Limit order books, diffusion approximations and reflected SPDEs: from microscopic to macroscopic models

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Abstract

Motivated by a zero-intelligence approach, the aim of this paper is to connect the microscopic (discrete price and volume), mesoscopic (discrete price and continuous volume) and macroscopic (continuous price and volume) frameworks for the modelling of limit order books, with a view to providing a natural probabilistic description of their behaviour in a high to ultra high-frequency setting. Starting with a microscopic framework, we first examine the limiting behaviour of the order book process when order arrival and cancellation rates are sent to infinity and when volumes are considered to be of infinitesimal size. We then consider the transition between this mesoscopic model and a macroscopic model for the limit order book, obtained by letting the tick size tend to zero. The macroscopic limit can then be described using reflected SPDEs which typically arise in stochastic interface models. We then use financial data to discuss a possible calibration procedure for the model and illustrate numerically how it can reproduce observed behaviour of prices. This could then be used as a market simulator for short-term price prediction or for testing optimal execution strategies.

1 Introduction

The rising prevalence of order-driven markets in recent years has generated a significant interest in the modelling of limit order books, for an overview see the survey paper [14]. In such markets, three specific types of orders can be submitted. Firstly, limit orders are orders to buy or sell a designated number of shares at a specified price or better. Secondly, market orders are orders to immediately buy or sell a certain number of shares at the best available price. Finally, cancellation orders enable a market participant to cancel an existing limit order. Whilst market orders are instantly matched against the best available limit orders of the opposite quote, the collection of unexecuted and uncancelled limit orders is recorded in the limit order book (LOB), according to price and time priority. In limit order book terminology, the bid refers to the price of the best limit buy order, whereas the ask designates the price of the best sell order. The average of the bid and the ask is referred to as the mid. Two other quantities of interest are the spread, which corresponds to the difference between the ask and the bid, and the tick size, which is the smallest price increment in the market.

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There is now a large amount of high-quality financial time series of orders enabling one to conduct statistical analyses of various limit order book features and to guide the development of models. A good model for the order book should describe the evolution of prices and capture some of the stylised facts that are observed in financial time series. It can then be used for examining strategies for order placement and for the optimal execution of large orders.

Limit order book models have been developed by two independent schools of thought. The first one, initiated by economists, has been based on a ‘perfect-rationality’ approach, where market participants all employ optimal strategies to place limit orders. The second one, led by econophysicists and mathematicians, has been associated with a ‘zero-intelligence’ framework, that is limit order arrivals can be viewed as a purely random process and no strategic order placement is taken into account.

Within the realm of perfect-rationality, where order flow is considered as static, the central issue concerns the strategic trading decisions of agents in which they maximise their individual utility. Notable models in the perfect-rationality literature include those due to Mendelson [22], who analysed the statistical behaviour of the market from a clearing house perspective, Kyle [20], where the question of insider trading with sequential auctions is addressed, Roşu [25], who introduced the notion of optimal choice between market orders and limit orders, and Almgren and Chriss [2], where a model for the optimal execution of large orders is developed.

In the zero-intelligence approach the order flow is treated as dynamic and the focus is shifted to the random nature of order arrivals. One of the first models dealing with this was developed by Kruk [19], where he established a functional limit theorem for the order flow in a continuous double-auction setting. More recently, there has been a significant interest in modelling the book as a multiclass queueing system, [1], [5], [9], [24], [10]. In order to deal with a market where orders are submitted at high frequency, Cont and Larrard [8] considered a heavy traffic approximation of the order book process from a queueing theory perspective. The order book is reduced to the best bid and ask queues: once the bid (or ask) queue has been depleted, it takes a new value drawn from a stationary distribution representing the depth of the order book after a price change.

Over the years, there have also been a number of attempts to establish scaling limits for the full order book. One of the first papers to explore this direction is [6], where the order book is modelled as a two-species interacting particle system and a hydrodynamic limit is obtained for the associated empirical process. This particle system approach is also used by [10], where their limit is actually an ODE with a constant price. The work of [21] obtains a limit for one side of the order book as a measure-valued process. In [4] and [15] Horst et al. establish functional limit theorems for two-sided order books and obtain PDE or SPDE limits depending on the initial scaling procedure. Finally, Zheng [29] (using the results of Kim et al. [18] on stochastic Stefan problems) and Müller [24] and [17] developed models for the order book as a stochastic free boundary problem.

1.1 From micro to macro models

Motivated by a zero-intelligence approach, the aim of this paper is to bridge the gap between microscopic (discrete volume and price), mesoscopic (continuous volume and discrete price) and macroscopic (continuous volume and price) models of limit order books. The financial context of our study is the following: we consider an order-driven market where orders and cancellations are submitted at very high frequency. Starting with a discrete-space model describing the microscopic evolution of the order book, we prove that by sending order arrival and cancellation rates to infinity and by rescaling order volumes, the behaviour of the book can be described in terms of a system of coupled stochastic differential equations. This is what we call the mesoscopic limiting process. Next, by sending the tick size to zero, we derive a macroscopic SPDE limit from the mesoscopic
process.

Even though we send the tick size to zero we wish to capture the fact that in a high frequency trading environment the price changes are comparatively rare in the evolution of the order book. Thus we will consider our model for the book as generating price changes at a much lower frequency, so there is a natural separation of time scales. Our price changes will be a macroscopic tick movement occurring as a result of imbalances created in the book by the order flow. Our main mathematical result is Theorem 3.5 showing that the queueing system converges weakly to a reflected SPDE for the dynamics of the order book along with a discrete price process evolving in a realistic way.

An outline of the paper is as follows. In the next section we will develop our microscopic model. We allow quite a degree of flexibility in the arrival rates and cancellation rates of orders and show in Theorem 2.3 that by letting the volume size of orders go to 0 as their rate of arrival goes to infinity that the microscopic system has a scaling limit which is a coupled system of stochastic differential equations. The initial result is for the static order book and we then show how to incorporate price changes which are functions of the two sides of the order book. In Section 3 we let the tick size go to zero and show that this system converges weakly to a reflected SPDE. In our main result (Theorem 3.5) we also incorporate price changes to get a full model for the order book and the discrete price dynamics it generates. In Section 4 we illustrate with some examples how our general framework can incorporate some natural models. In Section 5 we develop the numerical application for our framework. By considering data from LOBSTER we show how to determine some of the parameters in a simple version of the model and how it will produce realistic order book profiles and prices series. The proofs of the Theorems are then given in the Appendix.

2 Diffusion Approximations: From Microscopic to Mesoscopic Models

We begin this section by introducing a simple microscopic order book model, where order arrivals and cancellations are driven by Poisson processes. The rates for order arrivals, market orders and cancellations will be allowed to depend on the current mid price, the price relative to the mid and the number of orders currently on the book at the point in question. We will first consider a model for the evolution of the book in between price changes, and will then add price changes by introducing stopping times which depend on how the book has evolved. This will allow us to easily maintain a separation of time-scales between the evolution of the order book profiles and the corresponding price process when passing to mesoscopic and macroscopic limits, where we will scale time and space for our models. Maintaining this separation is sensible since, typically, price movements occur on a significantly slower time-scale to order book events. For example, examining order book data for the SPDR Trust Series I from June 21 2012 between 9:30am and 10:30am, we see that there were 1154736 order book events and approximately 3835 price changes.

Once our microscopic model has been described, our goal is then to establish a diffusion approximation for the model. This is done initially for the static model, where price changes are not included, and then for the dynamic case. We fix here a terminal time \( T > 0 \), and will prove convergence on the finite time interval \([0, T]\).

2.1 The Discrete Order Book Process in a Static Setting

In this section we describe the dynamics of our microscopic order book model when it is static i.e. we describe its behaviour in between price changes. The index \( n \) here will be used later in
order to take a diffusive scaling of the model.

We will work on a relative price grid i.e. at any given time, the \( i \)th price point of the bid/ask side of the book refers to the price which is \( i \) ticks away from the best bid/ask respectively. The grid is given by \( \{0, 1, 2, \ldots, N\} \) for some \( N \in \mathbb{N} \). For every \( n \in \mathbb{N} \), we consider two \( N - 1 \) dimensional processes, \( Z^b_n = (Z^{b,1}_n, \ldots, Z^{b,N-1}_n) \) and \( Z^a_n = (Z^{a,1}_n, \ldots, Z^{a,N-1}_n) \), each taking values in \( \mathbb{Z}^N \) and representing the limit order volumes currently on the bid and ask sides respectively of the static discrete order book process. The mid price \( m \in \mathbb{R} \) is taken to be fixed here, and by convention we think of the spread as being constantly equal to two ticks. For each \( i \in \{1, 2, \ldots, N-1\} \), \( Z^b_n \) then represents the number of outstanding orders to buy at price \( m - i \), whilst \( Z^a_n \) represents the number of outstanding limit orders to sell at price \( m + i \). Order and cancellation sizes are assumed to be \( 1 \) in the static setting, although the results here can easily be adapted to the case where order sizes assumed only to be bounded. We choose the rates at which different orders arrive in our model such that they possess the following three features.

1. At each price level, there is a common high frequency rate for limit orders and cancellation/market orders. We allow for these rates to be dependent on the relative price (with respect to the mid), the current position of the mid and the number of offers currently at that price. These terms are intended to capture the effects of high frequency trading.

2. Residual imbalance between limit orders and cancellations/market orders at different price levels gives lower frequency terms. These are once again allowed to be dependent on price, the current midprice and the number of offers at that price.

3. There is a lower frequency “smoothing” which takes place. If there are more orders on average at the two neighbouring prices in the book, then the rate of limit orders at that price point increases if the current midprice and the number of offers at that price. These terms are intended to capture the effects of high frequency trading.

Altogether, this motivates the following description for the dynamics of the bid side of the order book in our model.

(i) For \( i \in \{1, \ldots, N-1\} \), \( Z^b_n \rightarrow Z^b_n + e_i \) at exponential rate

\[
\frac{1}{2} \sigma^2_{b,m,n}(i, Z^b_n) \left( 1 + \mathbb{1}_{\{Z^b_n = 0\}} \right) + f_{b,m,n}(i, Z^b_n) + \alpha_{b,n}(Z^{b,i+1}_n + Z^{b,i-1}_n - 2Z^b_n) \mathbb{1}_{\{Z^{b,i+1}_n + Z^{b,i-1}_n - 2Z^b_n \geq 0\}}.
\]

(ii) For \( i \in \{1, \ldots, N-1\} \), \( Z^b_n \rightarrow Z^b_n - e_i \) at exponential rate

\[
\frac{1}{2} \sigma^2_{b,m,n}(i, Z^b_n) \mathbb{1}_{\{Z^b_n \geq 1\}} + g_{b,m,n}(i, Z^b_n) \mathbb{1}_{\{Z^b_n \geq 1\}} - \alpha_{b,n}(Z^{b,i+1}_n + Z^{b,i-1}_n - 2Z^b_n) \mathbb{1}_{\{Z^{b,i+1}_n + Z^{b,i-1}_n - 2Z^b_n \leq 0\}}.
\]

Similarly, the dynamics of the ask side of the book are given by:

(i) For \( i \in \{1, \ldots, N-1\} \), \( Z^a_n \rightarrow Z^a_n + e_i \) at exponential rate

\[
\frac{1}{2} \sigma^2_{a,m,n}(i, Z^a_n) \left( 1 + \mathbb{1}_{\{Z^n_a = 0\}} \right) + f_{a,m,n}(i, Z^a_n) + \alpha_{a,n}(Z^{a,i+1}_n + Z^{a,i-1}_n - 2Z^a_n) \mathbb{1}_{\{Z^{a,i+1}_n + Z^{a,i-1}_n - 2Z^a_n \geq 0\}}.
\]
(ii) For $i \in \{1, \ldots, N-1\}$, $Z_n^a \rightarrow Z_n^a - e_i$ at exponential rate

$$\frac{1}{2} \sigma_{a,m,n}(i, Z_n^{b,i}) \mathbb{I}_{\{Z_n^{b,i} \geq 1\}} + g_{a,m,n}(i, Z_n^{b,i}) \mathbb{I}_{\{Z_n^{b,i} \geq 1\}} - \alpha_{a,n}(Z_n^{b,i+1} + Z_n^{b,i-1} - 2Z_n^{b,i}) \mathbb{I}_{\{Z_n^{b,i+1} + Z_n^{b,i-1} - 2Z_n^{b,i} \leq 0\}}.$$

In the above we have that:

(a) For $k \in \{b, a\}$, every $n$ and every $m \in \mathbb{R}$, $\sigma_{k,m,n}$ is a map from $\{1, 2, \ldots N - 1\} \times \mathbb{N} \rightarrow \mathbb{R}^+.$

(b) For $k \in \{b, a\}$, every $n$ and every $m \in \mathbb{R}$, $f_{k,m,n}$ and $g_{k,m,n}$ are maps from $\{1, 2, \ldots N - 1\} \times \mathbb{N} \rightarrow \mathbb{R}^+.$

Remark 2.1. We remark here that market orders have the same impact on the profile of the book as cancellations at the best price levels. Market orders are therefore accounted for in these static dynamics.

Remark 2.2. Our model here only accounts for placement of small orders, which we have taken without loss of generality to be of size one. In Section 4 we give an example of how to include larger orders on a longer timescale in our dynamic model.

