Stefan Problems for Reflected SPDEs Driven by Space-Time White Noise

Ben Hambly* and Jasdeep Kalsi †

Mathematical Institute, University of Oxford

August 14, 2018

Abstract

We prove the existence and uniqueness of solutions to a one-dimensional Stefan Problem for reflected SPDEs which are driven by space-time white noise. The solutions are shown to exist until almost surely positive blow-up times. Such equations can model the evolution of phases driven by competition at an interface, with the dynamics of the shared boundary depending on the derivatives of two competing profiles at this point. The novel features here are the presence of space-time white noise; the reflection measures, which maintain positivity for the competing profiles; and a sufficient condition to make sense of the Stefan condition at the boundary. We illustrate the behaviour of the solution numerically to show that this sufficient condition is close to necessary.

1 Introduction

Stefan problems have been extensively studied since the original work by Josef Stefan in 1888, and have a number of applications in physics, engineering, biology and finance. Broadly speaking, they describe situations where an interface moves with pressure or relative pressure due to competition from two types. In this paper, we will study stochastic, reflected versions of this problem in one-dimension, which take the form

$$\begin{aligned}\frac{\partial u^1}{\partial t} &= \Delta u^1 + f_1(p(t)-x,t,u^1(t,p(t)-\cdot)) + \sigma_1(p(t)-x,t,u^1(t,p(t)-\cdot))\dot{W} + \eta^1 \\ \frac{\partial u^2}{\partial t} &= \Delta u^2 + f_2(x-p(t),t,u^2(t,\cdot-p(t))) + \sigma_2(x-p(t),t,u^2(t,\cdot-p(t)))\dot{W} + \eta^2,\end{aligned} \tag{1.1}$$

where u^1 and u^2 satisfy Dirichlet boundary conditions enforcing that they are zero at $p(t)$, with the point $p(t)$ evolving according to the equation

$$p'(t) = h\left(\frac{\partial u^1}{\partial x}(t,p(t)^-), \frac{\partial u^2}{\partial x}(t,p(t)^+)\right).$$

Here, $\dot{W}$ is a space-time white noise; f_i and σ_i, for $i = 1,2$, are the drift and volatility for each type, which are determined by the distance to the moving boundary $p(t)$; and (η^1,η^2) are reflection measures for the functions u^1 and u^2 respectively, keeping the profiles positive and satisfying the conditions

*ben.hambly@maths.ox.ac.uk
†jasdeep.kalsi@maths.ox.ac.uk

(i) $\int_0^\infty \int_{\mathbb{R}} u^1(t,x)\, \eta^1(\mathrm{d}t, \mathrm{d}x) = 0$, and

(ii) $\int_0^\infty \int_{\mathbb{R}} u^2(t,x)\, \eta^2(\mathrm{d}t, \mathrm{d}x) = 0$.

The equation describes the evolution of two reflected SPDEs which share a moving boundary. The derivative of the moving boundary is then determined by h, a locally Lipschitz function of the spatial derivatives of the two SPDEs at the shared boundary. We note here that, in general, reflected SPDEs of this type will only be up to 1/2-Hölder continuous in space. We will therefore require the functions σ_i to satisfy suitable conditions which will ensure that the volatility decays at least linearly at the interface, which will be shown to be sufficient for the existence of a spatial derivative there.

We recall the classical one-sided version of the Stefan problem, which takes the form

$$\frac{\partial u}{\partial t} = \Delta u, \qquad p'(t) = -\frac{\partial u}{\partial x}(t, p(t)^+).$$

The profile of u at a given time t has support in the set $[p(t), \infty)$, and we have the Dirichlet condition $u(t, p(t)) = 0$. Typically, we have that $u_0 \geq 0$, from which it follows that $u \geq 0$. The equation for p' then implies that the boundary recedes with "pressure" from u. This can be thought of as a model for the melting of a block of ice when in contact with a body of water. The point $p(t)$ represents the interface between the water and ice, and u the temperature profile of the body of water.

