,k}_{\tau}). \]

Thus

\[ \Delta V_t^{*,m,k} = \max_{t \leq \tau \leq T} E_{t} \left[ h_{\tau} + A^{*,m-1,\min(k\tau-1,m-1)}_{\tau+1} - (D^{*,m-1,k}_{\tau+1} - D^{*,m-1,k}_{\tau}) \right] + V_t^{*,m-1,k} \]

\[ + (M^{*,m-1,k}_{\tau+1} - M^{*,m-1,k}_{\tau}) + \left( M^{*,m-1,k}_{\tau} - M^{*,m-1,k}_{\tau-1} \right) \]

\[ - V_t^{*,m-1,k} \]

We are now ready to prove

Proposition 3 The marginal value is a decreasing function of the number of exercise opportunities i.e. for \( m \geq 1 \)

\[ \Delta V_t^{*,m+1,k} \leq \Delta V_t^{*,m,k}, \quad \forall t. \]

Proof: We will use induction by \( m \). First step \( m = 1 \).

\[ A^{*,1,\min(k\tau-1,1)} - D^{*,1,k}_{\tau+1} + D^{*,1,k}_{\tau} = \left\{ \begin{array}{ll} -D^{*,1,k}_{\tau+1} + D^{*,1,k}_{\tau}, & k\tau = 1 \\ -D^{*,1,k}_{\tau+1} + D^{*,1,k}_{\tau}, & k\tau > 1 \end{array} \right. \]

but in both cases it is a negative quantity. Thus

\[ \Delta V_t^{*,2,k} = \max_{t \leq \tau \leq T} E_{t} \left[ h_{\tau} + A^{*,1,\min(k\tau-1,1)}_{\tau+1} - D^{*,1,k}_{\tau+1} + D^{*,1,k}_{\tau} \right] \]

\[ \leq \max_{t \leq \tau \leq T} E_{\tau} \left[ h_{\tau} \right] \]

\[ = \Delta V_t^{*,1,k} \]
and the first step is proved. By the inductive hypothesis we can assume that
\( \Delta V^t_{*,l,k} \leq \Delta V^t_{*,l-1,k} \), for any \( l \leq m \). Now we will prove that \( \Delta V^t_{*,m+1,k} \leq \Delta V^t_{*,m,k} \). First we will show that
\[
D^t_{*,m+1,k} - D^t_{*,m,k} \geq D^t_{*,m-1,k} - D^t_{*,m-1,k}, \quad \forall t,
\]
which is equivalent to
\[
A^t_{*,m,min(k_t,m)} \geq A^t_{*,m-1,min(k_t,m-1)}.
\]
From the induction hypothesis we have
\[
A^t_{*,m-1,min(k_t,m-1)} \geq A^t_{*,m-1,min(k_t,m-1)}.
\]
Also the summation in \( A^t_{*,m,min(k_t,m)} \) is 'deeper' than the summation \( A^t_{*,m,min(k_t,m-1)} \) and the desired inequality follows. From there
\[
\begin{align*}
D^t_{*,m,k} - D^t_{*,m,k} &\leq (D^t_{*,m-1,k} - D^t_{*,m-1,k}) \\
&\geq D^t_{*,m,k} - D^t_{*,m,k} - (D^t_{*,m-1,k} - D^t_{*,m-1,k}) \\
&= A^t_{*,m,min(k_t,m)} - A^t_{*,m-1,min(k_t,m-1)},
\end{align*}
\]
Now we will show that
\[
\begin{align*}
A^t_{*,m,min(k_t,m)} - A^t_{*,m-1,min(k_t,m-1)} &\geq 0, \\
A^t_{*,m,min(k_t,m-1)} - A^t_{*,m-1,min(k_t,m-1)} &\geq 0,
\end{align*}
\]
For \( m < k_t \) we have equality, for \( m = k_t \) the inequality is equivalent to
\[
A^t_{*,m,m} \geq A^t_{*,m,m-1},
\]
which is again trivially true. When \( m > k_t \) the inequality is equivalent to
\[
A^t_{*,m,k_t} - A^t_{*,m-1,k_t} \geq A^t_{*,m,k_t} - A^t_{*,m-1,k_t}, \quad \forall t,
\]
which follows from the fact that
\[
\Delta V^t_{*,m-k_t+1,k_t} \leq \Delta V^t_{*,m-k_t,k_t}, \quad \forall t.
\]
Thus
\[
\begin{align*}
D^t_{*,m,k} - D^t_{*,m,k} &\leq (D^t_{*,m-1,k} - D^t_{*,m-1,k}) \\
&\geq A^t_{*,m,min(k_t,m-1)} - A^t_{*,m-1,min(k_t,m-1)}, \quad \forall t \geq t.
\end{align*}
\]
The last step is now
\[
\begin{align*}
\Delta V^t_{*,m+1,k} &= \max_{t \leq \tau \leq T} E_t \left[ h_t + A^t_{*,m,min(k_t-1,m)} - D^t_{*,m,k} \right] + D^t_{*,m,k} \\
&\leq \max_{t \leq \tau \leq T} E_t \left[ h_t + A^t_{*,m-1,min(k_t-1,m-1)} - D^t_{*,m-1,k} \right] + D^t_{*,m-1,k} \\
&= \Delta V^t_{*,m,k},
\end{align*}
\]
and the proof is completed. \( \square \)
Remark 1 In the proof of the Proposition 3 we proved that
\[ D_{t+1}^{*,m,k} - D_t^{*,m,k} \geq D_{t+1}^{*,m-1,k} - D_t^{*,m-1,k} , \]
for any \( m > 1 \). This inequality is equivalent to
\[ \Delta D_{t+1}^{*,m,k} \geq \Delta D_t^{*,m,k} . \]

Remark 2 From
\[ E_t[\Delta V_{t+1}^{*,m,k}] \leq E_t[\Delta V_{t+1}^{*,m-1,k}] \leq \ldots \leq E_t[\Delta V_{t+1}^{*,m-k+1,k}] \]
we can determine the optimal exercise strategy as