2.2 Heavy Traffic Diffusion Approximation in a Static Setting

We now switch our attention to the heavy traffic approximation of the suitably rescaled static microscopic order book process. Time is accelerated by a factor of $n$ and volumes are divided by $\sqrt{n}$. Therefore, we are considering the limits of the processes

$$\tilde{Z}_n^b(t) := \frac{Z_n^b(nt)}{\sqrt{n}} \quad \text{and} \quad \tilde{Z}_n^a(t) := \frac{Z_n^a(nt)}{\sqrt{n}}.$$

These processes therefore take values in $\frac{1}{\sqrt{n}} \mathbb{Z}^{N-1}$ and the limiting process will take values in $\mathbb{R}^{N-1}$. In order to obtain convergence, we need that various quantities in our microscopic model converge suitably. We will assume that

(i) For $k \in \{b, a\}$, every $m \in \mathbb{R}$ and every $n \geq 1$

$$\sigma_{k,m,n}(i, u) = \sigma_{k,m}(i, \frac{u}{\sqrt{n}})$$

for every $i \in \{1, 2, \ldots N - 1\}$ and every $u \in \mathbb{N}$.

(ii) For $k \in \{b, a\}$, every $m \in \mathbb{R}$ and every $n \geq 1$

$$f_{k,m,n}(i, u) = \frac{1}{\sqrt{n}} f_{k,m}(i, \frac{u}{\sqrt{n}}),$$

$$g_{k,m,n}(i, u) = \frac{1}{\sqrt{n}} g_{k,m}(i, \frac{u}{\sqrt{n}}),$$

for every $i \in \{1, 2, \ldots N - 1\}$ and every $u \in \mathbb{N}$.

(iii) For $k \in \{b, a\}$, $\alpha_{k,n} = \frac{1}{n} \alpha_k > 0.$
Here, the functions $\sigma_{k,m}, f_{k,m}, g_{k,m}$ and $h_{k,m} := f_{k,m} - g_{k,m}$ are all measurable from $\{1, 2, \ldots, N - 1\} \times \mathbb{R}^+ \to \mathbb{R}^+$. For technical reasons, we further assume that these functions are bounded on compact sets, Lipschitz continuous in the second argument and have linear growth in the second argument.

**Theorem 2.3.** The $\frac{1}{n} \mathbb{Z} \times \frac{1}{n} \mathbb{Z}$-valued process $(\tilde{Z}_{n}^{b}, \tilde{Z}_{n}^{a})$ converges weakly in \(\mathcal{M}(\mathbb{D}([0, \infty), \mathbb{R}^{N-1}) \times \mathbb{D}([0, \infty), \mathbb{R}^{N-1}))\) as $n \to \infty$ to the unique $\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}$-valued strong Markov diffusion process $(X^{b}, X^{a})$ which satisfies the following system of reflected SDEs:

\[
dX_{t}^{b,i} = \alpha_{b}(X_{t}^{b,i+1} + X_{t}^{b,i-1} - 2X_{t}^{b,i}) dt + h_{b,m}(i, X_{t}^{b,i}) dt + \sigma_{b,m}(i, X_{t}^{b,i}) dW_{t}^{b,i} + d\eta^{b,i}_{t},
\]

\[
dX_{t}^{a,i} = \alpha_{a}(X_{t}^{a,i+1} + X_{t}^{a,i-1} - 2X_{t}^{a,i}) dt + h_{a,m}(i, X_{t}^{a,i}) dt + \sigma_{a,m}(i, X_{t}^{a,i}) dW_{t}^{a,i} + d\eta^{a,i}_{t},
\]

for $i = 1, \ldots, N - 1$ with the pinning conditions that $X_{\tau}^{k,0} = X_{\tau}^{k,N} = 0$, where $W^{k,i}$ are independent Brownian motions. The $\eta^{k,i}$ are reflection measures which maintain positivity of the $X^{k,i}$.

### 2.3 Dynamic Discrete Order Book Process

We now describe the mechanism for price movements in the model to give our microscopic dynamic model. Price changes in both directions will be assumed to occur at positive rates which depend on the state of the book at any given time (including the current position of the mid). Our motivating example is the case where these rates are dependent on the imbalance of the number of bid limit orders compared to ask limit orders currently on the book near the mid. Here, relatively more offers to buy near the mid make a price increase more likely and relatively more offers to sell near the mid make a price decrease more likely.

In order to formalise this, we introduce functions $\theta_{u,m}^{n}$ and $\theta_{d,m}^{n}$ for $n \geq 1$ and $m \in \mathbb{R}$, which map $\mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \to \mathbb{R}_{>0}$. These will determine the rate of upward and downward price movements respectively as a function of the profiles of the bid and ask sides of the book. We also define for $n \geq 1$ functions $R^{n}$. These map $\mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \times \{u, d\} \to \mathcal{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})$ and will determine the distribution of the new profiles of the bid and ask sides of the book following price changes as a function of the profiles at the time of the price change and the direction of the price change. We fix some $\epsilon > 0$, which determines the size of price changes. Typically, we take $\epsilon \in \mathbb{N}$, so that the price jumps are multiples of the tick size, which is 1 in our current scaling as each queue represents one tick. We additionally introduce i.i.d. rate one exponential random variables $(Y_{n,u}^{i})_{i=1}^{\infty}$ and $(Y_{n,d}^{i})_{i=1}^{\infty}$ which will be used in the construction. These are independent of each other and of the driving Poisson processes in the model. With this in place, we can start to construct our dynamic process. Let $Z_{n,1}^{b}(0), Z_{n,1}^{a}(0) \in \mathbb{N}^{N-1}$ be the initial profiles for the bid and ask sides of the book respectively, and let $m_{1}^{b}$ be the initial mid for the $n$th microscopic order book. We denote by $Z_{n,1}^{b}, Z_{n,1}^{a}$ the processes evolving according to our dynamics for the microscopic order book with mid $m_{1}^{b}$ and initial profiles $Z_{n,1}^{b}(0), Z_{n,1}^{a}(0)$. We define the following stopping times.

\[
\tau_{n,u}^{1} := \inf \left\{ t \geq 0 \mid \int_{0}^{t} \theta_{u,m}^{n}(Z_{n,1}^{b}(s), Z_{n,1}^{a}(s)) ds \geq Y_{1,u}^{1} \right\} \wedge (T + 1),
\]

\[
\tau_{n,d}^{1} := \inf \left\{ t \geq 0 \mid \int_{0}^{t} \theta_{d,m}^{n}(Z_{n,1}^{b}(s), Z_{n,1}^{a}(s)) ds \geq Y_{1,d}^{1} \right\} \wedge (T + 1).
\]

The truncation of the stopping times at $T + 1$ is done for technical reasons only and has no effect on the model, as we will be proving convergence on $[0, T]$. We also define $\tau_{n}^{1} := \tau_{n,u}^{1} \wedge \tau_{n,d}^{1}$. The
time $\tau_n^1$ triggers a price change. If $\tau_n^1 = \tau_{n,u}$, we have an upward price change, and set $m_n^2 = m_n^1 + \epsilon$. Similarly, if $\tau_n^1 = \tau_{n,d}$, we have a downward price change, and set $m_n^2 = m_n^1 - \epsilon$. We then let $Z_{n,2}^b, Z_{n,2}^a$ be the new processes which follow the dynamics of the bid and ask sides of the static microscopic order book with mid $m_n^2$. The initial profile has the law of $R^n(Z_{n,1}^b(\tau_n^1), Z_{n,1}^a(\tau_n^1), u)$ if the price change was upward and $R^n(Z_{n,1}^b(\tau_n^1), Z_{n,1}^a(\tau_n^1), d)$ if the price change was downward, and is conditionally independent of the past behaviour of the order book given $(Z_{n,1}^b(\tau_n^1), Z_{n,1}^a(\tau_n^1))$ and the direction of the price change. The processes $Z_{n,2}^b, Z_{n,2}^a$ are taken to be conditionally independent of the past of the dynamic order book given the initial profile and $m_n^2$. We can iterate this procedure to define further stopping times, price points and processes $Z_{n,i}, Z_{n,i}^a$, describing the dynamics after the $i$th price change. Having defined $(Z_{n,i}^b)_{i=1}^M$, $(Z_{n,i}^a)_{i=1}^M$, $(\tau_i)_{i=1}^{M-1}$ and $(m_i^j)_{i=1}^M$, we set

$$\tau_{n,u}^M := \inf \left\{ t \geq 0 \mid \int_0^t \theta_{u,m_n^1}^n (Z_{n,M}^b(s), Z_{n,M}^a(s)) \, ds \geq Y_{n,u}^M \right\},$$

$$\tau_{n,d}^M := \inf \left\{ t \geq 0 \mid \int_0^t \theta_{d,m_n^1}^n (Z_{n,M}^b(s), Z_{n,M}^a(s)) \, ds \geq Y_{n,d}^M \right\}.$$

As before, we define $\tau_n^M := \tau_{n,u}^M \wedge \tau_{n,d}^M$, with the time $\tau_n^M$ triggering a price change. If $\tau_n^M = \tau_{n,u}$, we set $n_n^{M+1} = n_n^M + \epsilon$ and if $\tau_n^M = \tau_{n,d}$, we set $m_n^{M+1} = m_n^M - \epsilon$. We then let $Z_{n,M+1}^b, Z_{n,M+1}^a$ be new processes which follows the dynamics of the bid and ask sides of the static microscopic order book with the mid now given by $m_n^{M+1}$, and the initial profile having the law of $R^n(Z_{n,M}^b(\tau_n^M), m_n^M, m_n^{M+1})$ and being conditionally independent of the past behaviour of the order book given $(Z_{n,M}^b(\tau_n^M), Z_{n,M}^a(\tau_n^M))$ and the direction of the price change. Once again, $Z_{n,M+1}^b, Z_{n,M+1}^a$ are taken to be conditionally independent of the past of the dynamic order book given the initial profile and the new mid position. Our dynamic microscopic order book is then described by the processes $(\hat{Z}_{n,i}^b(t), \hat{Z}_{n,i}^a(t), m_n(t))$ describing the evolution of the two sides of the book and the mid through time, where

$$\hat{Z}_{n,i}^b(t) := \sum_{i=1}^{\infty} Z_{n,i}^b \left( t - \sum_{j=1}^{i-1} \tau_n^j \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau_n^j \leq t < \sum_{j=1}^i \tau_n^j \right\}},$$

$$\hat{Z}_{n,i}^a(t) := \sum_{i=1}^{\infty} Z_{n,i}^a \left( t - \sum_{j=1}^{i-1} \tau_n^j \right) \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau_n^j \leq t < \sum_{j=1}^i \tau_n^j \right\}},$$

$$m_n(t) := \sum_{i=1}^{\infty} m_i \mathbb{1}_{\left\{ \sum_{j=1}^{i-1} \tau_n^j \leq t < \sum_{j=1}^i \tau_n^j \right\}}.$$

### 2.4 Heavy Traffic Diffusion Approximation in a Dynamic Setting

We now present the convergence of our dynamic microscopic model to a dynamic mesoscopic model. We should first, of course, define the dynamic mesoscopic model. This is done in essentially the same way as in the microscopic case. We therefore only give an overview here and refer to the previous section for the precise details. The static mesoscopic model determines the behaviour of the dynamic model in between price changes. The functions determining rates of upward price changes and downward price changes as functions of the book profile are now denoted by $\theta_{n,u}$ and $\theta_{d,m}$ respectively. They are now bounded maps from $([R^+)^{N-1} \times ([R^+)^{N-1} \to \mathbb{R}_{>0}$, which we
Theorem 2.4. Let \( M \) correspond to the microscopic model in the following ways. Our regenerative function which determines the distribution of the new profile following a price change is denoted by \( R \) and now maps \( (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)\times \{u,d\} \to \mathcal{M}( (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)\times \{u,d\}) \). This function is also assumed to be continuous, where \( \mathcal{M}( (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)\times \{u,d\}) \) is equipped with the topology of weak convergence. For ease of notation, we introduce the maps \( P_n : \mathbb{R}^2 \to \mathbb{R}^2 \), which simply divide each coordinate by \( \sqrt{n} \). The functions here are then approximated by the corresponding functions in the microscopic model in the following ways.

(i) For \( k \in \{u,d\} \), \( v_1, v_2 \in \mathbb{N}^{N-1} \) and \( u_1, u_2 \in (\mathbb{R}^+)^{N-1} \),

\[
|n\theta_{k,m}^i(v_1, v_2) - \theta_{k,m}(u_1, u_2)| \leq r(\|P_n((v_1, v_2)) - (u_1, u_2)\|),
\]

where \( \lim_{x \to 0} r(x) = 0 \).

(ii) For \( k \in \{u,d\} \), if \( (v_1^n, v_2^n) \) is a sequence in \( \mathbb{N}^{N-1} \times \mathbb{N}^{N-1} \) such that \( P_n((v_1^n, v_2^n)) \to (u_1, u_2) \), then

\[
R^n(v_1^n, v_2^n, k) \circ P_n^{-1} \Rightarrow R(u_1, u_2, k)
\]

in law in \( \mathcal{M}( (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)\times \{u,d\}) \).