Recently, stochastic perturbations of this classical problem have received significant attention. In [6], existence and uniqueness for solutions to a Stefan problem where the two sides satify SPDEs driven by spatially coloured noise is proved. A particular case of the corresponding problem when the SPDE is driven by space-time white noise was then studied in [10]. It is assumed here that the volatility, $\sigma(x)$ vanishes faster than $x^{3/2}$ as $x \downarrow 0$ i.e. as the moving interface is approached. More recent work on such problems include the models in [7] and in [5]. In these papers, as in [6], the two sides satisfy SPDEs driven by spatially coloured noise, as this makes it easier to establish the existence of the spatial derivative at the interface. Motivated by modelling the limit order book in financial markets, the authors also include a Brownian noise term in the moving boundary.

In addition to stochastic moving boundary problems, the study of reflected SPDEs has also attracted interest. Equations of the type

$$\frac{\partial u}{\partial t} = \Delta u + f(x, u) + \sigma(x, u)\dot{W} + \eta,$$

where $\dot{W}$ is space-time white noise and η is a reflection measure, were initially studied in [8] in the case of constant volatility i.e. $\sigma \equiv 1$. Existence for the equation in the case when $\sigma = \sigma(x, u)$ satisfies Lipschitz and linear growth conditions in its second argument was then proved in [3] using a penalization method. Uniqueness for varying volatility $\sigma = \sigma(x, u)$ on compact spatial domains was then shown in [9], with the authors achieving this by decoupling the obstacle and SPDE components of the problem.

In this paper, we aim to take a first step towards studying reflected stochastic Stefan problems, by proving an existence and uniqueness result. The main condition required is that the volatility decays at least linearly close to the boundary, with the remaining conditions on the coefficients being mild. We also characterise the blow-up time for such equations, and demonstrate that it coincides with blow-up of one of the derivatives of the profiles at the shared boundary.

The results here extend the work in [10] by incorporating reflection and allowing for drift and volatility coefficients which depend on the spatial variable as well as the solution itself, and are Lipschitz in the solution. The condition on the decay of the volatility at the boundary is also relaxed, and required to be linear only. In addition, we are able to choose a general Lipschitz function h to determine the boundary evolution. This paper can also be considered an extension of [4], in which the corresponding moving boundary problems were considered in the case when the interface is driven by functions of the profiles in the space of continuous paths. These equations, however, did not require the profiles to have derivatives at the shared interface, and were shown to exist without the need for decay of the volatility at the shared interface.

The outline for this paper is as follows. We begin in Section 2 by defining our notion of solutions, motivating this by some simple calculations. Section 2 is then concluded by a statement of our main existence and uniqueness theorem. In Section 3 we turn our attention to the deterministic obstacle problem which corresponds to our equations here, and prove a key result which will allow us to ensure that the reflection component will not prevent our SPDE solutions from having derivatives at the boundary. Section 4 is dedicated to establishing some key estimates for the main proof, with some of the technical details deferred to the appendix. In Section 5 we prove our main result, presenting the arguments for existence and uniqueness for our problem. This is done by first truncating the problem suitably and then performing a Picard iteration, making use of the estimates from Sections 3 and 4. We conclude in Section 6 with a simple simulations of the equations, illustrating the appearance of derivatives when the coefficients fall within our framework and briefly exploring cases where this is not the case.

2 Notion of Solution and Statement of Main Theorem

We begin this section by defining the space $\mathscr{H}$. The solutions to our Stefan problem will be $\mathscr{H}$-valued processes.

Definition 2.1.

$$\mathscr{H} := \left\{ f \in C([0,1]) \mid f(0) = f(1) = 0, f'(0) \text{ exists} \right\}.$$

We equip $\mathscr{H}$ with the norm

$$\|f\|_{\mathscr{H}} := \sup_{x \in (0,1]} \left| \frac{f(x)}{x} \right|.$$

Remark 2.2. *$(\mathscr{H}, \|\cdot\|_{\mathscr{H}})$ is a Banach space. In addition, we can characterise $\mathscr{H}$ as follows*

$$\mathscr{H} = \left\{ f \in C([0,1]) \mid \exists g \in C([0,1]) \text{ s.t. } f(x) = xg(x) \right\}$$