- if \( h_t \geq E_t[\Delta V_{t+1}^{*,m-k+1,k}] \) - exercise \( k_t \) times
- \ldots
- if \( E_t[\Delta V_{t+1}^{*,m-1,k}] > h_t \geq E_t[\Delta V_{t+1}^{*,m,k}] \) - exercise once
- if \( E_t[\Delta V_{t+1}^{*,m,k}] > h_t \) - do not exercise

Proposition 4 The marginal value process is a supermartingale i.e. for \( m \geq 1 \)
\[ \Delta V_{t}^{*,m,k} \geq E_t[\Delta V_{t+1}^{*,m,k}] . \]

Proof: Using the Doob decomposition
\[ V_{t}^{*,m,k} = V_0^{*,m,k} + M_{t}^{*,m,k} - D_{t}^{*,m,k} \]
and
\[ V_{t}^{*,m-1,k} = V_0^{*,m-1,k} + M_{t}^{*,m-1,k} - D_{t}^{*,m-1,k} \]
we have
\[ \Delta V_{t}^{*,m,k} = \Delta V_0^{*,m,k} + \Delta M_{t}^{*,m,k} - \Delta D_{t}^{*,m,k} . \tag{1} \]

Analogously
\[ \Delta V_{t+1}^{*,m,k} = \Delta V_0^{*,m,k} + \Delta M_{t+1}^{*,m,k} - \Delta D_{t+1}^{*,m,k} . \tag{2} \]

Using the fact that \( \Delta M_{t+1}^{*,m,k} \) is a martingale by taking the conditional expectation \( E_t \) of (2) and subtracting from it (1) we have
\[ \Delta V_{t}^{*,m,k} - E_t[\Delta V_{t+1}^{*,m,k}] = \Delta D_{t+1}^{*,m,k} - \Delta D_{t}^{*,m,k} . \]

From Remark 1 the last quantity is positive and the proposition follows. \( \square \)
4 Degenerate case

In this section we will consider the case, when the number of exercises of the option allowed is the same at each time point i.e. \( k_0 = k_1 = k_2 = \ldots = k_T = k \). In this case we will prove that the option is equivalent to \( k \) options which can only be exercised one at a time.

**Proposition 5** For any \( i \geq 0 \)

\[
\Delta V_{i}^{*,k_1+1,k} = \Delta V_{i}^{*,k_1+2,k} = \ldots = \Delta V_{i}^{*,k_1+k,k}, \quad \forall i.
\]

**Proof:** We will prove this proposition by induction on \( i \). Let \( i = 0 \). We will use backward induction with respect to \( t \). For \( t = T \) we have

\[
V_T^{*,j,k} = jh_T, \quad j = 1, 2, \ldots, k
\]

and

\[
\Delta V_T^{*,1,k} = \Delta V_T^{*,2,k} = \ldots = \Delta V_T^{*,k,k}.
\]

Now from the induction hypothesis for \( t \) we have that

\[
\Delta V_{t+1}^{*,1,k} = \Delta V_{t+1}^{*,2,k} = \ldots = \Delta V_{t+1}^{*,k,k}.
\]

Using this for any \( m \leq k \) we have that

\[
V_t^{*,m,k} = \begin{cases} mh_t, & h_t \geq \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] \\ \mathcal{E}_t[V_t^{*,m,k}], & \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] > h_t \end{cases}
\]

and

\[
V_t^{*,m-1,k} = \begin{cases} (m-1)h_t, & h_t \geq \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] \\ \mathcal{E}_t[V_t^{*,m-1,k}], & \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] > h_t \end{cases}
\]

Thus

\[
\Delta V_t^{*,m,k} = \begin{cases} h_t, & h_t \geq \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] \\ \mathcal{E}_t[\Delta V_{t+1}^{*,m,k}], & \mathcal{E}_t[\Delta V_{t+1}^{*,1,k}] > h_t \end{cases}
\]

From here clearly

\[
\Delta V_t^{*,1,k} = \Delta V_t^{*,2,k} = \ldots = \Delta V_t^{*,k,k}.
\]

Thus the induction on \( t \) is completed and we have the first step of the induction on \( m \).

Now from the induction hypothesis we have

\[
\Delta V_{t}^{*,k_1+1} = \Delta V_{t}^{*,k_1+2} = \ldots = \Delta V_{t}^{*,k_1+k}, \quad \forall t.
\]

We will prove that

\[
\Delta V_{t}^{*,k_1+1} = \Delta V_{t}^{*,k_1+2} = \ldots = \Delta V_{t}^{*,k_1+k}, \quad \forall t.
\]
Again we will use backward induction on \( t \). For \( t = T \)
\[
\Delta V^{*,k(i+1)+1,k}_T = \Delta V^{*,k(i+1)+2,k}_T = \ldots = \Delta V^{*,k(i+1)+k,k}_T = 0.
\]

Now suppose we have
\[
\Delta V^{*,k(i+1)+1,k}_{t+1} = \Delta V^{*,k(i+1)+2,k}_{t+1} = \ldots = \Delta V^{*,k(i+1)+k,k}_{t+1},
\]
we shall prove that
\[
\Delta V^{*,k(i+1)+1,k}_t = \Delta V^{*,k(i+1)+2,k}_t = \ldots = \Delta V^{*,k(i+1)+k,k}_t.
\]

For \( m > k \) we have
\[
V^{*,m,k}_t = \begin{cases} 
  kh_t + E_t[V^{*,m-k,k}_{t+1}], & h_t \geq E_t[\Delta V^{*,m-k+1,k}_{t+1}] \\
  \ldots \\
  h_t + E_t[V^{*,m-1,k}_{t+1}], & E_t[\Delta V^{*,m-1,k}_{t+1}] > h_t \geq E_t[\Delta V^{*,m,k}_{t+1}] \\
  E_t[V^{*,m,k}_{t+1}], & E_t[\Delta V^{*,m,k}_{t+1}] > h_t
\end{cases}
\]
and
\[
V^{*,m-1,k}_t = \begin{cases} 
  kh_t + E_t[V^{*,m-k-1,k}_{t+1}], & h_t \geq E_t[\Delta V^{*,m-k,k}_{t+1}] \\
  \ldots \\
  h_t + E_t[V^{*,m-2,k}_{t+1}], & E_t[\Delta V^{*,m-2,k}_{t+1}] > h_t \geq E_t[\Delta V^{*,m-1,k}_{t+1}] \\
  E_t[V^{*,m-1,k}_{t+1}], & E_t[\Delta V^{*,m-1,k}_{t+1}] > h_t
\end{cases}
\]