With this in place, we define the processes \( X_i \), stopping times \( \tau^i_u, \tau^i_d, \tau^i \) and price sequence \( m^i \) analogously to the previous section, with price jumps once again of size \( \epsilon \). The underlying collections of exponential random variables which are used in the construction of the stopping times \( \tau^i_u \) and \( \tau^i_d \) are now denoted \( Y^i_u \) and \( Y^i_d \). Hence, \( X_i \) is the sequence of static mesoscopic models with suitable mids, \( (\tau^i_u, \tau^i_d) \) are stopping times determining the times in between price changes as well as the direction of these price changes, and \( m^i \) is the sequence of mids. Our dynamic mesoscopic model is then given by \((\hat{X}^b(t), \hat{X}^a(t), m(t))\), where

\[
\hat{X}^b(t) := \sum_{i=1}^{\infty} X_i^b \left( t - \sum_{j=1}^{i-1} \tau^j \right) 1 \left\{ \sum_{j=1}^{i-1} \tau^j \leq t < \sum_{j=1}^{i} \tau^j \right\},
\]

\[
\hat{X}^a(t) := \sum_{i=1}^{\infty} X_i^a \left( t - \sum_{j=1}^{i-1} \tau^j \right) 1 \left\{ \sum_{j=1}^{i-1} \tau^j \leq t < \sum_{j=1}^{i} \tau^j \right\},
\]

\[
m(t) := \sum_{i=1}^{\infty} m^i 1 \left\{ \sum_{j=1}^{i-1} \tau^j \leq t < \sum_{j=1}^{i} \tau^j \right\}.
\]

**Theorem 2.4.** Let \( T > 0 \). Suppose that \( \left( \frac{Z_{n,1}^b(0)}{\sqrt{n}}, \frac{Z_{n,1}^a(0)}{\sqrt{n}} \right) \Rightarrow (X^b_1(0), X^a_1(0)) \) weakly in \( \mathcal{M}( (\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)\times \{u,d\}) \). Let \((\hat{Z}^b_n(t), \hat{Z}^a_n(t), m_n(t))\) be dynamic microscopic models with initial data \( \left( \frac{Z_{n,1}^b(0)}{\sqrt{n}}, \frac{Z_{n,1}^a(0)}{\sqrt{n}}, m^1 \right) \), and let \((\hat{X}^b(t), \hat{X}^a(t), m(t))\) be the dynamic mesoscopic model with initial data \( (X^b_1(0), X^a_1(0), m^1) \). Then

\[
\left( \frac{\hat{Z}^b_n(nt)}{\sqrt{n}}, \frac{\hat{Z}^a_n(nt)}{\sqrt{n}}, m_n(nt) \right) \Rightarrow (\hat{X}^b(t), \hat{X}^a(t), m(t))
\]

weakly in \( \mathcal{M}( \mathbb{D}([0,T], \mathbb{R}^{N-1}) \times \mathbb{D}([0,T], \mathbb{R}^{N-1}) \times \mathbb{D}([0,T], \mathbb{R})) \).
3 Reflected SPDEs: from mesoscopic to macroscopic models

In this section we rescale our previously obtained dynamic mesoscopic system and bridge the gap between mesoscopic and macroscopic models of limit order books. We show convergence to a dynamic macroscopic model limit by letting the tick size tend to zero and rescaling suitably. As with the dynamic result in the previous section, the proof of this relies heavily on convergence of the static models, which we present first. We are able to obtain convergence in the static setting by applying Theorem 2.1 in T. Zhang [28].

Remark 3.1. Note that, although we take the tick size to zero for the bid and ask sides of the book, the price process moves according to macroscopic price jumps. This both simplifies the analysis, avoiding the need to consider a moving boundary model, and allows us to maintain a natural separation of time-scales for the order book evolution and price changes.

3.1 SPDE limit in a static setting

We begin this section by describing the sequence of rescaled static mesoscopic models which we will consider. For every $N \geq 1$ we let $X_N^b(t), X_N^a(t)$ satisfy the dynamics of the bid and ask sides of the static mesoscopic model on the price grid $\{0, 1, 2, \ldots N\}$ with mid $m \in \mathbb{R}$. These models are now indexed by $N$, which was suppressed in the previous section. We wish to emphasise this here since we will be taking $N$ to infinity. We will rescale space and time appropriately and map the coordinates of $X_N^b(t)$ and $X_N^a(t)$ to equally spaced points on $[0, 1]$. Define here the function $Q_N : \mathbb{R}^{N-1} \to C_0((0, 1))$, such that

(i) For $i = 1, 2, \ldots, N - 1$, $Q_N(x) \left( \frac{i}{N} \right) = \frac{x_i}{\sqrt{N}}$.

(ii) For $i = 0, 1, 2, \ldots, N - 1$, $Q_N(x)$ is linear between the points $\left[ \frac{i}{N}, \frac{i+1}{N} \right]$.

The aim will be to take the limit of $(Q_N(X_N^b(N^2t)), Q_N(X_N^a(N^2t)))$. We should have that the parameters for our SDE systems are consistent in some way. Given the rescaling, they are chosen such that for $k \in \{b, a\}, m \in \mathbb{R}$, $N \geq 1$, $i = 0, 1, \ldots, N - 1$ and $u \in (\mathbb{R}^+)^{N-1}$:

(i) $h_{k,m}^N(i, u) := N^{-\frac{3}{2}} h_{k,m} \left( \frac{i}{N}, \frac{u}{\sqrt{N}} \right)$,

(ii) $\sigma_{k,m}^N(i, u) := \sigma_{k,m} \left( \frac{i}{N}, \frac{u}{\sqrt{N}} \right)$,

where $h_{k,m}, \sigma_{k,m}$ are measurable maps from $[0, 1] \times [0, \infty) \to \mathbb{R}$. We further assume the following Lipschitz and linear growth conditions. These are required in order to prove convergence.

1. $|h_{k,m}(x, u) - h_{k,m}(y, v)| + |\sigma_{k,m}(x, u) - \sigma_{k,m}(y, v)| \leq C(|x - y| + |u - v|)$ and

2. $|h_{k,m}(x, u)| + |\sigma_{k,m}(x, u)| \leq C(1 + |u|)$.

We will show that our rescaled mesoscopic models converge to a reflected SPDE. Before doing so, we first give the definition of a solution to a reflected SPDE. This is the same as in T. Zhang [28].

Definition 3.2. We say that the pair $(u, \eta)$ is a solution the SPDE with reflection

$$\frac{\partial u}{\partial t} = \alpha \Delta u + h(x, u(t, x)) + \sigma(x, u(t, x)) \frac{\partial^2 W}{\partial x \partial t} + \eta(t, x)$$

(3.1)

with Dirichlet conditions $u(t, 0) = u(t, 1) = 0$ and initial data $u(0, x) = u_0 \in C_0((0, 1))^+$ if
(i) $u$ is a continuous adapted random field on $\mathbb{R}^+ \times [0,1]$ such that $u \geq 0$ almost surely.

(ii) $\eta$ is a random measure on $\mathbb{R}^+ \times (0,1)$ such that:

(a) For every $t \geq 0$, $\eta(\{1\} \times (0,1)) = 0$,

(b) For every $t \geq 0$, $\int_0^t \int_0^1 x(1-x)\eta(ds,dx) < \infty$,

(c) $\eta$ is adapted in the sense that for any measurable mapping $\psi$:

$$\int_0^t \int_0^1 \psi(s,x)\eta(ds,dx) \text{ is } \mathcal{F}_t - \text{measurable.}$$

(iii) For every $t \geq 0$ and every $\phi \in C^2([0,1])$ with $\phi(0) = \phi(1) = 0$,

$$\int_0^1 u(t,x)\phi(x)dx = \int_0^1 u(0,x)\phi(x)dx + \alpha \int_0^t \int_0^1 u(s,x)\phi''(x)dxds$$

$$+ \int_0^t \int_0^1 h(x,u(s,x))\phi(x)dxds + \int_0^t \int_0^1 \phi(x)\sigma(x,u(s,x))W(ds,dr)$$

$$+ \int_0^t \int_0^1 \phi(x)\eta(ds,dr)$$

almost surely.

(iv) $\int_0^\infty \int_0^1 u(t,x)\eta(dt,dr) = 0$.

The intuition for this equation is that the reflection measure here is analogous to the local time for a one dimensional diffusion. The solution follows the dynamics of a standard SPDE (without reflection), except when the solution meets the $x$-axis, where the profile is minimally pushed up by the reflection measure and kept positive.

Existence of strong solutions of to equations of the form (3.1), under our conditions on the coefficients, is proved by C. Donati-Martin and E. Pardoux in [12]. This means that, given a white noise process and the filtration that it generates, we can construct a solution which is adapted. Uniqueness was then proved by T. Xu and T. Zhang in [27].

We now pass to the macroscopic limit in the static setting. The following result can be shown by a direct application of Theorem 2.1 in T. Zhang [28].

**Theorem 3.3.** Fix some $T > 0$. Suppose that $(Q_N(X^b_N(0)),Q_N(X^a_N(0))) \Rightarrow (v^b_N, v^a_N)$ in law in $C_0((0,1)) \times C_0((0,1))$. Then $(Q_N(X^b_N(N^2t)),Q_N(X^a_N(N^2t))) \Rightarrow (v^b, v^a)$ in law in $C([0,T+1] \times [0,1])$, where $(v^b, v^a)$ is the unique solution to the pair of reflected stochastic heat equations

$$\frac{\partial v^b}{\partial t} = \alpha_b \Delta v^b + h_{b,m}(x,v^b) + \sigma_{b,m}(x,v^b)\frac{\partial^2 W^b}{\partial x \partial t} + \eta^b(t,x),$$

$$\frac{\partial v^a}{\partial t} = \alpha_a \Delta v^a + h_{a,m}(x,v^a) + \sigma_{a,m}(x,v^a)\frac{\partial^2 W^a}{\partial x \partial t} + \eta^a(t,x),$$

with pinning conditions $v^b(0,0) = v^b(t,1) = v^a(t,0) = v^a(t,1) = 0$ and initial data $v^b(0,x) = v^b_0(x) \in C_0((0,1))$ and $v^a(0,x) = v^a_0(x) \in C_0((0,1))$, where space-time white noises $W^b$ and $W^a$ are independent.

We call the limiting process $(v^b, v^a)$ our static macroscopic order book process with mid $m$. 
3.2 SPDE in a dynamic setting

Let \((\hat{X}_N^b(t), \hat{X}_N^a(t), m_N(t))\) be our \(N\)th dynamic mesoscopic model, with the parameters defined as in the previous section. The price change rates are now denoted by \(\theta_{b,m}^N\) and \(\theta_{a,m}^N\), and the regenerative functions by \(R^N\). We aim to prove convergence to a dynamic macroscopic model by taking the limit of the sequence \((Q_N(\hat{X}_N^b(N^2t)), Q_N(\hat{X}_N^a(N^2t)), m_N(N^2t))\).

We describe our dynamic macroscopic order book model. The principles are the same as for the dynamic microscopic and mesoscopic models, in that we assume the dynamics follow the static microscopic model, with the parameters defined \(\hat{X}_N\) simply applying \(Q\) to each coordinate. We connect the dynamic aspects of the mesoscopic and macroscopic models by assuming that \(\hat{X}_N^b, X^b\), \(\hat{X}_N^a, X^a\) are such that

\[
\left| N^2 \theta_{k,m}^N((X^1, X^2)) - \theta_{k,m}(u_1, u_2) \right| \leq r(\|Q_N((X^1, X^2)) - (u_1, u_2)\|),
\]

where \(\lim_{x \to 0} r(x) = 0\).

(ii) For every \(m \in \mathbb{R}\), if \((X^1, X^2) \in \mathbb{N}^{N-1} \times \mathbb{N}^{N-1}\) and \((u_1, u_2) \in C_0((0, 1)) \times C_0((0, 1))\) are such that

\[
\|Q_N((X_1, X_2)) - (u_1, u_2)\| \to 0,
\]

then, for \(k \in \{b, a\}\),

\[
\hat{R}^N(X^1, X^2, k) \circ \hat{Q}^{-1}_N \Rightarrow R(u_1, u_2, k)
\]

in law in \(C_0((0, 1)) \times C_0((0, 1))\).

Remark 3.4. For \(X \in \mathbb{N}^{N-1}\) and \(u \in C_0((0, 1))\),

\[
\|Q_N(X) - u\| \to 0
\]

is equivalent to the condition that

\[
\sup_{i=1,2,...,N-1} \left| X^i - u(i/N) \right| \to 0.
\]
**Theorem 3.5.** Let $C$ with initial data $(\text{initial data} (R_0, (\theta_u, \theta_d, \theta_c)))$ be our dynamic mesoscopic order book model with initial data $(X_N^b(0), X_N^a(0), m(0))$, and let $(u^b(t), u^a(t), m(t))$ be our dynamic macroscopic model with initial data $(u^b(0), u^a(0), m(0))$. Then we have that

$$(Q_N(X_N^b(0)), Q_N(X_N^a(0))) \rightarrow (u^b(0), u^a(0)) \text{ in law in } C_0((0, 1)) \times C_0((0, 1)).$$

**Remark 3.6.** We have assumed when proving convergence of our dynamic models that the functions determining the rates for price changes are bounded. This isn’t strictly necessary- it is sufficient to know that the model is chosen so that there are finitely many price changes in the time interval $[0, T + 1]$ almost surely. Failing this, it is still possible to prove convergence for the dynamic models if we terminate them after a predetermined finite number of price changes.

**4 Examples**

In this section we illustrate the flexibility of our set-up by discussing some ideas for different aspects of the model.

**Example 4.1.** Rate functions can be chosen such that

$$\theta_{u,m}(u^1, u^2) = \gamma F \left( \frac{\int_0^\epsilon (u^1(x) - u^2(x)) \, dx}{\epsilon} \right) + \delta$$

and

$$\theta_{d,m}(u^1, u^2) = \gamma F \left( \frac{\int_0^\epsilon (u^2(x) - u^1(x)) \, dx}{\epsilon} \right) + \delta$$

where $F$ is a continuous function, and $\delta > 0$. The motivation behind this example is that the rate at which price movements occur has two components which are natural from a modelling standpoint. The first is a function of the local imbalance (the difference between the number of offers to buy and the number of offers to sell in a region close to the mid). The second is a fixed rate, intended to represent price movements due to exogenous factors. We will see in Section 5 how one might fit these parameters to data.