Clearly, for any $f \in \mathscr{H}$, the function $g \in C([0,1])$ such that $f(x) = xg(x)$ is unique, and we have that

$$\|f\|_{\mathscr{H}} = \|g\|_\infty.$$

Definition 2.3. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Suppose that $\dot{W}$ is a space-time white noise defined on this space. Define for $A \in \mathscr{B}(\mathbb{R})$,*

$$W_t(A) := \dot{W}([0,t] \times A).$$

We say that $\dot{W}$ respects the filtration $\mathscr{F}_t$ if $(W_t(A))_{t \geq 0, A \in \mathscr{B}(\mathbb{R})}$ is an $\mathscr{F}_t$- martingale measure i.e. if for every $A \in \mathscr{B}(\mathbb{R})$, $(W_t(A))_{t \geq 0}$ is an $\mathscr{F}_t$-martingale.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a complete filtered probability space, and $\dot{W}$ space-time white noise on $\mathbb{R}^+ \times \mathbb{R}$. Let $\mathscr{F}_t$ be the complete filtration generated by the white noise, so that $\mathscr{F}_t = \sigma(\{W(s,x) \mid x \in \mathbb{R}, s \leq t\})$. Suppose that $(u^1, \eta^1, u^2, \eta^2, p)$ is an $\mathscr{F}_t$-adapted process solving (1.1). Then $p : \mathbb{R}^+ \times \Omega \mapsto \mathbb{R}$ is a

$\mathscr{F}_t$-adapted process such that the paths of $p(t)$ are C^1 almost surely (note that, in particular, p is $\mathscr{F}_t$-predictable). Let $\varphi \in C_c^\infty([0,T]\times(0,1))$, and define the function ϕ by setting $\phi(t,x) = \varphi(t, p(t)+x)$. By multiplying the equation for u^1 in (1.1) by such a ϕ and integrating over space and time, interpreting the derivatives in the usual weak sense, we obtain the expression

$$\begin{aligned}\int_{\mathbb{R}} u^1(t,x)\phi(t,x)\mathrm{d}x = &\int_{\mathbb{R}} u^1(0,x)\phi(0,x)\mathrm{d}x + \int_0^T \int_{\mathbb{R}} u^1(s,x)\frac{\partial \phi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{R}}\int_0^T u^1(s,x)\frac{\partial^2\phi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{R}}\int_0^T f_1(p(s)-x,s,u^1(s,p(s)-\cdot))\phi(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_{\mathbb{R}}\int_0^T \sigma_1(p(s)-x,s,u^1(s,p(s)-\cdot))\phi(s,x)W(\mathrm{d}x,\mathrm{d}s) \\ &+ \int_{\mathbb{R}}\int_0^T \phi(s,x)\;\eta^1(ds,dx).\end{aligned}$$

We now introduce a change in the spatial variable in order to associate our problem with a fixed boundary problem. Setting $v^1(t,x) = u^1(t,p(t)-x)$, the above equation becomes

$$\begin{aligned}\int_0^1 v^1(t,x)\phi(t,p(t)-x)\mathrm{d}x = &\int_0^1 v^1(0,x)\phi(0,p(0)-x)\mathrm{d}x + \int_0^T\int_0^1 v^1(s,x)\frac{\partial\phi}{\partial t}(s,p(s)-x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t v^1(s,x)\frac{\partial^2\phi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t f_1(x,s,v^1(s,\cdot))\phi(s,p(s)-x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t \sigma_1(x,s,v^1(s,\cdot))\phi(s,p(s)-x)W_p(\mathrm{d}x,\mathrm{d}s) \\ &+ \int_0^1\int_0^t \phi(s,p(s)-x)\;\eta_p^1(ds,dx).\end{aligned}$$

Here, $\dot{W}_p$ and η_p^1 are obtained by from W and η by shifting by $p(t)$. That is, for $t \in \mathbb{R}^+$ and $A \in \mathscr{B}(\mathbb{R})$,

$$\dot{W}_p([0,t]\times A) = \int_0^t\int_{A+p(s)} W(\mathrm{d}s,\mathrm{d}y), \qquad \eta_p^1([0,t]\times A) = \int_0^t\int_{A+p(s)} \eta(\mathrm{d}y,\mathrm{d}s),$$