The marginal value then can be written as
\[
\Delta V^{*,m,k}_t = \begin{cases} 
  E_t[\Delta V^{*,m-k,k}_{t+1}], & h_t \geq E_t[\Delta V^{*,m-k,k}_{t+1}] \\
  h_t, & E_t[\Delta V^{*,m-k,k}_{t+1}] > h_t \geq E_t[\Delta V^{*,m,k}_{t+1}] \\
  E_t[\Delta V^{*,m,k}_{t+1}], & E_t[\Delta V^{*,m,k}_{t+1}] > h_t
\end{cases}
\]

In this representation of \( \Delta V^{*,m,k}_t \), using both induction hypotheses, it is easy to see that
\[
\Delta V^{*,k(i+1)+1,k}_t = \Delta V^{*,k(i+1)+2,k}_t = \ldots = \Delta V^{*,k(i+1)+k,k}_t.
\]
Thus the inner and the outer inductions, together with the proof, are completed.

**Proposition 6** For any \( m \geq 1 \)
\[
V^{*,km,k}_t = kV^{*,m,1}_t.
\]
Proof: We will prove this proposition by induction on \( m \). We already have the first step of the induction \( m = 1 \) from Proposition 5. Now the induction hypothesis is

\[
V^{*,k(m-1),k}_{t} = kV^{*,m-1,1}_{t}, \quad \forall t.
\]

We also use inner backward induction on \( t \). For \( t = T \)

\[
V^{*,km,k}_{T} = kh_{T} = kV^{*,m,1}_{T}.
\]

By Proposition 5 all the middle cases in the right hand side of

\[
V^{*,mk,k}_{t} = \begin{cases} 
kh_{t} + E_{t}[V^{*,mk-k,k}_{t+1}], & h_{t} \geq E_{t}[\Delta V^{*,mk-k+1,k}_{t+1}]
\end{cases}
\]

\[
E_{t}[V^{*,mk,k}_{t+1}], & E_{t}[\Delta V^{*,mk,k}_{t+1}] > h_{t}\]

will collapse and we are left with

\[
V^{*,mk,k}_{t} = \begin{cases} 
kh_{t} + E_{t}[V^{*,(m-1)k,k}_{t+1}], & h_{t} \geq E_{t}[\Delta V^{*,mk,k}_{t+1}]
\end{cases}
\]

\[
E_{t}[V^{*,mk,k}_{t+1}], & E_{t}[\Delta V^{*,mk,k}_{t+1}] > h_{t}.
\]

We also have

\[
kV^{*,m,1}_{t} = \begin{cases} 
kh_{t} + kE_{t}[V^{*,(m-1)k,1}_{t+1}], & h_{t} \geq E_{t}[\Delta V^{*,m,1}_{t+1}]
\end{cases}
\]

\[
kE_{t}[V^{*,m,1}_{t+1}], & E_{t}[\Delta V^{*,m,1}_{t+1}] > h_{t}.
\]

We clearly have an equality by the induction hypotheses and bearing in mind that \( \Delta V^{*,mk,k}_{t} = \Delta V^{*,m,1}_{t+1} \). Thus we have completed the proof.

\( \square \)

5 Dual problem

In this section we will obtain a dual representation of the price of the American option, which can be exercised \( k_{t} \) times at time \( t \). We will follow the ideas of Rogers \([13]\) and Meinshausen and Hambly \([12]\). We will formulate the theorem, which gives this dual representation.

**Theorem 1** (Multiple exercise option price - Dual Representation). The option price \( V^{*,mk}_{0} \) at time \( t = 0 \) can be written as

\[
V^{*,mk}_{0} = \inf_{0 \leq \tau \leq T} \inf_{M \in \mathcal{H}_{0}} \max_{0 \leq t \leq \tau} \left( (h_{t} + A^{*,m-1,\min\{k_{t},m\}-1}_{t})1_{t<\tau} + \max\{h_{\tau} + A^{*,m-1,\min\{k_{\tau},m\}-1}_{t+1}, E_{t}[\Delta V^{*,m-1,k}_{t+1}], E_{t}[\Delta V^{*,m,k}_{t+1}] \} 1_{t=\tau} + E_{t}[V^{*,m-1,k}_{t+1} - \tilde{M}_{t}] \right),
\]

where the infima are taken over all stopping times \( \tau \) and over the set of integrable martingales \( \mathcal{H}_{0} \). The infimum here is attained for the optimal martingale \( \tilde{M}^{*,mk}_{t} \) and stopping time \( \tau \) defined as \( \tau^{*} = \min\{t : D^{*,m-1,k}_{t+1} > 0\} \).
To prove this theorem we will require several preliminary results. We start with the following lemma.

**Lemma 3** The marginal value \( \Delta V_{t}^{*,m,k} \) satisfies the inequality
\[
\Delta V_{t}^{*,m,k} \leq \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m,k}] \} - (D_t^{*,m-1,k} - D_t^{*,m-1,k}).
\]

**Proof:** Starting with
\[
\Delta D_{t+1}^{*,m,k} - \Delta D_{t}^{*,m,k} = \Delta V_{t}^{*,m,k} - E_t[\Delta V_{t+1}^{*,m,k}].
\]
We get
\[
D_{t+1}^{*,m,k} - D_{t+1}^{*,m-1,k} - (D_t^{*,m,k} - D_t^{*,m-1,k}) = \Delta V_{t}^{*,m,k} - E_t[\Delta V_{t+1}^{*,m,k}].
\]
Thus
\[
\Delta V_{t}^{*,m,k} = E_t[\Delta V_{t+1}^{*,m,k}] + D_{t+1}^{*,m,k} - D_{t}^{*,m,k} - (D_{t+1}^{*,m-1,k} - D_{t}^{*,m-1,k})
= E_t[\Delta V_{t+1}^{*,m,k}] + A_t^{*,m-1,\min(k_t,m)-1} - (D_{t+1}^{*,m-1,k} - D_{t}^{*,m-1,k})
= \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m,k}] \} - (D_{t+1}^{*,m-1,k} - D_{t}^{*,m-1,k})
\leq \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}] \} - (D_{t+1}^{*,m-1,k} - D_{t}^{*,m-1,k}).
\]
\[\square\]

We will also obtain a dual representation for the marginal value.