**Example 4.2.** We can incorporate large orders into our model. So far we have only directly considered small order sizes in our models, taking these wlog to be of size 1 in our microscopic models. We haven’t, however, mentioned larger order sizes which would appear as “jumps” in the macroscopic model profile in the limit. These can easily be incorporated by being assumed to appear (as one would expect) on a slower time scale to small orders. Large market orders which cause price changes are already accounted for in the existing set-up. We can also easily include extra stopping times $\tau_c^1$ into our models, which allow us to create a jump in the profile of the book without changing the price. These could then be used to model large cancellations or large limit orders for the book which do not trigger price changes. The third rate can then be given by some third rate function $\theta_{c,m}$, with the new profile again given by $R$ via the definition of $R(u^1, u^1, c)$, where we extend the definition of $R$ so that it now maps from $C_0((0, 1)) \times C_0((0, 1)) \times \{u, d, c\} \rightarrow \mathcal{M}(C_0((0, 1)) \times C_0((0, 1)))$.

We note here that in this model, the “static” models in between stopping times do not refer to the evolution of the book in between price changes, but rather the evolution of the book in between both price changes and large orders.
Example 4.3. As a particular case of the drift and volatility functions, \( h_{k,m} \) and \( \sigma_{k,m} \) we can take

1. \( h_{k,m}(x,u) = h_1(x) + h_2(x)u, \) and
2. \( \sigma_{k,m}(x,u) = \sigma_1(x) + \sigma_2(x)u. \)

This has a natural interpretation. The multiplicative terms \( h_2(x)u \) and \( \sigma_2(x)u \) can be thought of as self-exciting components for the order rates, whereby more orders on the book leads to faster trading. The remaining terms represent orders placed independently of the current order book profile.

Example 4.4. We note that our regeneration functions, which determine the profiles of the two sides of the order book following price changes, allow us to choose profiles which both depend on the state of the book when the price change occurs, and are random. Therefore, natural deterministic choices, such as suitably removing orders from the previous profiles when the price changes, are permitted. We are also able to choose random profiles, such as sampling from invariant measures, or some combination of these two mechanisms.

5 Numerical Investigation

The aim of this final section is to demonstrate that even simple versions of the model can reproduce features of the price series and order books seen in financial data. We describe the particular parameter choices for the model which will be used in this section, before giving an overview of the numerical scheme and introducing the LOBSTER dataset. Following this, we briefly explain the parameter estimation procedure used, before presenting numerical illustrations and results by plugging in the estimated parameters into the simulation algorithms. We would like to emphasise here that the intention behind this section is to demonstrate that sensible order book simulations can readily be obtained from the model - we do not claim that the numerical scheme implemented, or the methods used to fit the parameters, are optimal.

5.1 The Model

We will work with a special case of the models which fall within the framework described in the earlier sections. The two sides of the book, \( u^1 \) and \( u^2 \), will evolve in between price changes according to the reflected SPDEs

\[
\frac{\partial u^i}{\partial x} = \alpha \Delta u + f(x) + \sigma(x) \frac{\partial^2 W}{\partial x \partial t} + \eta^i.
\]

The implicit assumption here is that the order arrival rates, which give rise to the drift and volatility terms \( f \) and \( \sigma \) in our SPDE limit, depend on the distance from the mid only, and there is no dependence on the number of orders which are currently on the book at that price. We have also imposed that the coefficients \( f \) and \( \sigma \) here do not depend on \( i \). This is due to the symmetry in the order arrival rates for the bid and ask sides of the book, which can be seen in the data.

We will now describe the form of the rate functions which will be used in this section, \( \theta_u(u) \) and \( \theta_d(u) \), which determine the rates at which the price moves up and down respectively. In a particular case of Example 1 of Section 4, the rate at time \( t \) for an upward price jump will be given by

\[
\theta_u(u^a(t,\cdot), u^b(t,\cdot)) = \gamma \max \left( \int_0^x \left( u^1(t,x) - u^2(t,x) \right) \, dx, 0 \right) + \delta
\]
and the rate of a downward price movement will similarly be given by

$$
\theta_d(u^a(t, \cdot), u^b(t, \cdot)) = \gamma \max \left( \int_0^\pi (u^2(t, x) - u^1(t, x)) \, dx, 0 \right) + \delta. \quad (5.2)
$$

The first terms in these rates represent the contribution of order imbalance close to the mid as a driving factor for price movement, whilst the second terms represent price movements due to additional exogenous factors.

Our regeneration functions $R(u^1, u^2, d)$, which determine the profiles of the two sides of the book following price changes, will simply shift the profiles of the two sides of the book by the size of a price jump, in the relevant direction. That is, $R$ is given by

$$
R\left(u^1, u^2, 1\right) = \begin{cases}
(0, u^2(x + \epsilon)) & \text{for } x \in [0, \epsilon], \\
(u^1(x - \epsilon), u^2(x + \epsilon)) & \text{for } x \in [\epsilon, 1 - \epsilon], \\
(u^1(x - \epsilon), 0) & \text{for } x \in [\epsilon, 1], \\
(0, u^2(x + \epsilon)) & \text{for } x \in [0, \epsilon], \\
(u^1(x - \epsilon), u^2(x + \epsilon)) & \text{for } x \in [\epsilon, 1 - \epsilon], \\
(u^1(x - \epsilon), 0) & \text{for } x \in [\epsilon, 1],
\end{cases} \quad (5.3)
$$

We note here that, strictly speaking, some of these new profiles do not satisfy the Dirichlet conditions imposed earlier in the analysis. However, the equations can be shown to have solutions when started from a general continuous initial profile, with the solution in $C_0(0, 1)$ at all positive times. In addition, the regeneration function as described here has a natural interpretation simply as the best bid/ask queue having been depleted to trigger the price change.

### 5.2 The Numerical Scheme

We use a simple forward time-stepping scheme in order to simulate our equations in between price movements. The equation is discretised into $M$ time steps and $N$ space steps, and we define $t_i := iT/M$, $x_i := i/N$. We denote the simulated height of the bid/ask sides of the book at time $t_j$ and position $x_j$ by $u^b(t_j, x_j)$ and $u^a(t_j, x_j)$ respectively. The simulated price process at time $t_j$ is denoted by $p(t_j)$. Given our simulated solution up to the $j$th time step, we begin our approximation of the following time-step by first determining whether there should be a price movement at this time-step. These are approximations of the probabilities of price jumps in that time period given by the rates $\theta_b(u^a(t, \cdot), u^b(t, \cdot))$ and $\theta_b(u^a(t, \cdot), u^b(t, \cdot))$ as in (5.1) and (5.2). Writing these explicitly, we have that

$$
\pi^+[t_j] = \max\left( \frac{\gamma}{2} \left( u^b(t_j, x_1) - u^a(t_j, x_1) \right) \times T/M, 0 \right) \times T/M
$$

and

$$
\pi^-[t_j] = \max\left( \frac{\gamma}{2} \left( u^a(t_j, x_1) - u^b(t_j, x_1) \right) \times T/M, 0 \right) \times T/M.
$$

We simulate a uniform random variable on $[0, 1]$, which we denote by $Y[t_j]$. If $Y[t_j] < \pi^+[t_j]$ we take this as an indication that the price has moved up in that time-step. Similarly, if $\pi^+[t_j] \leq Y[t_j] < \pi^+[t_j] + \pi^-[t_j]$, we take this as an indication that there has been a downward price change at this time-step. If we are in neither of these cases, the simulated order book does not change price during this time-step. If there has been a price increase, we update our price process by setting
\( p[t_{j+1}] = p[t_j] + 1 \), and adjust the profiles \( u^b \) and \( u^a \) by discrete approximations to (5.3). That is, we simply set

\[
    u^b(t_j, x_0) = u^a(t_j, x_N) = 0,
\]

and define

\[
    u^b(t_j, x_i) = u^b(t_j, x_{i-1})
\]

for \( i \in \{1, \ldots, N\} \), and

\[
    u^a(t_j, x_i) = u^a(t_j, x_{i+1})
\]

for \( i \in \{0, \ldots, N - 1\} \). We analogously update \( u^b \), \( u^a \) and \( p \) in the event of a downward price movement. If there is no price change at this time-step, we do not update \( u^b \), \( u^a \) at this point, and set \( p(t_{j+1}) = p(t_{j}) \). At the end of this price updating procedure, we then simulate the profiles of \( u^b \) and \( u^a \) at the next time-step by setting

\[
    u^b(t_{j+1}, x_i) := \max \left\{ u^b(t_j, x_i) + \frac{1}{MN^2} \left( u^b(t_j, x_{i+1}) + u^b(t_j, x_{i-1}) - 2u^b(t_j, x_i) \right) + \frac{1}{M} f(x_i) + \frac{\sqrt{TN}}{\sqrt{M}} \sigma(x_i) Z_{i,j}^b, 0 \right\},
\]

and similarly

\[
    u^a(t_{j+1}, x_i) := \max \left\{ u^a(t_j, x_i) + \frac{1}{MN^2} \left( u^a(t_j, x_{i+1}) + u^a(t_j, x_{i-1}) - 2u^a(t_j, x_i) \right) + \frac{1}{M} f(x_i) + \frac{\sqrt{TN}}{\sqrt{M}} \sigma(x_i) Z_{i,j}^a, 0 \right\}.
\]

The \( Z_{i,j} \) here are simply simulated unit normal random variables, and appear due to the discretisation of the space-time white noise component of our equations. We take the maximum with zero at each time-step in order to capture the influence of the reflection measures in the equations. This step completes our forward time-step, and we return the profiles \( u^b(t_{j+1}, \cdot) \), \( u^a(t_{j+1}, \cdot) \) and the price \( p(t_{j+1}) \) as our simulated order book at time \( t_{j+1} \).

5.3 Dataset Description

The data we have at our disposal originates from the LOBSTER (Limit Order Book System, The Efficient Reconstructor) database project initiated by the Humboldt University of Berlin, which gives access to reconstructed limit order book data for all NASDAQ traded stocks between June 2007 up to the present day. For each trading day of a given ticker, LOBSTER generates two distinct files. On the one hand, a message file, which lists indicators for the different kinds of events which cause an update of the book (limit order arrivals and cancellations, executions or market orders, trading halts) within a prespecified price range. On the other hand, an order book file, which displays the evolution of the book up to a chosen number of occupied price levels (which can go up to 200, depending on the selected ticker). Order book events are timestamped according to seconds after midnight, and the decimal precision available ranges from milliseconds to nanoseconds.

Our sample consists of data from the SPDR Trust Series I, and covers the 50 best levels on each side of the book on June 21 2012 between 09:30:00.000 and 10:30:00.000 EST. Over this period, there were a total of 520284 limit orders, 496858 cancellations and 31433 market orders. As remarked earlier, the number of price changes is only 3835.
5.4 Parameter Fitting

In this section, we aim to obtain input parameters for our model based on the data described in the previous section. We would like to emphasise here that we do not claim that our fitting methods in this section are particularly robust - we simply aim to fit the parameters in a reasonable way so that we can demonstrate that our model can provide sensible simulations of the evolution of the order book. In addition, we do not fit the smoothing parameter, $\alpha$ and instead choose this value such that the profiles obtained are on the correct scale. In order to account for the reflection measure, we will only use the data of limit orders placed in price levels which are currently occupied. This is done since we should only take into consideration the order arrivals when the order volumes are away from zero, so as not to bias our drift estimates with components which could be attributed to the reflection measure.

Throughout this section, we will denote by $X_{j,i}^b$ the order volume in the $i$th price point below the best bid at the $j$th time-step for our dataset. We similarly denote by $X_{j,i}^a$ the order volume in the $i$th price point above the best ask at the $j$th time-step for our dataset.

The scaling used will measure order volumes in units of $10^4$ orders, and the 50 queues of the order book will map to a spatial interval of length 1. This simply results in the SPDEs for each side of the book each being on a spatial interval of length 1. Time units for the SPDEs will be measured in seconds. In this scale, our choice of smoothing parameter, $\alpha$ will be given by $\alpha = 0.1$.

5.4.1 Volatility and Drift Estimates

We begin this section by fitting the volatility and drift estimates for our SPDE. Let $\hat{\sigma}(i/50)$ denote our estimate for $\sigma(i)$, the volatility at spatial position $i/50$, corresponding to $i$ price points away from the best bid/ask. We calculate $\hat{\sigma}$ by equating

$$\frac{1}{50} \times 3600 \times \sigma^2(i/50) = \frac{1}{2} \sum \left[ \text{Order Size} \times 10^{-4} \right]^2,$$

where the sum is taken over all orders on the bid/ask side of the book of all types which are $i$ price points away from the best bid/ask price respectively, with the omission of limit orders placed in unoccupied queues. This simply matches the quadratic variation of different spatial intervals in the model with the corresponding value from the dataset. The factors of 50 and 3600 appear here due to our spatial and time scaling respectively, whilst the $1/2$ is present simply as we are summing data from both sides of the book, rather than just one side. The factor of $10^{-4}$ appears as we are working with units of $10^4$ orders.