Note that, since the process $p(t)$ is $\mathscr{F}_t$-predictable, $\dot{W}_p$ is then also a space time white noise which respects the filtration $\mathscr{F}_t$. Also, η_p^1 is a reflection measure for v, so that

$$\int_0^T\int_0^1 v^1(t,x)\;\eta_p^1(\mathrm{d}t,\mathrm{d}x) = 0.$$

Differentiating ϕ in time gives

$$\frac{\partial\phi}{\partial t}(s,x) = \frac{\partial\varphi}{\partial t}(t,p(t)+x) + p'(t)\frac{\partial\varphi}{\partial x}(t,p(t)+x).$$

It therefore follows that

$$\begin{aligned}\int_0^1 v^1(t,x)\varphi(t,x)\mathrm{d}x = &\int_0^1 v^1(0,x)\varphi(0,x)\mathrm{d}x + \int_0^t\int_0^1 v^1(s,x)\frac{\partial\varphi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^t\int_0^1 v^1(s,x)p'(s)\frac{\partial\varphi}{\partial x}(s,x)\mathrm{d}x\mathrm{d}s + \int_0^1\int_0^t v^1(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t f_1(x,s,v^1(s,\cdot))\varphi(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t \sigma_1(x,s,v^1(s,\cdot))\varphi(s,x)W_p(\mathrm{d}x,\mathrm{d}s) \\ &+ \int_0^1\int_0^t \varphi(s,x)\ \eta_p^1(ds,dx).\end{aligned}$$

We can perform similar manipulations to obtain a weak form for $v^2(t,x) := u^2(t,p(t)+x)$. This yields that for test functions $\varphi \in C_c^\infty([0,T]\times(0,1))$, we should have that

$$\begin{aligned}\int_0^1 v^2(t,x)\varphi(t,x)\mathrm{d}x = &\int_0^1 v^2(0,x)\varphi(0,x)\mathrm{d}x + \int_0^t\int_0^1 v^2(s,x)\frac{\partial\varphi}{\partial t}(s,x)\mathrm{d}x\mathrm{d}s \\ &- \int_0^t\int_0^1 v^2(s,x)p'(s)\frac{\partial\varphi}{\partial x}(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t v^2(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\mathrm{d}x\mathrm{d}s + \int_0^1\int_0^t f_2(x,s,v^2(s,\cdot))\varphi(s,x)\mathrm{d}x\mathrm{d}s \\ &+ \int_0^1\int_0^t \sigma_2(x,s,v^2(s,\cdot))\varphi(s,x)W_p^-(\mathrm{d}x,\mathrm{d}s) + \int_0^1\int_0^t \varphi(s,x)\ \eta_p^2(ds,dx),\end{aligned}$$

where $\dot{W}_p^-$ is given by

$$\dot{W}_p^-([0,t]\times A) = \dot{W}_p([0,t]\times(-A)).$$

Note also that, since $(u^1,\eta^1,u^2,\eta^2,p)$ is $\mathscr{F}_t$-adapted, we know that $(v^1,\eta_p^1,v^2,\eta_p^1,p)$ is also $\mathscr{F}_t$-adapted.

We now define our notion of solution for a class of reflected SPDEs. This will prove useful when defining solutions to our moving boundary problems, and when proving existence for these via a sequence of solutions to truncated versions of the equations.

Definition 2.4. *Let $(\Omega,\mathscr{F},\mathscr{F}_t,\mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. Suppose that $\tilde{v}$ is a continuous $\mathscr{F}_t$-adapted process taking values in $\mathscr{H}$. Let $F:\mathscr{H}\times\mathscr{H}\to\mathscr{H}$. For the $\mathscr{F}_t$-stopping time τ, we say that the pair (v,η) is a local solution to the reflected SPDE*

$$\frac{\partial v}{\partial t} = \Delta v + h(v(t,\cdot),\tilde{v}(t,\cdot))\frac{\partial F(v)}{\partial x} + f(x,t,v(t,\cdot)) + \sigma(x,t,v(t,\cdot))\dot{W} + \eta$$

on $[0,\infty)\times[0,1]$ with Dirichlet boundary conditions $v(t,0)=v(t,1)=0$ and initial data $v_0\in\mathscr{H}$, until time τ, if

(i) For every $x\in[0,1]$ and every $t\geq 0$, $v(t,x)$ is an $\mathscr{F}_t$-measurable random variable.

(ii) $v\geq 0$ almost surely.