**Theorem 2** *(Marginal value - Dual Representation).* The marginal value \( \Delta V_{0}^{*,m,k} \) is equal to
\[
\Delta V_{0}^{*,m,k} = \inf_{0 \leq t \leq T} \inf_{M \in H_0} E_0\left[ \max_{0 \leq \tau \leq \tau^*} \left( (h_t + A_t^{*,m-1,\min(k_t,m)-1})1_{t<\tau} \right. \right.
+ \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}] \} 1_{t=\tau} - D_{t+1}^{*,m-1,k} - M_t \),
\]
where the infima are taken over all stopping times \( \tau \) and over the set of integrable martingales \( H_0 \). The infimum is attained for the martingale \( \Delta M_{t}^{*,m,k} \) and stopping time \( \tau^* \) defined as \( \tau^* = \min\{t : D_{t+1}^{*,m-1,k} > 0 \} \).

We divide the proof of this theorem in two parts.

**Proposition 7** For all stopping times \( \tau \leq T \) and all integrable martingales \( M_t \)
\[
\Delta V_{0}^{*,m,k} \leq E_0\left[ \max_{0 \leq \tau \leq \tau^*} \left( (h_t + A_t^{*,m-1,\min(k_t,m)-1})1_{t<\tau} \right. \right.
+ \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}] \} 1_{t=\tau} - D_{t+1}^{*,m-1,k} - M_t \).
\]

Proof: Here we use Proposition 2 and split the interval for the stopping time, introducing stopping times $\theta$ and $\theta'$

\[
\Delta V^{\ast, m, k}_\theta = \max_{0 \leq \theta \leq T} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta - D^{\ast, m-1, k}_{\theta+1} \right]
\]

\[
= \max_{0 \leq \theta \leq \tau} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta - D^{\ast, m-1, k}_{\theta+1} \right] 1_{\theta < \tau}
+ \max_{\tau \leq \theta' \leq T} \left[ h_{\theta'} + A^{\ast, m-1, \min\{k, m\} - 1}_{\theta'} - D^{\ast, m-1, k}_{\theta'+1} \right] 1_{\theta' = \tau}
\]

\[
= \max_{0 \leq \theta \leq \tau} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta - D^{\ast, m-1, k}_{\theta+1} \right] 1_{\theta < \tau}
+ \left( \max_{\tau \leq \theta' \leq T} \left[ h_{\theta'} + A^{\ast, m-1, \min\{k, m\} - 1}_{\theta'} - D^{\ast, m-1, k}_{\theta'+1} \right] + D^{\ast, m-1, k}_{\theta+1} \right) 1_{\theta' = \tau}
\]

\[
= D^{\ast, m-1, k}_{\theta+1} 1_{\theta < \tau} - D^{\ast, m-1, k}_{\theta+1} 1_{\theta = \tau}.
\]

Here again by Proposition 2 we use $\Delta V^{\ast, m, k}_\theta$ instead of its representation and then by Lemma 3 we have

\[
\Delta V^{\ast, m, k}_\theta = \max_{0 \leq \theta \leq \tau} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta \right] 1_{\theta < \tau}
+ \Delta V^{\ast, m, k}_\theta 1_{\theta = \tau} - D^{\ast, m-1, k}_{\theta+1} 1_{\theta < \tau} - D^{\ast, m-1, k}_{\theta+1} 1_{\theta = \tau}
\]

\[
= \max_{0 \leq \theta \leq \tau} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta \right] 1_{\theta < \tau}
+ \left( \Delta V^{\ast, m, k}_\theta + D^{\ast, m-1, k}_{\theta+1} - D^{\ast, m-1, k}_{\theta+1} \right) 1_{\theta = \tau} - D^{\ast, m-1, k}_{\theta+1}
\]

\[
\leq \max_{0 \leq \theta \leq \tau} E_0 \left[ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta \right] 1_{\theta < \tau}
+ \max \left\{ h_\theta + A^{\ast, m-1, \min\{k, m\} - 1}_\theta, E_\theta [\Delta V^{\ast, m-1, k}_{\theta+1}] \right\} 1_{\theta = \tau} - D^{\ast, m-1, k}_{\theta+1}.
\]

Now by introducing a martingale $M_t$ vanishing at zero and using the inequality for interchanging maximum and expectation
\[ 
\Delta V_{0}^{*,m,k} = \max_{0 \leq \theta \leq \tau} E_0 \left[ (h_\theta + A_\theta^{*,m-1,\min(k_\theta,m)-1})1_{\theta < \tau} \right. \\
\left. + \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\}1_{t = \tau} \right. \\
\left. - D_{t+1}^{*,m-1,k} - M_\theta \right] \\
\leq E_0 \left[ \max_{0 \leq t \leq \tau} \left( (h_t + A_t^{*,m-1,\min(k_t,m)-1})1_{t < \tau} \right. \\
\left. + \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\}1_{t = \tau} \right. \\
\left. - D_{t+1}^{*,m-1,k} - M_t \right] 
\] 

and the proof is completed.

The second part of the proof of Theorem 2 is the following inequality.

**Proposition 8** The following inequality holds for the marginal value \( \Delta V_{0}^{*,m,k} \)

\[ \Delta V_{0}^{*,m,k} \geq \inf_{0 \leq \tau \leq T} \inf_{M \in H_0} E_0 \left[ \max_{0 \leq t \leq \tau} \left( (h_t + A_t^{*,m-1,\min(k_t,m)-1})1_{t < \tau} \right. \\
\left. + \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\}1_{t = \tau} \right. \\
\left. - D_{t+1}^{*,m-1,k} - M_t \right] \right]. \]

The infimum is attained for \( \Delta M_{t}^{*,m,k} \) and \( \tau^* = \min\{t : D_{t+1}^{*,m-1,k} > 0\} \).