We now fit the drift term. Let $d_i^b$ be the total number of bid limit orders placed in the hour long period in queues which are $i$ price points away from the best bid, once again disregarding those orders placed in unoccupied queues. We similarly define $d_i^a$, with $c_i^b$ and $c_i^a$ the corresponding values for market/cancellation orders at the different relative price points for the bid and ask sides of the book respectively. Denote by $\hat{f}(i/50)$ our estimate for the drift at spatial position $i$. We can then estimate the drift parameter by equating

$$\hat{f}(i/50) = \frac{1}{2} \left( d_i^a + d_i^b - c_i^a - c_i^b \right) \times 10^{-4} \times \frac{1}{3600}. \quad (5.4)$$

Figure 2 displays the estimated drift and volatility functions obtained from the data by the techniques described above. We note that the volatility is at its largest close to the mid, representing that there was significantly more activity at these price points over the trading period. A consequence of this will be that our simulated order book profiles will have realistic shapes, with highest average order volumes at price points which are close to the mid.
5.4.2 Estimates for Rates of Price Changes

It is only left to produce estimates for $\gamma$, the rate at which the model changes price due to imbalance of the bid and ask queues, and $\delta$, the rate at which the model changes price due to exogenous factors. Let $P(t)$ denote the price process of our dataset, measured in cents. Our estimate of $\gamma$, $\hat{\gamma}$, is chosen such that it satisfies the equation

$$P(1) - P(0) = \hat{\gamma} \times 3600 \times I.$$ 

The value $I$ here is the average local imbalance of the data over the entire period, given by

$$I = \frac{1}{10^6} \sum_{j=1}^{J} \left[ \int_{0}^{1/50} \left( \tilde{X}_j^b(x) - \tilde{X}_j^a(x) \right) \, dx \right],$$

where $\tilde{X}_j^b(x)$ and $\tilde{X}_j^a(x)$ are obtained from $X_{j,i}^b$ and $X_{j,i}^a$ by setting $\tilde{X}_j^b(i/50) = 10^{-4}X_{j,i}^b$ and $\tilde{X}_j^a(i/50) = 10^{-4}X_{j,i}^a$, and then linearly interpolating in between these points. Writing $I$ in terms of $X^b$ and $X^a$, we have

$$I = \frac{1}{10^6} \sum_{j=1}^{J} \left( X_{j,1}^b - X_{j,1}^a \right).$$

The rate for price movements in either direction due to exogenous factors, $\hat{\delta}$, is then fitted so that the quadratic variation of the price process from the dataset matches with the expected quadratic variation had the price moved due to our price changing mechanism with parameters $\hat{\gamma}$ and $\hat{\delta}$. The expected number of price changes due to the local imbalance component is given by

$$\tilde{I} = \frac{1}{10^6} \sum_{j=1}^{J} \left| X_{j,1}^b - X_{j,1}^a \right|.$$ 

For a given $\delta$ we would expect an extra $2 \times 3600 \times \delta$ price changes over the time period. Our parameter $\hat{\delta}$ is therefore chosen such that it solves

$$7200\hat{\delta} = \left[ \sum_{j=1}^{J} \left( P(j/J) - P((j-1)/J) \right)^2 \right] - \hat{\gamma} \tilde{I}.$$ 

Figure 1: Estimated Drift (a), and volatility (b).
Implementing the procedures described, we obtain the following values for $\hat{\delta}$ and $\hat{\gamma}$.

\[
\begin{array}{cc}
\hat{\gamma} & \hat{\delta} \\
36.344 & 0.47992
\end{array}
\]

Table 1: Estimated Price Change Rates.

5.5 Simulation Results

In this section, we will present the results of our numerical simulation of the order book under our model. In addition to the parameters which were fitted in the previous section, we fix the following additional parameters for the simulation.

\[
\begin{array}{ccccc}
\alpha & L & T & N & M \\
0.1 & 1 & 3600 & 50 & 5000000
\end{array}
\]

Table 2: Additional Model Parameters.

Recall that $\alpha$ refers to the smoothing parameter, $L$ to the length of the spatial intervals on which each side of the book is modelled, $T$ the time period in seconds for which we run the simulation and $N, M$ the number of space and time steps used respectively.

We will now present graphs illustrating the outcome of our simulations. In order to emphasise the strength of the fit, we present graphs of our simulations next to the corresponding graphs from the dataset.

![Figure 2: Price process of SPDR1 (a), simulated price process from fitted model (b).](image)
Figure 3: Price process of SPDR1 (a), simulated price process from fitted model (b) over the first 36 seconds of the trading period.

We note that the quadratic variation of the first path over the hour long time period is 0.421 (where the y-axis is measured in dollars), and the quadratic variation of the simulated path is 0.4790. When fitting, we attributed 0.3455 of the quadratic variation of the price from our dataset to exogenous price movements, with the remaining considered to be due to local imbalance. Based on this fitting, 0.3455 of the quadratic variation for our simulated price process can also be attributed to exogenous movements, and once again the remainder here is due to local imbalance. Therefore, price movements due to local imbalance were produced at a not unreasonable rate by the simulation, although perhaps slightly too frequently. The overall volatility of the path is a very good fit with our original data. The reasonable fit here is very much compatible with the fact that the average order book profile over the hour long period appears to match very well with that which was observed in the original data, which would mean that in particular we might expect the local imbalance for the real data and the simulated process to be of similar magnitude on average. The following figures show the average profiles of the ask side of the book for the real and simulated processes, together with snapshots of these profiles.

Figure 4: Time averaged order book profiles for the ask side of SPDR1 (a) and our simulation (b).
We note that the average profile from our simulation is certainly on the correct scale, and agrees with the average profile from the data in terms of having most mass concentrated close to the mid. The snapshots from the simulation also demonstrate this, and clearly have some realistic features. The nature of the decay away from the mid, however, is not quite as good a fit, with higher average order volumes being apparent further into the book. Another feature which doesn’t appear to have been captured by the simulation is the infrequent appearance of very high order volume at prices deep into the book as seen in Figure ?? This is perhaps not so important from a modelling perspective, however.

A Appendix

A.1 Proof of Theorem 1

We begin this section by noting that the static microscopic system decouples into two independent problems on either side of the mid. We therefore prove only the convergence for the system on the bid side of the book, with the proof for the ask side the same.

Recall that the dynamics were given by

(i) For $i \in \{1, ..., N - 1\}$, $Z^b_n \rightarrow Z^b_n + e_i$ at exponential rate

$$\frac{1}{2} \sigma^2_{b,m,n}(i, Z^b_n) \left( 1 + \mathbb{1}_{\{Z^b_n = 0\}} \right) + f_{b,m,n}(i, Z^b_n)$$

$$+ \alpha_{b,n}(Z^b_{n+1} + Z^b_{n-1} - 2Z^b_n) \mathbb{1}_{\{Z^b_{n+1} + Z^b_{n-1} - 2Z^b_n \geq 0\}}.$$

(ii) For $i \in \{1, ..., N - 1\}$, $Z^b_n \rightarrow Z^b_n - e_i$ at exponential rate

$$\frac{1}{2} \sigma^2_{b,m,n}(i, Z^b_n) \mathbb{1}_{\{Z^b_n \geq 1\}} + g_{b,m,n}(i, Z^b_n) \mathbb{1}_{\{Z^b_n \geq 1\}}$$

$$- \alpha_{b,n}(Z^b_{n+1} + Z^b_{n-1} - 2Z^b_n) \mathbb{1}_{\{Z^b_{n+1} + Z^b_{n-1} - 2Z^b_n \leq 0\}}.$$

It follows that the rescaled processes, $\tilde{Z}^b_n(t)$, have dynamics given by:
Then we have that, for $F \in L_{\infty}$ at exponential rate
\[
\frac{n}{2} \sigma_{b,m}^2(i, \hat{Z}_n^{b,i}) \left( 1 + \mathbb{1}_{\{\hat{Z}_n^{b,i} = 0\}} \right) + \sqrt{n} f_{b,m} \left( i, \hat{Z}_n^{b,i} \right) + \alpha_b (\hat{Z}_n^{b,i+1} + \hat{Z}_n^{b,i-1} - 2 \hat{Z}_n^{b,i}) \mathbb{1}_{\{\hat{Z}_n^{b,i+1} + \hat{Z}_n^{b,i-1} - 2 \hat{Z}_n^{b,i} \geq 0\}}.
\]

(ii) $\hat{Z}_n \to \tilde{Z}_n - \frac{e_i}{\sqrt{n}}$ at exponential rate
\[
\frac{n}{2} \sigma_{b,m}^2(i, \hat{Z}_n^{b,i}) \left( 1 + \mathbb{1}_{\{\hat{Z}_n^{b,i} = 0\}} \right) + \sqrt{n} g_{b,m} \left( i, \hat{Z}_n^{b,i} \right) \mathbb{1}_{\{\hat{Z}_n^{b,i} \geq \frac{1}{\sqrt{n}}\}} - \alpha_b (\hat{Z}_n^{b,i+1} + \hat{Z}_n^{b,i-1} - 2 \hat{Z}_n^{b,i}) \mathbb{1}_{\{\hat{Z}_n^{b,i+1} + \hat{Z}_n^{b,i-1} - 2 \hat{Z}_n^{b,i} \leq 0\}}.
\]

In order to prove convergence of these processes, we will argue that their infinitesimnal generators converge (see Ethier and Kurtz [13]). We start by calculating the infinitesimal generators for the $\tilde{Z}_n$. Define:

\[
\Delta^{0}_{n,k} F(y) := \sqrt{n} \left[ F(y) - F \left( y - \frac{e_k}{\sqrt{n}} \right) \right],
\]

\[
\Delta^{1}_{n,k} F(y) := \sqrt{n} \left[ F \left( y + \frac{e_k}{\sqrt{n}} \right) - F(y) \right],
\]

\[
\Delta^{2}_{n,k} := n \left[ F \left( y + \frac{e_k}{\sqrt{n}} \right) + F \left( y - \frac{e_k}{\sqrt{n}} \right) - 2F(y) \right].
\]

Then we have that, for $F \in C_0([0, \infty)^{N-1})$, the continuous functions on $[0, \infty)^{N-1}$ which vanish at $\infty$,

\[
\frac{1}{t} \mathbb{E} \left[ F(\hat{Z}_n(t)) | \hat{Z}_n(0) = y \right] = \sum_{k=1}^{N-1} \Delta^{r}_{n,k} F(y) \left( \frac{\sqrt{n}}{2} \sigma_{b,m}^2(k, y_k) \left( 1 + \mathbb{1}_{\{y_k = 0\}} \right) \right) + \sum_{k=1}^{N-1} \Delta^{r}_{n,k} F(y) f_{b,m} (k, y_k)
\]

\[
+ \sum_{k=1}^{N-1} \Delta^{r}_{n,k} F(y) \alpha_b (y_{k+1} + y_{k-1} - 2y_k) \mathbb{1}_{\{y_{k+1} + y_{k-1} - 2y_k \geq 0\}} + \sum_{k=1}^{N-1} \Delta^{l}_{n,k} F(y) \left( -\frac{\sqrt{n}}{2} \sigma_{b,m}^2(k, y_k) \mathbb{1}_{\{y_k \geq \frac{1}{\sqrt{n}}\}} \right)
\]

\[
+ \sum_{k=1}^{N-1} \Delta^{l}_{n,k} F(y) \left( -g_{b,m} (k, y_k) \mathbb{1}_{\{y_k \geq \frac{1}{\sqrt{n}}\}} \right) + \sum_{k=1}^{N-1} \Delta^{l}_{n,k} F(y) \alpha_b (y_{k+1} + y_{k-1} - 2y_k) \mathbb{1}_{\{y_{k+1} + y_{k-1} - 2y_k \leq 0\}} + R_{y,t},
\]
Recall that our candidate limiting process, $X^b$, satisfies the system of reflected SDEs

$$dX^b_t = \alpha_b(X^b_t) + \gamma_b(X^b_t)dt + \sigma_b(X^b_t)dW^b_t, \quad t \geq 0,$$

where $R_{y,t} \to 0$ as $t \to 0$ uniformly in compact sets for $y$. Note that the local boundedness of $\sigma$, $f$ and $g$ ensures that the jump rate of the process after the first jump from $y$ can be bounded, which is essential in the above calculation. If we further assumed that $F$ were smooth with compact support, so $F \in C_c^\infty([0, \infty)^{N-1})$, we obtain that the remainder term $R_{y,t}$ converges uniformly to zero for $y \in (\frac{1}{\sqrt{n}}\mathbb{Z}^N)^{N-1}$. Therefore, writing $A_n$ for the generator of $\tilde{Z}_n$ and rearranging gives that for $F \in C_c^\infty([0, \infty)^{N-1})$,

$$A_nF(y) = \sum_{k=1}^{N-1} \frac{1}{2} \Delta^2_{n,k}F(y) \mathbb{1}_{\{y_k \geq \frac{1}{\sqrt{n}}\}} \sigma^2_{b,m}(k, y_k) + \sum_{k=1}^{N-1} \Delta^r_{n,k}F(y) \sqrt{n} \sigma^2_{b,m}(k, y_k) \mathbb{1}_{\{y_k = 0\}}$$

$$+ \sum_{k=1}^{N-1} \Delta^l_{n,k}F(y) \left[ f_{b,m}(k, y_k) + \alpha_b y_k + y_k - 2y_k \mathbb{1}_{\{y_k+1+y_k-2y_k \geq 0\}} \right]$$

$$+ \sum_{k=1}^{N-1} \Delta^l_{n,k}F(y) \left[ -g_{b,m}(k, y_k) \mathbb{1}_{\{y_k \geq \frac{1}{\sqrt{n}}\}} + \alpha_b(y_k+1+y_k-2y_k) \mathbb{1}_{\{y_k+1+y_k-2y_k \leq 0\}} \right].$$

Recall that our candidate limiting process, $X^b$, satisfies the system of reflected SDEs

$$dX^b_t = \alpha_b(X^b_t) + \gamma_b(X^b_t)dt + \sigma_b(X^b_t)dW^b_t, \quad t \geq 0,$$

for $i = 1, 2, ... N-1$, with $X^0 = X^N = 0$ and $h_{b,m} := f_{b,m} - g_{b,m}$. Note that by Theorem 4.1 in [26], we have existence of a strong solution and pathwise uniqueness for this system of SDEs. Inspection of the proof also reveals that the solution here, where the diffusion and drift coefficients do not have any explicit time-dependence, is continuous. We can calculate the corresponding generator to be

$$AF(x) = \frac{1}{2} \sum_{k=1}^{N-1} \sigma_{b,m}(k, x_k) \frac{\partial^2 F}{\partial x_k^2} + \sum_{k=1}^{N-1} \left[ h_{b,m}(k, x_k) + \alpha_b(x_{k+1} + x_{k-1} - 2x_k) \right] \frac{\partial F}{\partial x_k},$$

acting on the domain

$$\mathcal{D}(A) = \left\{ F \in C^2_0([0, \infty)^{N-1}) \text{ s.t. } \forall k, \quad \frac{\partial F}{\partial x_k} \bigg|_{x_k = 0} = 0 \right\}.$$ 