(iii) $v\big|_{[0,\tau)\times[0,1]} \in C([0,\tau);\mathscr{H})$ *almost surely.*

(iv) $v(t,x)=\infty$ *for every* $t\geq\tau$ *and* $x\in[0,1]$ *almost surely.*

(v) η *is a measure on* $(0,1)\times[0,\infty)$ *such that*

(a) For every measurable map $\psi:[0,1]\times[0,\infty)\to\mathbb{R}$,

$$\int_0^t\int_0^1 \psi(x,s)\,\eta(dx,ds)$$

is $\mathscr{F}_t$*-measurable.*

(b) $\int_0^\infty\int_0^1 v(t,x)\,\eta(dx,dt)=0.$

(vi) There exists a localising sequence of stopping times $\tau_n\uparrow\tau$ *almost surely, such that for every* $\varphi\in C_c^{1,2}([0,\infty)\times[0,1])$ *such that* $\varphi(s,0)=\varphi(s,1)=0$ *for every* $s\in[0,\infty)$,

$$\begin{aligned}\int_0^1 v(t\wedge\tau_n,x)\varphi(t,x)\,dx &= \int_0^1 v(0,x)\varphi(0,x)\,dx+\int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial\varphi}{\partial t}(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 v(s,x)\frac{\partial^2\varphi}{\partial x^2}(s,x)\,dx\,ds\\ &-\int_0^{t\wedge\tau_n}\int_0^1 h(v(s,\cdot),\tilde{v}(s,\cdot))F(v(s,x))\frac{\partial\varphi}{\partial x}(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 f(x,s,v(s,\cdot))\varphi(s,x)\,dx\,ds\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \sigma(x,s,v(s,\cdot))\varphi(s,x)W(dx,ds)\\ &+\int_0^{t\wedge\tau_n}\int_0^1 \varphi(s,x)\,\tilde{\eta}(ds,dx).\end{aligned}\tag{2.1}$$

for every $t\geq 0$ *almost surely.*

We say that a local solution is *maximal* if there does not exist a solution to the equation on a larger stochastic interval, and we say that a local solution is *global* if we can take $\tau_n=\infty$ in (2.1).

The following definitions provide us with notation which will allow us to easily move between the relative frame (measured with respect to the current position of the boundary) and the fixed frame when discussing solutions to our equations.

Definition 2.5. *For a function* $p:[0,T]\to\mathbb{R}$ *we define* $\theta_p^1:[0,T]\times\mathbb{R}\to[0,T]\times\mathbb{R}$ *such that*

$$\theta_p^1(t,x):=(t,p(t)-x).$$

We similarly define $\theta_p^2:[0,T]\times\mathbb{R}\to[0,T]\times\mathbb{R}$ *such that*

$$\theta_p^2(t,x):=(t,p(t)+x).$$

Definition 2.6. *For a space time white noise* $\dot{W}$*, we denote by* $\dot{W}^-$ *the space time white noise such that* $\dot{W}^-([0,t]\times A)=\dot{W}([0,t]\times(-A))$.

Before finally stating our definition for solutions to (1.1), we first outline the conditions on our coefficients f_i, σ_i and h.

(i) For $i = 1, 2,$ $f_i, \sigma_i : [0, 1] \times [0, \infty) \times \mathscr{H} \to \mathbb{R}$ are measurable mappings.

(ii) $h : \mathscr{H} \times \mathscr{H} \to \mathbb{R}$ is a measurable map.

(iii) For $i = 1, 2,$ f_i is such that for every $T > 0$ there exists C_T such that for every $t \in [0, T]$,

$$|f_i(x, t, 0)| \leq C_T.$$

(iv) For $i = 1, 2,$ f_i satisfies the local Lipschitz condition that for every $M, T > 0$, there exists $C_{T,M}$ such that for every $t \in [0, T]$ and every $u, v \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}, \|v\|_{\mathscr{H}} \leq M$,

$$|f_i(x, t, u) - f_i(x, t, v)| \leq C_{T,M}\|u - v\|_{\mathscr{H}}.$$

(v) For $i = 1, 2,$ σ_i is such that for every $T > 0$ there exists C_T such that for every $t \in [0, T]$

$$|\sigma_i(x, t, 0)| \leq C_T x$$

(vi) For $i = 1, 2,$ σ_i satisfies the local Lipschitz condition that for every $M, T > 0$, there exists $C_{T,M}$ such that for every $t \in [0, T]$ and every $u, v \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}, \|v\|_{\mathscr{H}} \leq M$,

$$|\sigma_i(x, t, u) - \sigma_i(x, t, v)| \leq C_{T,M}x\|u - v\|_{\mathscr{H}}.$$

(vii) h is bounded on bounded sets.