**Proof:** Define a stopping time \( \tau^* = \min\{t : D_{t+1}^{*,m-1,k} > 0\} \). Substituting \( \tau^* \) for \( \tau \) and \( \Delta M_{t}^{*,m,k} \) for \( M_t \), we have

\[ \inf_{0 \leq \tau \leq T} \inf_{M \in H_0} E_0 \left[ \max_{0 \leq t \leq \tau} \left( (h_t + A_t^{*,m-1,\min(k_t,m)-1})1_{t < \tau} \right. \\
\left. + \max\{h_t + A_t^{*,m-1,\min(k_t,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\}1_{t = \tau} \right. \\
\left. - D_{t+1}^{*,m-1,k} - M_t \right] \right]. \]

By the definition of \( \tau^* \) we have that \( D_{\tau^*+1}^{*,m-1,k} = 0 \). From

\[ D_{\tau^*+1}^{*,m-1,k} - D_{\tau^*+1}^{*,m-1,k} = A_{\tau^*+1}^{*,m-1,\min(k_{\tau^*+1}^*,m)-1} \]

the positivity of \( D_{\tau^*+1}^{*,m-1,k} \), and Proposition 3 it follows that

\[ E_{\tau^*}[\Delta V_{\tau^*+1}^{*,m-1,k}] = h_{\tau^*} + A_{\tau^*}^{*,m-2,\min\{k_{\tau^*},m-1\}-1} - D_{\tau^*+1}^{*,m-1,k} \]

\[ \leq h_{\tau^*} + A_{\tau^*}^{*,m-1,\min(k_{\tau^*},m)-1}. \]
Thus
\[
\max\{h_{\tau^*}, + A_t^{*,m-1,\min\{k_t, m\}-1}, E_{\tau^*}[\Delta V_{t+1}^{*,m-1,k}]\}
= h_{\tau^*} + A_t^{*,m-1,\min\{k_t, m\}-1}.
\]

Then we have
\[
\inf \inf \limits_{0 \leq \tau \leq T M \in H_0} E_0 \left[ \max \limits_{0 \leq t \leq \tau} \left( h_t + A_t^{*,m-1,\min\{k_t, m\}-1} \right) \mathbf{1}_{t < \tau} + \max\{h_t + A_t^{*,m-1,\min\{k_t, m\}-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\} \mathbf{1}_{t = \tau} - D_{t+1}^{*,m-1,k} - M_t \right] \\
\leq E_0 \left[ \max \limits_{0 \leq t \leq \tau^*} \left( h_t + A_t^{*,m-1,\min\{k_t, m\}-1} \right) \mathbf{1}_{t < \tau^*} + (h_t + A_t^{*,m-1,\min\{k_t, m\}-1} \mathbf{1}_{t = \tau^*} - D_{t+1}^{*,m-1,k} - \Delta M_t^*) \right] \\
= E_0 \left[ \max \limits_{0 \leq \tau \leq \tau^*} \left( h_t + A_t^{*,m-1,\min\{k_t, m\}-1} - D_{t+1}^{*,m-1,k} - \Delta M_t^* \right) \right].
\]

From the representation
\[
\Delta V_{t+1}^{*,m,k} = \max \limits_{t \leq \tau \leq T} E_t \left[ h_{\tau^*} + A_{\tau^*}^{*,m-1,\min\{k_{\tau^*}, m\}-1} - D_{\tau^*+1}^{*,m-1,k} \right] + D_{t+1}^{*,m-1,k}
\]
and the positivity of $D_{t+1}^{*,m-1,1,k}$ we have
\[
\Delta V_{t+1}^{*,m,k} \geq \max \limits_{t \leq \tau \leq T} E_t \left[ h_{\tau^*} + A_{\tau^*}^{*,m-1,\min\{k_{\tau^*}, m\}-1} - D_{\tau^*+1}^{*,m-1,k} \right] \\
\geq h_{\tau^*} + A_{\tau^*}^{*,m-1,\min\{k_{\tau^*}, m\}-1} - D_{\tau^*+1}^{*,m-1,k}.
\]

On the other hand
\[
\Delta V_{t+1}^{*,m,k} = \Delta V_0^{*,m,k} + \Delta M_t^{*,m,k} - \Delta D_{t+1}^{*,m,k}.
\]

Hence
\[
h_{\tau^*} + A_{\tau^*}^{*,m-1,\min\{k_{\tau^*}, m\}-1} - D_{\tau^*+1}^{*,m-1,k} - \Delta M_t^{*,m,k} \leq \Delta V_0^{*,m,k} - \Delta D_{t+1}^{*,m,k}.
\]

So
\[
\inf \inf \limits_{0 \leq \tau \leq T M \in H_0} E_0 \left[ \max \limits_{0 \leq t \leq \tau} \left( h_t + A_t^{*,m-1,\min\{k_t, m\}-1} \right) \mathbf{1}_{t < \tau} + \max\{h_t + A_t^{*,m-1,\min\{k_t, m\}-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\} \mathbf{1}_{t = \tau} - D_{t+1}^{*,m-1,k} - M_t \right] \\
\leq E_0 \left[ \max \limits_{0 \leq \tau \leq \tau^*} \left( \Delta V_0^{*,m,k} - \Delta D_{t+1}^{*,m,k} \right) \right] \\
\leq \Delta V_0^{*,m,k}.
\]
For the last step we used that by Remark 1 $\Delta D_t^{*,m,k}$ is a positive process. Using Proposition 7 we can see that equality is attained for $\tau^*$ and $\Delta M_t^{*,m,k}$, and the proof is completed.

With this we also finished the proof of Theorem 2. Now we can proceed to the proof of Theorem 1.