This has a core given by

$$C(A) = \left\{ F \in C^\infty_c([0, \infty)^{N-1}) \text{ s.t. } \forall k, \quad \frac{\partial F}{\partial x_k} \bigg|_{x_k = 0} = 0 \right\}.$$ 

It is enough to prove (see [13]) that for $F \in C(A)$

$$\sup_{y \in \left(\frac{1}{\sqrt{n}}\mathbb{Z}^N\right)^{N-1}} |A_nF(y) - AF(y)| \to 0.$$

First suppose that $y_k \geq \frac{1}{\sqrt{n}}$. Then, by Taylor’s Theorem we have that

$$\frac{1}{2} \sigma_{b,m}(k, y_k) \left| \Delta^2_{n,k}F(y) - \frac{\partial^2 F}{\partial y_k^2} \right| \leq \frac{1}{6\sqrt{n}} \left\| \frac{\partial^3 F}{\partial y_k^3} \right\|_\infty \|\sigma_{b,m}\|_{F,\infty},$$
where \(\|\sigma_{b,m}\|_{F,\infty}\) is the supremum of \(\sigma_{b,m}\) over the support of \(F\). Similarly
\[
\left| \Delta_{n,k}^r F(y) - \frac{\partial F}{\partial y_k}(y) \right| f_{b,m}(k, y_k) \leq \frac{1}{2\sqrt{n}} \left\| \frac{\partial^2 F}{\partial y_k^2} \right\|_{\infty} \|f_{b,m}\|_{F,\infty},
\]
and
\[
\left| \Delta_{n,k}^l F(y) - \frac{\partial F}{\partial y_k}(y) \right| g_{b,m}(k, y_k) \leq \frac{1}{2\sqrt{n}} \left\| \frac{\partial^2 F}{\partial y_k^2} \right\|_{\infty} \|g_{b,m}\|_{F,\infty}.
\]

Now suppose that \(y_k = 0\). Then, again by Taylor’s theorem, (extending \(F\) by reflection about the axes and using that the first derivatives are zero there) we have that
\[
\sigma_{b,m}^2(k, y_k) \left| \sqrt{n} \Delta_{n,k}^r F(y) - \frac{1}{2} \frac{\partial^2 F}{\partial y_k^2}(y) \right| \leq \frac{1}{6\sqrt{n}} \left\| \frac{\partial^2 F}{\partial y_k^2} \right\|_{\infty} \|\sigma_{b,m}^2\|_{F,\infty},
\]
and
\[
\left| \Delta_{n,k}^r F(y) - \frac{\partial F}{\partial y_k}(y) \right| f_{b,m}(k, y_k) = \left| \Delta_{n,k}^r F(y) \right| f_{b,m}(k, y_k) \leq \frac{1}{2\sqrt{n}} \left\| \frac{\partial^2 F}{\partial y_k^2} \right\|_{\infty} \|f_{b,m}\|_{F,\infty}.
\]
Putting these cases together and letting \(n \to \infty\) gives that \(\text{(A.1)}\) holds.

**Remark A.1.** Since the limiting process \(X^b\) is continuous, convergence in law in \(\mathbb{D}([0, \infty); \mathbb{R}^{N-1})\) implies convergence in law in \(\mathbb{D}([0, \tilde{T}]; \mathbb{R}^{N-1})\) for any \(\tilde{T} > 0\).

### A.2 Proof of Theorem 2

We use Theorem 1 as a basis for the proof of Theorem 2. We start by proving some continuity type results for certain maps. These will be used in an inductive argument which will allow us to prove that
\[
\left( \left( Z_{n,i}^b(nt)/\sqrt{n} \right)_{i=1}^{\infty}, \left( Z_{n,i}^a(nt)/\sqrt{n} \right)_{i=1}^{\infty}, (\tau^i/n)_{i=1}^{\infty}, (m^i)_{i=1}^{\infty} \right)
\]

in law in \(\mathbb{D}([0, T + 1], \mathbb{R}^{N-1})_{\mathbb{N}} \times \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})_{\mathbb{N}} \times [0, T + 1]_{\mathbb{N}} \times \mathbb{R}^N\). For a metric space \(M\), we use the topology of pointwise convergence for \(M^N\), which is itself metrizable. Finally, we conclude by an application of the Skorohod representation theorem.

**Proposition A.2.** Fix some \(m \in \mathbb{R}\). Let \(w_n : [0, \infty) \to \mathbb{Z}^{N-1} \times \mathbb{Z}^{N-1}\) and \(w : [0, T + 1] \to \mathbb{R}^{N-1} \times \mathbb{R}^{N-1}\) be such that
\[
\sup_{t \in [0, T + 1]} \left| \frac{1}{\sqrt{n}} (w^1_n(nt), w^2_n(nt)) - (w^1(t), w^2(t)) \right| \to 0.
\]

Then we have that
\[
\sup_{t \in [0, T + 1]} \left| n\theta_{b,m}^n(w_n(nt)) - \theta_{b,m}(w(t)) \right| \to 0,
\]
and
\[
\sup_{t \in [0, T + 1]} \left| n\theta_{a,m}^n(w_n(nt)) - \theta_{a,m}(w(t)) \right| \to 0.
\]
Proof. This is a direct consequence of assumption (i) in Section 2.4. \qed

Proposition A.3. Suppose \((Z_n^1, Z_n^2)\) is a sequence in \(\mathcal{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})\) such that \((Z_n^1, Z_n^2) \Rightarrow (X^1, X^2)\) in \(\mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\). Define the maps \(\tilde{R}_n : \mathcal{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1}) \times \{u, d\} \to \mathcal{M}(\mathbb{N}^{N-1} \times \mathbb{N}^{N-1})\) and \(\tilde{R} : \mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}) \times \{u, d\} \to \mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\) such that

(i) \(\tilde{R}_n(x_1, x_2, k)(A) := \mathbb{E} [R_n(x_1, x_2, k)(A)]\)

(ii) \(\tilde{R}(y_1, y_2, k)(B) := \mathbb{E} [R(y_1, y_2, k)(B)].\)

Then, for \(k \in \{u, d\}\),

\[ \tilde{R}_n(Z_n^1, Z_n^2, k) \circ P_n^{-1} \Rightarrow \tilde{R}(X^1, X^2, k) \]

in \(\mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\).

Proof. We Skorohod represent the convergence of \((Z_n^1, Z_n^2)\), so we assume that \((Z_n^1, Z_n^2) \to (X^1, X^2)\) almost surely. We then have that

\[ R_n(Z_n^1, Z_n^2, k) \circ P_n^{-1} \to R(X^1, X^2, k) \]

in \(\mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\) almost surely. Therefore,

\[ R_n(Z_n^1, Z_n^2, k) \circ P_n^{-1} \Rightarrow R(X^1, X^2, k) \]

as \(\mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\)-valued random variables. Fix some open set \(A \in \mathcal{B}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1})\). Let \(F_A : \mathcal{M}((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}) \to [0, 1]\) such that

\[ F_A(\mu) := \mu(A). \]

Then \(F_A\) is a bounded continuous function. So we have that

\[ \mathbb{E} \left[ F_A(R_n(Z_n^1, Z_n^2, k) \circ P_n^{-1}) \right] \to \mathbb{E} \left[ F_A(R(X^1, X^2, k)) \right]. \]

But this is just the statement that

\[ \left( \tilde{R}_n(Z_n^1, Z_n^2, k) \circ P_n^{-1} \right)(A) \to \tilde{R}(X^1, X^2, k)(A). \]

So we have the result. \qed

Proposition A.4. Suppose that \(f_n : [0, T + 1] \to \mathbb{R}\) and \(f : [0, T + 1] \to \mathbb{R}_{>0}\) are such that \(f\) is continuous and

\[ \sup_{t \in [0, T + 1]} |f_n(t) - f(t)| \to 0. \]

Define

\[ \tau^N := \inf \left\{ t \geq 0 \mid x_n \leq \int_0^t f_n(s) \, ds \right\} \wedge (T + 1) \]

and

\[ \tau := \inf \left\{ t \geq 0 \mid x \leq \int_0^t f(s) \, ds \right\} \wedge (T + 1). \]

Then \(\tau^N \to \tau\).
Proof. Suppose that $\tau \leq t$ with $t \in [0, T + 1]$. Fix $\delta > 0$.

$$\left| \left( x_n - \int_0^t f_n(s)ds \right) - \left( x - \int_0^t f(s)ds \right) \right| \leq |x_n - x| + (T + 1) \sup_{t\in[0,T+1]} |f_n(t) - f(t)|.$$  

Let $r := \inf_{t\in[0,T+1]} f(t) > 0$. Then for $n$ large enough, we have that $r_n := \inf_{t\in[0,T+1]} f_n(t) > \frac{r}{2}$. For $n$ large enough,

$$|x_n - x| + (T + 1) \sup_{t\in[0,T+1]} |f_n(t) - f(t)| < \frac{\delta r}{2},$$

Therefore,

$$x_n - \int_0^t f_n(s)ds < x - \int_0^t f(s)ds + \frac{\delta r}{2} = \frac{\delta r}{2}$$

for large enough $n$. So

$$x_n - \int_0^{t+\delta} f_n(s)ds \leq x_n - \int_0^t f_n(s)ds - \frac{\delta r}{2} < 0.$$ 

Therefore $\tau_n \leq \tau + \delta$ for large enough $n$. Similarly, $\tau_n \geq \tau - \delta$ for large enough $n$. So $\tau_n \to \tau$. \qed

**Theorem A.5.** Suppose that

$$\left( \frac{Z_{n,1}^b(0)}{\sqrt{n}}, \frac{Z_{n,1}^a(0)}{\sqrt{n}} \right) \Rightarrow (X^b(0), X^a(0))$$

in law in $(\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1} \times \mathbb{R}$. Let $(\tilde{Z}_n^b(t), \tilde{Z}_n^a(t), m_n(t))$ be dynamic microscopic models with initial data $(Z_{n,1}^b(0), Z_{n,1}^a(0), m_1^1)$, and let $(\tilde{X}^b(t), \tilde{X}^a(t), m(t))$ be the dynamic mesoscopic model with initial data $(X_{1,1}^b(0), X_{1,1}^a(0), m_1^1)$. Then

$$\left( \left( \frac{Z_{n,i}^b(nt)/\sqrt{n}}{\sqrt{n}} \right)_{i=1}^\infty, \left( \frac{Z_{n,i}^a(nt)/\sqrt{n}}{\sqrt{n}} \right)_{i=1}^\infty, (\tau_i/n)_{i=1}^\infty, (m_i/n)_{i=1}^\infty \right)$$

$$\Rightarrow \left( (X_{1,i}^b)_{i=1}^\infty, (X_{1,i}^a)_{i=1}^\infty, (\tau_i)_{i=1}^\infty, (m_i)_{i=1}^\infty \right)$$

in law in $\mathbb{D}([0,T + 1], \mathbb{R}^{N-1})^N \times \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^N \times [0, T + 1]^N \times \mathbb{R}^N$.

**Proof.** We begin the proof by introducing the following notation.

(i) $\mathcal{Z}_n^{b,M}(t) := (Z_{n,i}^b(nt)/\sqrt{n})_{i=1}^M$, $\mathcal{Z}_n^{b,M}(t) := (X^b_i(t))_{i=1}^M$.

(ii) $\mathcal{Z}_n^{a,M}(t) := (Z_{n,i}^a(nt)/\sqrt{n})_{i=1}^M$, $\mathcal{Z}_n^{a,M}(t) := (X^a_i(t))_{i=1}^M$.

(iii) $\tilde{\tau}_{n,u} := (\tau_{n,u}/n)_{i=1}^M$, $\tilde{\tau}^M := (\tau_i)_{i=1}^M$.

(iv) $\tilde{\tau}_{n,d} := (\tau_{n,d}/n)_{i=1}^M$, $\tilde{\tau}^M := (\tau_i)_{i=1}^M$.

(v) $\tilde{m}_n^M := (m_{n,i})_{i=1}^M$, $\tilde{m}^M := (m_i)_{i=1}^M$. 