(viii) h satisfies the local Lipschitz condition that for every $M > 0$, there exists C_M such that for every $u, v \in \mathscr{H}$ with $\|u\|_{\mathscr{H}}, \|v\|_{\mathscr{H}} \leq M$,

$$|h(u_1, v_1) - h(u_2, v_2)| \leq C_M(\|u_1 - u_2\|_{\mathscr{H}} + \|v_1 - v_2\|_{\mathscr{H}}).$$

Remark 2.7. *An immediate consequence of the conditions on f_i and σ_i is that they satisfy local linear growth conditions. That is, for every $T, M > 0$ there exists a constant $C_{T,M}$ such that for every $t \in [0, T]$ and every $u \in \mathscr{H}$ with $\|u\|_{\mathscr{H}} \leq M$,*

$$f_i(x, t, u) \leq C_{T,M}\left(1 + \|u\|_{\mathscr{H}}\right),$$

$$\sigma_i(x, t, u) \leq C_{T,M}\left(1 + \|u\|_{\mathscr{H}}\right).$$

Remark 2.8. *The class of permitted functions for h includes as a subclass functions of the form*

$$h(u, v) = \tilde{h}\left(\frac{\partial u}{\partial x}(0), \frac{\partial v}{\partial x}(0)\right),$$

where $\tilde{h}$ is a Lipschitz function from $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Remark 2.9. *The x term in the linear growth and Lipschitz conditions for σ ensure that the volatility will decay at least linearly as we approach the boundary i.e. as $x \downarrow 0$. We note that, in the case when $\sigma(x, t, u) = \tilde{\sigma}(x, t, u(x))$, the typical Lipschitz condition*

$$|\tilde{\sigma}(x, t, u) - \tilde{\sigma}(x, t, v)| \leq C|u - v|$$

implies the Lipschitz condition (vi). In particular, volatility functions of the form

$$\sigma(x, t, u) = \sigma_1(x) + \sigma_2(x)u(x),$$

where $\frac{\sigma_1}{x}, \sigma_2 \in L^\infty([0, 1])$, are permitted.

We are now in position to state our definition for a solution to problem (1.1). Following this, we state the main result, namely existence and uniqueness for maximal solutions to these equations.

Definition 2.10. *Let $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$ be a complete filtered probability space. Let $\dot{W}$ be a space time white noise on this space which respects the filtration $\mathscr{F}_t$. We say that the quintuple $(u^1, \eta^1, u^2, \eta^2, p)$ satisfies the reflected stochastic Stefan problem with initial data $(u_0^1, u_0^2) \in \mathscr{H}^2$, up to the $\mathscr{F}_t$-stopping time τ if*

(i) $(v^1, \tilde{\eta}^1) := (u^1 \circ \theta_p^1, \eta^1 \circ (\theta_p^1)^{-1})$ *solves, in the sense of Definition 2.4, the reflected SPDE*

$$\frac{\partial v^1}{\partial t} = \Delta v^1 - p'(t)\frac{\partial v^1}{\partial x} + f_1(x,t,v^1(t,\cdot)) + \sigma_1(x,t,v^1(t,\cdot))\dot{W} + \tilde{\eta}^1$$

with Dirichlet boundary conditions $v^1(0) = v^1(1) = 0$ and initial data $v^1(0,x) = u_0^1(x)$ until time τ.

(ii) $(v^2, \tilde{\eta}^2) := (u^2 \circ \theta_p^2, \eta^2 \circ (\theta_p^1)^{-1})$ *solves, in the sense of Definition 2.4, the reflected SPDE*

$$\frac{\partial v^2}{\partial t} = \Delta v^2 + p'(t)\frac{\partial v^2}{\partial x} + f_2(x,t,v^2(t,\cdot)) + \sigma_2(x,t,v^2(t,\cdot))\dot{W}^- + \tilde{\eta}^2$$

with Dirichlet boundary conditions $v^2(0) = v^2(1) = 0$ and initial data $v^2(0,x) = u_0^2(x)$ until time τ.