**Proof (of Theorem 1):** We have

$$D_t^{*,m-1,k} - D_t^{*,m-1,k} = V_t^{*,m-1,k} - E_t[V_{t+1}^{*,m-1,k}].$$

From here

$$D_t^{*,m-1,k} + E_t[V_{t+1}^{*,m-1,k}] = D_t^{*,m-1,k} + V_t^{*,m-1,k}$$

$$= V_0^{*,m-1,k} + M_t^{*,m-1,k}.$$  

Thus

$$D_t^{*,m-1,k} = -E_t[V_{t+1}^{*,m-1,k}] + V_0^{*,m-1,k} + M_t^{*,m-1,k}.$$  

Substituting this in the dual representation for the marginal value $\Delta V_0^{*,m,k}$ (Theorem 2) we obtain

$$\Delta V_0^{*,m,k} = \inf_{0 \leq \tau \leq T} \inf_{M \in M_0} E_0[\max_{0 \leq t \leq \tau} ((h_t + A_t^{*,m-1,\min(k,m)-1}) E_t[\Delta V_{t+1}^{*,m-1,k}] + \max\{h_t + A_t^{*,m-1,\min(k,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\} 1_{t<\tau}$$

$$+ E_t[V_{t+1}^{*,m-1,k}] - V_0^{*,m-1,k} - M_t^{*,m-1,k}]].$$

Now by combining the last two martingales in $\tilde{M}_t = M_t^{*,m-1,k} + M_t$ we have

$$V_0^{*,m,k} = \inf_{0 \leq \tau \leq T} \inf_{M \in M_0} E_0[\max_{0 \leq t \leq \tau} ((h_t + A_t^{*,m-1,\min(k,m)-1}) E_t[\Delta V_{t+1}^{*,m-1,k}] + \max\{h_t + A_t^{*,m-1,\min(k,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\} 1_{t<\tau}$$

$$+ E_t[V_{t+1}^{*,m-1,k}] - (\tilde{M}_t)]$$

The equality in the dual representation of the marginal value was attained for the martingale $\Delta M_t^{*,m,k}$ and the stopping time $\tau^* = \min\{t : D_t^{*,m-1,k} > 0\}$, so here the equality will be attained for the martingale $\tilde{M}_t = M_t^{*,m-1,k} + \Delta M_t^{*,m,k} = M_t^{*,m,k}$ and the same stopping time.

**Remark 3** From Theorem 1 and Theorem 2 we also have that

$$V_0^{*,m,k} = E_0[\max_{0 \leq t \leq \tau} ((h_t + A_t^{*,m-1,\min(k,m)-1}) 1_{t<\tau}$$

$$+ \max\{h_t + A_t^{*,m-1,\min(k,m)-1}, E_t[\Delta V_{t+1}^{*,m-1,k}]\} 1_{t=\tau}$$

$$+ E_t[V_{t+1}^{*,m-1,k}] - M_t^{*,m,k}]]$$  

(3)
Proof: We have that
\[ V^*_{\tau} \text{ is constant for every path and is equal to } V_0^{*,m,k} \]

So for \( t = \tau^* \) the left side of (5) will be strictly positive and hence the right side will be strictly positive. However the right side is strictly positive only if
\[ h_t > E_t[\Delta V_{t+1}^{*,m,k}] \]
Thus an equivalent definition for \( \tau^* \) would be \( \tau^* = \min \{ t : h_t > E_t[\Delta V_{t+1}^{*,m,k}] \} \). Then
\[ h_{\tau^*} + A_t^{*,m-1,\min(k_i,m)-1} < E_{\tau^*}[\Delta V_{\tau^*+1}^{*,m,k}] \]

Using this we have
\[
E_0\left[ \max_{0 \leq t \leq \tau^*} ((h_t + A_t^{*,m-1,\min(k_i,m)-1})1_{t<\tau^*} + (h_t + A_t^{*,m-1,\min(k_i,m)-1})1_{t=\tau^*} + E_t[V_{t+1}^{*,m-1,k} - M_t^{*,m,k}] \right] \\
= E_0\left[ \max_{0 \leq t \leq \tau^*} (h_t + A_t^{*,m-1,\min(k_i,m)-1}) + E_t[V_{t+1}^{*,m-1,k} - M_t^{*,m,k}] \right].
\]

Proposition 9 The value of
\[ \max_{0 \leq t \leq \tau^*} (h_t + A_t^{*,m-1,\min(k_i,m)-1}) + E_t[V_{t+1}^{*,m-1,k} - M_t^{*,m,k}] \]
is constant for every path and is equal to \( V_0^{*,m,k} \).

Proof: We have that
\[ h_t + A_t^{*,m-1,\min(k_i,m)-1} + E_t[V_{t+1}^{*,m-1,k}] \leq V_t^{*,m,k} \]
and
\[ V_t^{*,m,k} = V_0^{*,m,k} + M_t^{*,m,k} - D_t^{*,m,k}. \]
So
\[
\max_{0 \leq t \leq \tau^*} (h_t + A_t^{*,m-1,\min(k_i,m)-1} + E_t[V_{t+1}^{*,m-1,k}] - M_t^{*,m,k}) \\
\leq \max_{0 \leq t \leq \tau^*} (V_t^{*,m,k} - M_t^{*,m,k}) \\
\leq \max_{0 \leq t \leq \tau^*} (V_t^{*,m,k} - D_t^{*,m,k}) \\
\leq V_0^{*,m,k}.
\]
At the same time

\[ V^{*,m,k}_0 = \max_{0 \leq t \leq \tau^*} \left( (h_t + A_t^{*,m-1,\min(k_t,m)} - 1) + E_t[V^{*,m-1,k}_t] - M_t^{*,m,k} \right) \]

so a.s.

\[ V^{*,m,k}_0 = \max_{0 \leq t \leq \tau^*} \left( (h_t + A_t^{*,m-1,\min(k_t,m)} - 1) + E_t[V^{*,m-1,k}_t] - M_t^{*,m,k} \right). \]

\[ \square \]

6 Practical application

In this section we will introduce methods to compute lower and upper bounds for the price of the multiple exercise claims, using some of the theoretical properties we have derived. The lower bound is calculated by a generalization of the regression method introduced by Longstaff and Schwartz[11] and the upper bound calculation relies on the dual representation in Theorem 1.

With the ordering of the continuation values

\[ E_t[\Delta V^{*,m,k}_{t+1}] \leq E_t[\Delta V^{*,m-1,k}_{t+1}], \quad \forall m, \forall t \]

we can determine the optimal exercise strategy at time level \( t \) as

- if \( h_t \geq E_t[\Delta V^{*,m-1,k_t+1}_{t+1}] \) - exercise \( k_t \) times,
- if \( E_t[\Delta V^{*,m-1,k_t+1}_{t+1}] > h_t \geq E_t[\Delta V^{*,m-1,k_t+2}_{t+1}] \) - exercise \( k_t - 1 \) times,
- \[ \text{...} \]
- if \( E_t[\Delta V^{*,m,k}_{t+1}] > h_t \) - do not exercise.