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We will prove by induction that for every \( M \geq 1 \),
\[
(\mathcal{U}_n^{b,M}, \mathcal{U}_n^{a,M}, \mathcal{Z}_n^{M-1}, \mathcal{Z}_n^{M-1}, \mathcal{m}_n^{M}) \implies (\mathcal{U}_n^{b,M}, \mathcal{U}_n^{a,M}, \mathcal{Z}_n^{M-1}, \mathcal{Z}_n^{M-1}, \mathcal{m}_n^{M})
\]  
(A.1)
in law in \( \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^{M} \times \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^{M} \times [0, T + 1]^{M-1} \times [0, T + 1]^{M-1} \times \mathbb{R}^{M} \), from which the result follows. Note that by applying Theorem \([2.3]\) we have
\[
(\mathcal{U}_n^{b,1}, \mathcal{U}_n^{a,1}, \mathcal{m}_n^{1}) \implies (\mathcal{U}_n^{b,1}, \mathcal{U}_n^{a,1}, \mathcal{m}_n^{1})
\]
in \( \mathbb{D}([0, T + 1], \mathbb{R}^{N-1}) \times \mathbb{D}([0, T + 1], \mathbb{R}^{N-1}) \times \mathbb{R} \). Suppose we had for some \( M \geq 1 \) that (A.1) holds. Then we have that
\[
(\mathcal{U}_n^{b,M}, \mathcal{U}_n^{a,M}, \mathcal{Z}_n^{M-1}, \mathcal{Z}_n^{M-1}, \mathcal{m}_n^{M}, Y_n^{M}, Y_n^{M}) \implies (\mathcal{U}_n^{b,M}, \mathcal{U}_n^{a,M}, \mathcal{Z}_n^{M-1}, \mathcal{Z}_n^{M-1}, \mathcal{m}_n^{M}, Y_n^{M}, Y_n^{M}).
\]
By the Skorohod convergence theorem, we can assume that this convergence holds almost surely.
Recall that
\[
\tau_{n,a}^M := \inf \left\{ t \geq 0 \mid \int_0^t \theta_{n,a}^n (Z_{n,a}^b, Z_{n,a}^a) \, ds \geq Y_{n,a}^M \right\},
\]
\[
\tau_{n,b}^M := \inf \left\{ t \geq 0 \mid \int_0^t \theta_{n,b}^n (Z_{n,b}^b, Z_{n,b}^a) \, ds \geq Y_{n,b}^M \right\}.
\]
By a change of variables,
\[
\int_0^t \theta_{n,a}^n (Z_{n,a}^b, Z_{n,a}^a) \, ds = \int_0^{t/n} \frac{n}{n} \theta_{n,a}^n (Z_{n,a}^b, Z_{n,a}^a) \, dr.
\]
Therefore,
\[
\frac{1}{n} \tau_{n,a}^M = \frac{1}{n} \inf \left\{ t \geq 0 \mid \int_0^{t/n} \frac{n}{n} \theta_{n,a}^n (Z_{n,a}^b, Z_{n,a}^a) \, dr \geq Y_{n,b}^M \right\}.
\]
We have that \( m_n^M = m^M \) for large enough \( n \) almost surely. Note that since the limits \( X_n^b, X_n^a \) are continuous, convergence of \( Z_{n,a}^b(nt)/\sqrt{n} \) to \( X_M^b(t) \) and \( Z_{n,a}^a(nt)/\sqrt{n} \) to \( X_M^a(t) \) in the Skorohod topology implies that
\[
\sup_{t \in [0, T + 1]} \left| \frac{1}{\sqrt{n}} (Z_{n,a}^b(nt), Z_{n,a}^a(nt) - (X_M^b(t), X_M^a(t)) \right| \to 0.
\]
Since \( \theta_{n,a} \) was assumed to be continuous and \( X_n^b, X_n^a \) are continuous, \( \theta_{n,a}(X_n^b(t), X_n^a(t)) \) is continuous almost surely. We can therefore apply Propositions \([2.2]\) and \([2.4]\) to deduce that
\[
\frac{1}{n} \tau_{n,b}^M \to \inf \left\{ t \geq 0 \mid \int_0^t \theta_{n,a} (X_M^b(s), X_M^a(s)) \, ds \geq Y_b^M \right\} = \tau_b^M
\]
almost surely. Similarly, \( \frac{1}{n} \tau_{n,a}^M \to \tau_a^M \) almost surely. Note that \( \tau_b^M \neq \tau_a^M \) almost surely. This implies that \( m_n^{M+1} \to m^{M+1} \) almost surely. Therefore, we have deduced that
\((\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \to (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M)\). \quad (A.2)

For \(i = 1, \ldots, M + 1\), let \(A_i\) and \(B_i\) be open subsets of \(\mathbb{D}([0, T]; \mathbb{R}^{N-1})\) and \(p_i \in (m^3 + \epsilon \mathbb{R})\). For \(i = 1, \ldots, M\), let \(C_i, D_i\) be open subsets of \([0, T + 1]\). For \(K \in \mathbb{N}\), define

\[
\tilde{A}_K := \prod_{i=1}^K A_i, \quad \tilde{B}_K := \prod_{i=1}^K B_i, \quad \tilde{C}_K := \prod_{i=1}^K C_i, \quad \tilde{D}_K := \prod_{i=1}^K D_i, \quad P_K := \prod_{i=1}^K \{p_i\}.
\]

Suppose that

\[
P \left[ (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right] > 0.
\]

Then we have that, for large enough \(n\)

\[
P \left[ (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right]
= \mathbb{P} \left[ (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right]
\times \mathbb{P}_1 \left[ \frac{1}{\sqrt{n}} (Z_{n,M+1}(nt), Z_{n,M+1}^a(nt)) \in A_{M+1} \times B_{M+1} \right],
\]

where \(\mathbb{P}_1\) here denotes the conditional probability law given the event \((\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\). By (A.2), we know that

\[
P \left[ (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right]
\rightarrow P \left[ (\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right].
\]

Let \(Q_n\) be the law on \((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}\) induced by \(R^n(Z_{n,M}^b(\tau_n^M), Z_{n,M}^a(\tau_n^M), k) \circ P_{-n}\) and the probability measure \(\mathbb{P}_1\), where \(k\) is the direction of the last price change. That is

\[
Q_n(A) = \mathbb{E}^{\mathbb{P}_1} \left[ (R^n(Z_{n,M}^b(\tau_n^M), Z_{n,M}^a(\tau_n^M), k) \circ P_{-n})(A) \right].
\]

Similarly, we define the measure \(Q\) on \((\mathbb{R}^+)^{N-1} \times (\mathbb{R}^+)^{N-1}\) by setting

\[
Q(A) = \mathbb{E}^{\mathbb{P}_2} \left[ (R^b(X_M^b(\tau_M^b), X_M^a(\tau_M^a), k))(A) \right],
\]

where \(\mathbb{P}_2\) here is the conditional probability law given the event \((\mathcal{N}_b^M, \mathcal{N}_a^M, \tilde{\mathcal{N}}_b^M, \tilde{\mathcal{N}}_a^M, \tilde{\mathcal{N}}_n^M) \in \tilde{A}_M \times \tilde{B}_M \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1}\). By (A.2) and an application of Proposition A.3, we see that \(Q_n \Rightarrow Q\). We can now apply our result for convergence in a static setting, Theorem 2.3, to deduce that

\[
\mathbb{P}_1 \left[ \frac{1}{\sqrt{n}} (Z_{n,M+1}^b(nt), Z_{n,M+1}^a(nt)) \in A_{M+1} \times B_{M+1} \right] \rightarrow \mathbb{P}_2 \left[ (X_{M+1}^b, X_{M+1}^a) \in A_{M+1} \times B_{M+1} \right].
\]

We have therefore deduced that

\[
P \left[ (\mathcal{N}_{b,M+1}^b, \mathcal{N}_{a,M+1}^a, \tilde{\mathcal{N}}_{b,M+1}^b, \tilde{\mathcal{N}}_{a,M+1}^b, \tilde{\mathcal{N}}_{n,M+1}^n) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right]
\rightarrow \mathbb{P} \left[ (\mathcal{N}_{b,M+1}^b, \mathcal{N}_{a,M+1}^a, \tilde{\mathcal{N}}_{b,M+1}^b, \tilde{\mathcal{N}}_{a,M+1}^b, \tilde{\mathcal{N}}_{n,M+1}^n) \in \tilde{A}_{M+1} \times \tilde{B}_{M+1} \times \tilde{C}_M \times \tilde{D}_M \times P_{M+1} \right].
\]

So we have that \((\mathcal{N}_{b,M+1}^b, \mathcal{N}_{a,M+1}^a, \tilde{\mathcal{N}}_{b,M+1}^b, \tilde{\mathcal{N}}_{a,M+1}^b, \tilde{\mathcal{N}}_{n,M+1}^n) \Rightarrow (\mathcal{N}_{b,M+1}^b, \mathcal{N}_{a,M+1}^a, \tilde{\mathcal{N}}_{b,M+1}^b, \tilde{\mathcal{N}}_{a,M+1}^b, \tilde{\mathcal{N}}_{n,M+1}^n)\) which concludes our inductive argument.

\(\square\)
Proposition A.6. Let \( g : \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^N \times [0, T + 1]^N \to \mathbb{D}([0, T], \mathbb{R}^{N-1}) \) such that

\[
g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)(t) := \sum_{j=1}^\infty u_j \left( t - \sum_{i=1}^{j-1} t_i \right) \mathbb{1}_{\left\{ \sum_{i=1}^{j-1} t_i \leq t < \sum_{i=1}^j t_i \right\}}.
\]

Suppose that

(i) \( t_i > 0 \) for every \( i \).

(ii) \( \sum_{i=1}^K t_i \neq T + 1 \) for every \( K \).

(iii) \( \exists K \geq 1 \) such that \( \sum_{i=1}^K t_i > T + 1 \).

(iv) \( u_i \) is continuous for every \( i \).

Then \( g \) is continuous at the point \( ((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty) \).

Proof. Suppose that the sequence \( ((u_i^n)_{i=1}^\infty, (t_i^n)_{i=1}^\infty) \) converges to \( ((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty) \) in \( \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^N \times [0, T + 1]^N \). Since the functions \( u_i \) are continuous, we have that

\[
\sup_{t \in [0, T + 1]} |u_i^n(t) - u_i(t)| \to 0
\]

for each \( i \). Recall that \( f_n \to f \) in \( \mathcal{D}_{[0,T]}(M) \) for a metric space \( M \) iff \( \exists \lambda_n : [0, T] \to [0, T] \), strictly increasing continuous bijections such that

(a) \( \sup_{t \in [0,T]} |\lambda_n(t) - t| \to 0 \) and

(b) \( \sup_{t \in [0,T]} |f_n(\lambda_n(t)) - f(t)| \to 0 \).

For our case, we define the continuous bijections \( \lambda_n : [0, T] \to [0, T] \) such that for \( m \geq 1 \) we set

\[
\lambda_n \left( \sum_{i=1}^m t_i \right) := \sum_{i=1}^m t_i^n,
\]

and linearly interpolate between these points, together with the endpoints \( \lambda_n(0) = 0 \) and \( \lambda_n(T) = T \) (this may define the function for values greater than \( T \) as well but we only care about the restriction). Then, for large enough \( n \) so that the \( t_i^n \) are all strictly positive, \( \lambda_n \) is a strictly increasing bijection from \( [0, T] \) to \( [0, T] \), and the \( \lambda_n \) satisfy (a) above. Note that this can only be done if we have the conditions stated for the \( t_i \). We have that

\[
g((u_i^n)_{i=1}^\infty, (t_i^n)_{i=1}^\infty)(\lambda_n(t)) = \sum_{j=1}^\infty u_j^n \left( \lambda_n(t) - \sum_{i=1}^{j-1} t_i^n \right) \mathbb{1}_{\left\{ \sum_{i=0}^{j-1} t_i^n \leq \lambda_n(t) < \sum_{i=1}^j t_i^n \right\}}
\]

\[
= \sum_{j=1}^\infty u_j^{n-1} \left( \lambda_n(t) - \sum_{i=0}^{j-1} t_i^n \right) \mathbb{1}_{\left\{ \sum_{i=0}^{j-1} t_i \leq t < \sum_{i=1}^j t_i \right\}}.
\]

Then, by uniform convergence of the \( u_i^n \) to \( u_i \) and by construction of the \( \lambda_n \), noting the uniform continuity of the \( u_i \), we have that this converges uniformly for \( t \in [0, T] \) to \( g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty) \). So (b) holds and we have the result. \( \square \)
We are now in position to conclude the proof of Theorem 2.4.

Proposition A.7. Let \( h : [0, T + 1]^N \times \mathbb{R} \to \mathbb{D}([0, T + 1], \mathbb{R}) \) such that

\[
h((t_i)_{i=1}^\infty, (m_i)_{i=1}^\infty) := \sum_{i=1}^\infty m_i \mathbb{1}_{ \left\{ \sum_{j=1}^{i-1} t_j \leq t < \sum_{j=1}^i t_j \right\} }.
\]

Suppose that

(i) \( t_i > 0 \) for every \( i \).

(ii) \( \sum_{i=1}^K t_i \neq T + 1 \) for every \( K \).

(iii) \( \exists K \geq 1 \) such that \( \sum_{i=1}^K t_i > T + 1 \).

Then \( h \) is continuous at the point \((t_i)_{i=1}^\infty, (m_i)_{i=1}^\infty\).

Proof. This is essentially the same as the proof of Proposition A.6. \( \square \)

We are now in a position to conclude the proof of Theorem 2.4. We begin by Skorohod representing the convergence in Theorem A.5. So we have that

\[
\left( Z^b_{n,i}(nt)/\sqrt{n} \right)_{i=1}^\infty, \left( Z^a_{n,i}(nt)/\sqrt{n} \right)_{i=1}^\infty, (\tau^i_n/n)_{i=1}^\infty, (m^i_n)_{i=1}^\infty \to \left( (X^b)_{i=1}^\infty, (X^a)_{i=1}^\infty, (\tau^i)_{i=1}^\infty, (m^i)_{i=1}^\infty \right)
\]

in \( \mathbb{D}([0, T + 1], \mathbb{R}^{N-1})^N \times \mathbb{D}([0, T + 1], \mathbb{R}^N) \times [0, T + 1]^N \times \mathbb{R}^N \) almost surely, where the vectors represent the key objects for the dynamic microscopic and mesoscopic models respectively. By Proposition 4, we clearly have that \((X^b)_{i=1}^\infty, (\tau^i)_{i=1}^\infty\) and \((X^a)_{i=1}^\infty, (\tau^i)_{i=1}^\infty\) are continuity points of the map \( g \) almost surely. Similarly, \((\tau^i)_{i=1}^\infty, (m^i)_{i=1}^\infty\) is a continuity point for the map \( h \) almost surely. The result follows.