In terms of the marginal continuation values we can write the strategy as

- if \( h_t \geq \Delta C^{*,m-1,k_t+1}_{t} \) - exercise \( k_t \) times,
- if \( \Delta C^{*,m-1,k_t+1}_{t} > h_t \geq \Delta C^{*,m-1,k_t+2}_{t} \) - exercise \( k_t - 1 \) times,
- \[ \text{...} \]
- if \( \Delta C^{*,m,k}_{t} > h_t \) - do not exercise.

We will start with the calculation of the lower bound. The idea here is to work backward in time, approximating the marginal continuation value with a linear combination of basis functions. In this way an approximation of the optimal exercise strategy is found and consequently a lower bound for the option price.

Let \( \psi_i : R^d \to R \) for \( i = 1, \ldots, l \) be the basis functions used for the regression.

**Definition 4** For all times \( t \in \{0,1,\ldots,T\} \) and at each point \( x \in R^d \) we define \( \Delta C^{m,k}_t(x) \), an approximation to the \( m \)-th marginal continuation value \( \Delta C^{*,m,k}_t \), by

\[ \Delta \hat{C}^{m}_t(x) = \sum_{i=1}^{l} c^{m}_{i,t} \psi_i(x). \]
Of course the optimal continuation values are not known. Thus we use non-optimal continuation values $C^m_t$ defined as follows.

**Definition 5** Suppose that, working backwards in time and forward from one exercise opportunity, approximations $\Delta C^m_{t+1}, \Delta C^m_{t+1}, \ldots, \Delta C^m_{t+1}$ to the $m$-th, $m-1$, ..., $m-k+1$ marginal continuation value functions have been obtained. Then for path $j$ define the approximate continuation value $C^m_{t}^{m,j}$ to be

$$C^m_{t}^{m,j} = \begin{cases} \hat{C}^m_{t+1}(X^{j}_{t+1}) + C^m_{t+1}(X^{j}_{t+1}), & \hat{h}_{t+1}(X^{j}_{t+1}) \leq \Delta \hat{C}^m_{t+1}(X^{j}_{t+1}) \\ h_{t+1}(X^{j}_{t+1}) \geq \Delta \hat{C}^m_{t+1}(X^{j}_{t+1}) > h_{t+1}(X^{j}_{t+1}) \geq \Delta \hat{C}^m_{t+1}(X^{j}_{t+1}) \geq \Delta \hat{C}^m_{t+1}(X^{j}_{t+1}) \geq \Delta \hat{C}^m_{t+1}(X^{j}_{t+1}) > \hat{h}_{t+1}(X^{j}_{t+1}) & \end{cases}$$

The non-optimal $m$-th marginal continuation values are also defined by

$$\Delta C^m_{t}^{m,j} = C^m_{t}^{m,j} - C^m_{t}^{m-1,j}.$$ 

Let $\psi = (\psi_1, \psi_2, \ldots, \psi_l)$ and $\hat{C}^m_{t} = (\hat{c}^m_{t,1}, \hat{c}^m_{t,2}, \ldots, \hat{c}^m_{t,l})$. If $n$ paths of the Markov chain are simulated, an estimate for the regression coefficients would be

$$\hat{c}^m_{t} = \arg\min_{c \in \mathbb{R}^l} \sum_{j=1}^{n} (\Delta C^m_{t}^{m,j} - \sum_{i=1}^{l} \hat{c}^m_{t,i} \psi_i(X^{j}_{t}))^2.$$

The explicit formulas for the coefficients are

$$\hat{c}^m_{t} = \psi^{-1} \hat{c}^m, \quad \psi_{t,p} = \frac{1}{n} \sum_{j=1}^{n} \psi_i(X^{j}_{t}) \psi_p(X^{j}_{t}), \quad \hat{c}^m_{t} = \frac{1}{n} \sum_{j=1}^{n} \psi_i(X^{j}_{t}) \Delta C^m_{t}^{m,j}.$$ 

Once the coefficients $\hat{c}^m_{t,1}, \hat{c}^m_{t,2}, \ldots, \hat{c}^m_{t,l}$ are obtained we can approximate the $m$-th marginal continuation value, and from there the stopping rule, at any point in the state space. We work backwards in time until we reach $t = 0$.

We can now move to the upper bound estimation. In order to use the dual representation from Theorem 1 we need approximations to the optimal martingale $M^{*,m,k}_{t}$ and the stopping time $\tau^*$. First we will show how an approximation to the stopping time $\tau^* = \min\{t : D^{*,m-1,k}_{t+1} > 0\}$ can be found. As we discussed after Remark 3 an equivalent definition of $\tau^*$ would be
\[ \tau^* = \min\{t : h_t > E_t[\Delta V_{t+1}^{*,m-1,k}]\} \] or, when written in terms of the marginal continuation value, \( \tau^* = \min\{t : h_t > \Delta C_{t+1}^{*,m-1,k}\} \). We will assume that we already have found an approximation to the marginal continuation value (here we will use the techniques described earlier for the lower bound approximation).

**Definition 6** (Approximation to the stopping time) Let \( \hat{\Delta} C_{t+1}^{m-1} \) be an approximation to the continuation value \( \Delta C_{t+1}^{*,m-1,k} \). Define \( \hat{\tau} \), an approximation to the stopping time \( \tau^* \), by

\[ \hat{\tau} = \min\{t : h_t > \hat{\Delta} C_{t+1}^{m-1}\} \] (6)

We will give an approximation to the optimal martingale \( M_{t+1}^{*,m,k} \) by using approximations to the first, second, \ldots, \( m \)-th marginal martingales. Thus we will start by approximating the marginal martingales.

We can write the martingale differences \( \Delta M_{t+1}^{*,m,k} = \Delta M_{t+1}^{*,m,k} - \Delta M_{t}^{*,m,k} \) as

\[
\Delta D M_{t+1}^{*,m,k} = \Delta V_{t+1}^{*,m,k} - \Delta V_{t}^{*,m,k} + \Delta D_{t+1}^{*,m,k} - \Delta D_{t}^{*,m,k} \\
= \Delta V_{t+1}^{*,m,k} - \hat{E}_t[\Delta V_{t+1}^{*,m,k}].
\]

Using \( \hat{\Delta} C_{t+1}^{m} \), our approximation to the marginal continuation value \( \Delta C_{t+1}^{*,m,k} \) using regression, we have

\[
\hat{\Delta} V_{t+1}^{m}(x) = \begin{cases} 
\Delta \hat{C}_{t+1}^{m-k}(x), & h_{t+1}(x) \geq \Delta \hat{C}_{t+1}^{m-k}(x) \\
\Delta \hat{C}_{t+1}^{m-k}(x), & \Delta \hat{C}_{t+1}^{m-k}(x) > h_{t+1}(x) \geq \Delta \hat{C}_{t+1}^{m}(x) \\
\Delta \hat{C}_{t+1}^{m}(x), & \Delta \hat{C}_{t+1}^{m}(x) > h_{t+1}(x)
\end{cases}
\] (7)

**Definition 7** (Approximation to \( \Delta D M_{t+1}^{*,m,k} \)) We define the approximation of the \( m \)-th marginal martingale difference by

\[
\hat{\Delta} D M_{t+1}^{m} = \hat{\Delta} V_{t+1}^{m} - \hat{E}_t[\hat{\Delta} V_{t+1}^{m}].
\] (8)

where \( \hat{\Delta} V_{t+1}^{m} \) is given by (7) and \( \hat{E}_t[\hat{\Delta} V_{t+1}^{m}] \) is calculated as an average over \( K \) i.i.d. subpaths advancing one time-step.

**Definition 8** (Approximation to \( \Delta M_{t+1}^{*,m,k} \)) We define the approximation of the \( m \)-th marginal martingale by

\[
\Delta M_{t+1}^{m} = \Delta D M_{t+1}^{m} + \Delta M_{t}^{m} + \ldots + \Delta M_{1}^{m}.
\] (9)

Finally we are ready to define an approximation to the martingale.

**Definition 9** (Approximation to \( M_{t+1}^{*,m,k} \)) We define the approximation of the martingale by

\[
M_{t+1}^{m} = \Delta M_{t+1}^{m} + \Delta M_{t+1}^{m-1} + \ldots + \Delta M_{1}^{m}.
\] (10)
Now suppose the expectations on the r.h.s. of
\[ V^*_0, m, k_0 = E_0 \left[ \max_{0 \leq t \leq \tau^*} \left( (h_t + A_{t, m}^{*, m-1, \min\{k_t, m\}} - 1) 1_{t < \tau^*} \right. \right. \]
\[ + \max \left\{ h_t + A_{t, m}^{*, m-1, \min\{k_t, m\}} - 1, E_t[\Delta V^*_t, m-1, k] \right\} 1_{t = \tau^*} + E_t[V^*_t, m-1, k] - M^*_t, m, k_0 ] \]

can be computed exactly. Using the approximations of the stopping time and the martingale instead of the optimal ones we obtain an upper bound for the option price. Of course the exact expectations appearing in the dual representation are not known and in order to obtain an upper bound they have to be approximated as well. The marginal continuation values that appear in \( A_t^{*, m-1, \min\{k_t, m\}} \) have negative signs, so in order to obtain upper bound for the option price we can replace them with lower bounds, calculated as before. Also in
\[ E_t[\Delta V^*_t, m-1, k] = E_t[V^*_t, m-1, k] - E_t[V^*_t, m-2, k]. \]

\( E_t[V^*_t, m-1, k] \) can be approximated by its upper bound and \( E_t[V^*_t, m-2, k] \) by its lower bound. In this way an upper bound for \( V^*_0, m, k \) can be calculated recursively, by calculating upper bounds for the corresponding \( E_t[V^*_t, m-1, k] \) and using the lower bound calculations from the regression algorithm. However the computational effort in this approach is exponential in the number of exercise times.

7 Numerical example

The numerical example we consider is pricing a swing contract with the following features:

1. The swing option has maturity \( T \) days and can be exercised on days 1, 2, ..., \( T \).
2. It can be exercised up to \( k_t \) times on day \( t \) and the total number of exercise rights is \( m \).
3. When exercising the option, its holder buys a certain number of units (usually 1MWh) of electricity for a prespecified fixed price \( K \).

The underlying process (electricity spot price) is the exponential of a discrete mean reverting process
\[ \log S_{t+1} = (1 - \alpha) \log S_t + \sigma W_t. \] (11)

With parameters \( S_0 = 1, \sigma = 0.5 \) and \( \alpha = 0.9 \). The payoff is taken to be the spot price itself. The basis functions used for approximating the marginal continuation values are
\[ \Psi = \{1, \log S\}. \] (12)
We look at options with a lifetime of one month or $T = 30$. The second column of Table 1 is for options with one exercise right per time (there is a tight confidence interval for this type of option and only the lower bounds are given). The third and fourth columns are with lower and upper bounds for options that can be exercised up to one time on each weekend day and up to two times on each weekday.

<table>
<thead>
<tr>
<th>Exercise possibilities</th>
<th>1 ex. right per time</th>
<th>1(weekend) and 2(weekday) ex. rights per time - lower</th>
<th>1(weekend) and 2(weekday) ex. rights per time - upper</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.56</td>
<td>2.56</td>
<td>2.57</td>
</tr>
<tr>
<td>2</td>
<td>4.71</td>
<td>4.99</td>
<td>5.12</td>
</tr>
<tr>
<td>3</td>
<td>6.65</td>
<td>7.14</td>
<td>7.39</td>
</tr>
<tr>
<td>4</td>
<td>8.49</td>
<td>9.21</td>
<td>9.62</td>
</tr>
<tr>
<td>5</td>
<td>10.14</td>
<td>11.14</td>
<td>11.68</td>
</tr>
<tr>
<td>6</td>
<td>11.69</td>
<td>12.97</td>
<td>13.69</td>
</tr>
</tbody>
</table>

The Longstaff-Schwarz lower bound can be implemented for options with greater maturity and many more exercise rights. The reason we only deal with such small parameter values is that we do not have as useful a representation for the dual in this multiple exercise case as was obtained in [12] for the single exercise at a time case. Thus the time taken for this numerical technique grows exponentially in the number of possible exercise opportunities. An open problem is to use the dual effectively to find a fast numerical upper bound approximation.

References