A.3 Proof of Theorem 3

As in the case of the proof of Theorem 1 we notice that the static mesoscopic systems decouple into two independent problems on either side of the mid. We therefore focus once more on proving the convergence on one side of the mid and wlog choose to prove convergence of the bid side. The key result which will allow us to do this is Theorem 2.1 in T.Zhang [28]. We remark here that we assume only that the initial data of our spatial discretisations converge almost surely rather than being directly sampled from the initial data of the limiting SPDE, as in the original statement in [28]. This version can easily be obtained by making minor changes to the original proof in [28].

Theorem A.8. Let \((u, \eta)\) be a solution of the reflected stochastic heat equation (3.1) with respect to a given white noise \( W \), with initial data \( u_0 \in \mathcal{M}(C_0((0,1))^+) \). Suppose that \( \sigma \) and \( h \) are Lipschitz in both variables and have linear growth in the second variable. For \( n \geq 1 \), let \((W^{n,k})_{k=1}^{n-1}\) be the independent Brownian motion given by

\[
W^{n,k} := \sqrt{n} \left[ W \left( t, \frac{k + 1}{n} \right) - W \left( t, \frac{k}{n} \right) \right],
\]
and let $u^n$ be the solution of the system of reflected SDEs

$$du_t^{n,k} = \alpha n^2 \left( u_t^{n,k+1} + u_t^{n,k-1} - 2u_t^{n,k} \right) dt + h \left( k/n, u_t^{n,k} \right) dt + \sqrt{n} \sigma \left( k/n, u_t^{n,k} \right) dW_t^{n,k} + d\eta_t^{n,k},$$

for $k = 1, \ldots, n-1$ with $u^{n,0} = u^{n,n} = 0$ and initial data $u_0^n \in \mathcal{M}(\mathbb{R}^+)$. For each $t \geq 0$, define the function $u^n(t, x)$ by setting $u^n(t, k/n) := u_t^{n,k}$ and linearly interpolating between these points. Suppose that

$$\sup_{x \in [0,1]} |u^n(0, x) - u_0(x)| \to 0$$

almost surely. Then for every $p \geq 1$ and every $t \in [0, T]$

$$\lim_{n \to \infty} \mathbb{E} \left[ \sup_{t \in [0, T]} \sup_{x \in [0,1]} |u^n(t, x) - u(t, x)|^p \right] = 0.$$

We are now able to prove Theorem 3.3 by a direct application of the above. Recall that the dynamics for the bid side of the $N$th static mesoscopic model were given by

$$dX^i_N(t) = \alpha_b (X^i_{N+1}(t) + X^i_{N-1}(t) - 2X^i_N(t)) dt + h_{b,m}(i, X^i_N(t)) dt + \sigma_{b,m}(i, X^i_N(t)) dW^i_t + d\eta^i_t$$

for $i = 1, \ldots, N-1$ with $X^0 = X^N = 0$. It follows that

$$dv_t^{N,k} = \alpha_b N^2 \left( v_t^{N,k+1} + v_t^{N,k-1} - 2v_t^{N,k} \right) dt + h_{b,m} \left( k/N, v_t^{N,k} \right) dt + \sqrt{N} \sigma_{b,m} \left( k/N, v_t^{N,k} \right) dW^i_t + d\eta^i_t,$$

where we define $v_t^{N,k} := X^k_N(N^2 t)$. Therefore, in order to apply Theorem A.8 to obtain convergence in law, it is only left to realise our SDE systems (A.3) on a common probability space to a solution of the limiting reflected SPDE, with the Brownian motions corresponding to the same white noise as in Theorem A.8. By Theorem 4.1 in [26], we have existence of a strong solution and pathwise uniqueness for the system of SDEs (A.3). Existence of strong solutions of the reflected stochastic heat equation, under our conditions on the coefficients, is proved by C. Donati-Martin and E. Pardoux in [12]. By strong here, we mean that a solution can be constructed with respect to any given white noise. Uniqueness was then proved by T. Xu and T. Zhang in [27]. Therefore we are able to place our processes in the correct framework to apply Theorem A.8. This concludes the proof of Theorem 3.

A.4 Proof of Theorem 4

The arguments here reflect those made when upgrading the proof of Theorem 2.3 to a proof of Theorem 2.4. We therefore refer to the proof of Theorem 2.4 to illustrate how to prove the key results in this section. Similarly to before, the main part of the proof is showing that

$$((Q_N((X^b_{N,i}(N^2 t, \cdot)))_{i=1}^{\infty}, (Q_N((X^0_{N,i}(N^2 t, \cdot))_{i=1}^{\infty}, (\tau^i_{N,u}/N^2)_{i=1}^{\infty}, (\tau^i_{N,d}/N^2)_{i=1}^{\infty}, (m^i_N)_{i=1}^{\infty})$$

$$\implies ((u^b_t(t, \cdot))_{i=1}^{\infty}, (u^0_t(t, \cdot))_{i=1}^{\infty}, (\tau^i_{u}(t))_{i=1}^{\infty}, (\tau^i_{d}(t))_{i=1}^{\infty}, (m^i)_{i=1}^{\infty})$$

in law in $\mathcal{D}([0, T + 1]; C_0([0, 1])) \times \mathbb{D}([0, T + 1]; C_0([0, 1])) \times [0, T + 1]^N \times \mathbb{R}^N$. Once again, for a metric space $M$ we equip $M^N$ with the topology of pointwise convergence, which is then also metrizable.
Proposition A.9. Fix some \( m \in \mathbb{R} \). Let \( w_N : [0, T + 1] \to \mathbb{R}^{N-1} \) and \( w : [0, T + 1] \to C_0((0, 1)) \) be such that
\[
\sup_{t \in [0,T+1]} |Q_N(w_N(t)) - w(t)| \to 0.
\]
Then we have that
\[
\sup_{t \in [0,T+1]} \left| N^2 \theta_{b,N,m}^N(w_N(t)) - \theta_{b,m}(w(t)) \right| \to 0,
\]
and
\[
\sup_{t \in [0,T+1]} \left| N^2 \theta_{a,N,m}^N(w_N(t)) - \theta_{a,m}(w(t)) \right| \to 0.
\]
Proof. This is a direct application of assumption (i) in Section 3.2.

Proposition A.10. Suppose \((X_N^1, X_N^2)\) is a sequence in \( \mathcal{M}(\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}) \) such that \((Q_N(X_N^1), Q_N(X_N^2)) \Rightarrow (u^1, u^2) \) in \( \mathcal{M}(C_0((0, 1)) \times C_0((0, 1))) \). Define the maps \( \tilde{R}^N : \mathcal{M}(\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}) \to \mathcal{M}(\mathbb{R}^{N-1} \times \mathbb{R}^{N-1}) \) and \( \bar{R} : \mathcal{M}(C_0((0, 1)) \times C_0((0, 1)) \times \{u, d\} \to \mathcal{M}(C_0((0, 1)) \times C_0((0, 1))) \) such that
\[(i) \quad \tilde{R}^N(X^1, X^2, k)(A) := \mathbb{E}\left[ R^N(X^1, X^2, k)(A) \right], \]
\[(ii) \quad \bar{R}(u^1, u^2, k)(B) := \mathbb{E}\left[ R(u^1, u^2, k)(B) \right].\]
Fix some \( k \in \{u, d\} \). Then
\[\tilde{R}^N(X^1, X^2, k) \circ Q_N^{-1} \Rightarrow \bar{R}(u^1, u^2, k)\]
in \( \mathcal{M}(C_0((0, 1)) \times C_0((0, 1))) \).
Proof. This is essentially the same as the proof of Proposition A.3

Theorem A.11. Suppose that
\[(Q_N(X_{N,1}^b(0)), Q_N(X_{N,1}^a(0))) \Rightarrow (u^b(0), u^a(0))\]
in law in \( C_0((0, 1)) \times C_0((0, 1)) \). Let \((X_{N,1}^b(t), X_{N,1}^a(t), m_N(t))\) be dynamic microscopic models with initial data \((X_{N,1}^b(0), X_{N,1}^a(0), m^1)\), and let \((u^b(t), u^a(t), m(t))\) be the dynamic macroscopic model with initial data \((u^b(0), u^a(0), m^1)\). Then
\[(Q_N(X_{N,i}^b(N^2t)))_{i=1}^{\infty}, (Q_N(X_{N,i}^a(N^2t)))_{i=1}^{\infty}, (\tau_{N,b}/N^2)_{i=1}^{\infty}, (\tau_{N,a}/N^2)_{i=1}^{\infty}, (m_N)_{i=1}^{\infty} \Rightarrow ((u^b)_{i=1}^{\infty}, (u^a)_{i=1}^{\infty}, (\tau_{b})_{i=1}^{\infty}, (\tau_{a})_{i=1}^{\infty}, (m)_{i=1}^{\infty})\] (A.4)
in law in \( \mathcal{D}([0, T+1]; C_0((0, 1)))^N \times \mathcal{D}([0, T+1]; C_0((0, 1)))^N \times [0, T+1]^N \times [0, T+1]^N \times \mathbb{R}^N \).
Proof. This follows the proof of Theorem A.2 and we refer to this for the details. We once again use an inductive argument, and show that for every \( M \geq 1, \)
\[(Q_N(X_{N,i}^b(N^2t)))_{i=1}^{M}, (Q_N(X_{N,i}^a(N^2t)))_{i=1}^{M}, (\tau_{N,b}/N^2)_{i=1}^{M-1}, (\tau_{N,a}/N^2)_{i=1}^{M-1}, (m_N)_{i=1}^{M} \Rightarrow ((u^b)_{i=1}^{M}, (u^a)_{i=1}^{M}, (\tau_{b})_{i=1}^{M-1}, (\tau_{a})_{i=1}^{M-1}, (m)_{i=1}^{M})\] (A.5)
in law in \( \mathcal{D}([0, T+1]; C_0((0, 1)))^N \times \mathcal{D}([0, T+1]; C_0((0, 1)))^N \times [0, T+1]^N \times [0, T+1]^N \times \mathbb{R}^N \), from which the result follows. Given the inductive hypothesis, we use the Skorohod representation theorem and Propositions A.9 and A.4 to obtain convergence of the next rescaled stopping times in the sequence. It then follows that the next mid in the sequence also converges. By conditioning as in the proof of Theorem A.2 and making use of Proposition A.10, we can obtain convergence of the process up to the next stopping time, concluding the inductive argument.
Proposition A.12. Let $g: \mathbb{D}([0, T + 1], C_0((0, 1)))^N \times [0, T + 1]^N \rightarrow \mathbb{D}([0, T], C_0((0, 1)))$ such that

$$g((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty) (t) := \sum_{j=1}^\infty u_j \left( t - \sum_{i=1}^{j-1} t_i \right) 1 \{ \sum_{i=1}^{j-1} t_i \leq t < \sum_{i=1}^j t_i \}.$$ 

Suppose that

(i) $t_i > 0$ for every $i$.

(ii) $\sum_{i=1}^K t_i \neq T + 1$ for every $K$.

(iii) $\exists K \geq 1$ such that $\sum_{i=1}^K t_i > T + 1$.

(iv) $u_i$ is continuous for every $i$

Then $g$ is continuous at the point $((u_i)_{i=1}^\infty, (t_i)_{i=1}^\infty)$.

Proof. This is essentially the same as the proof of Theorem A.6.

The following result is essentially the same as Proposition A.7.

Proposition A.13. Let $h: [0, T + 1]^N \times \mathbb{R} \rightarrow \mathbb{D}([0, T + 1]; \mathbb{R})$ such that

$$h((t_i)_{i=1}^\infty, (m_i)_{i=1}^\infty) := \sum_{i=1}^\infty m_i 1 \{ \sum_{j=1}^{i-1} t_j \leq t < \sum_{j=1}^i t_j \}.$$ 

Suppose that

(i) $t_i > 0$ for every $i$.

(ii) $\sum_{i=1}^K t_i \neq T + 1$ for every $K$.

(iii) $\exists K \geq 1$ such that $\sum_{i=1}^K t_i > T + 1$.

Then $h$ is continuous at the point $((t_i)_{i=1}^\infty, (m_i)_{i=1}^\infty)$.

We can now conclude the proof of Theorem 3.5. We begin by Skorohod representing the convergence in Theorem A.11. So we have that

$$((Q_N(X_{N,b}(N^2 t)))_{i=1}^\infty, (Q_N(X_{N,a}(N^2 t)))_{i=1}^\infty, (\tau_{N,b}^i/N^2)_{i=1}^\infty, (\tau_{N,a}^i/N^2)_{i=1}^\infty, (m_N^i)_{i=1}^\infty) \rightarrow ((u_i^b)_{i=1}^\infty, (u_i^a)_{i=1}^\infty, (\tau_i^b)_{i=1}^\infty, (\tau_i^a)_{i=1}^\infty, (m_i^\infty)_{i=1}^\infty) \quad (A.6)$$

in $\mathbb{D}([0, T + 1]; C_0((0, 1)))^N \times \mathbb{D}([0, T + 1]; C_0((0, 1)))^N \times [0, T + 1]^N \times [0, T + 1]^N \times \mathbb{R}^N$ almost surely. By Proposition A.12 we clearly have that $((u_i^b)_{i=1}^\infty, (\tau_i^b)_{i=1}^\infty)$ and $((u_i^a)_{i=1}^\infty, (\tau_i^a)_{i=1}^\infty)$ are continuity points of the map $g$ almost surely. Similarly, $((\tau_i^b)_{i=1}^\infty, (m_i^\infty)_{i=1}^\infty)$ is a continuity point for the map $h$ almost surely. The result follows.
References