(iii) $p'(t) = h(\frac{\partial v^1}{\partial x}(t, p(t)^-), \frac{\partial v^2}{\partial x}(p(t)^+))$.

We refer to $(v^1, \tilde{\eta}^1, v^2, \tilde{\eta}^2)$ as the solution to the moving boundary problem in the relative frame.

Theorem 2.11. *There exists a unique maximal solution to the reflected stochastic Stefan problem.*

Remark 2.12. *It is immediate from the definition that existence of a unique maximal solution to the reflected stochastic Stefan problem is equivalent to the existence of a unique maximal solution to the system of coupled reflected SPDEs*

$$\begin{aligned} \frac{\partial v^1}{\partial t} &= \Delta v^1 - p'(t)\frac{\partial v^1}{\partial x} + f_1(x,t,v^1(t,\cdot)) + \sigma_1(x,t,v^1(t,\cdot))\dot{W} + \eta^1, \\ \frac{\partial v^2}{\partial t} &= \Delta v^2 + p'(t)\frac{\partial v^2}{\partial x} + f_2(x,t,v^2(t,\cdot)) + \sigma_2(x,t,v^2(t,\cdot))\dot{W}^- + \eta^2, \\ p'(t) &= h\left(\frac{\partial v^1}{\partial x}(t,p(t)^-), \frac{\partial v^2}{\partial x}(t,p(t)^+)\right). \end{aligned} \tag{2.2}$$

3 The Deterministic Obstacle Problem and the Corresponding Bounds in $\mathscr{H}$

We begin this section by discussing some relevant work on a deterministic obstacle problem. The obstacle problem in the form given here was originally discussed in [8].

Definition 3.1. *Let $v \in C([0,T] \times [0,1])$ with $v(t,\cdot) \in C_0((0,1))$ for every $t \in [0,T]$. We say that the pair (z, η) satisfies the heat equation with obstacle v if:*

(i) $z \in C([0,T] \times [0,1])$, $z(t,0) = z(t,1) = 0$, $z(0,x) \equiv 0$ *and* $z \geq v$.

(ii) η *is a measure on* $(0,1) \times [0,T]$.

(iii) z *weakly solves the PDE*

$$\frac{\partial z}{\partial t} = \frac{\partial^2 z}{\partial x^2} + \eta$$

That is, for every $t \in [0,T]$ *and every* $\phi \in C^2((0,1)) \cap C_0((0,1))$,

$$\int_0^1 z(t,x)\phi(x)\,dx = \int_0^t \int_0^1 z(s,x)\phi''(x)\,dx\,ds + \int_0^t \int_0^1 \phi(x)\eta(dx,ds).$$

(iv) $\int_0^t \int_0^1 (z(s,x) - v(s,x))\, \eta(dx,ds) = 0$.

Existence and uniqueness for solutions to this problem was proved in [8]. It was also shown that the difference between two solutions can be bounded in the L^∞-norm by the difference in the L^∞-norm of the obstacles. We now adapt this result, and show that solutions to the obstacle problem in the case when the obstacle lies in $C([0,T];\mathscr{H})$ are themselves in the space $C([0,T];\mathscr{H})$. We also prove the corresponding estimate, that we can control the $\mathscr{H}$-norm of the difference in the solutions by the $\mathscr{H}$-norm of the difference of the obstacle. This will be required to prove existence for our reflected SPDEs via a Picard argument.

Theorem 3.2. *Suppose that* $v_1, v_2 \in C([0,T];\mathscr{H})$. *Let* z_1, z_2 *solve the corresponding parabolic obstacle problems with obstacles* v_1 *and* v_2. *Then we have that* $z_1, z_2 \in L^\infty([0,T];\mathscr{H})$ *and*

$$\|z_1 - z_2\|_{\mathscr{H},T} \leq \|v_1 - v_2\|_{\mathscr{H},T}.$$

Proof. Define $\phi(t) := \|v_1 - v_2\|_{\mathscr{H},t}$. For $i = 1,2$ and $\epsilon > 0$, let z_i solve the following penalised equations

$$\frac{\partial z_i^\epsilon}{\partial t} = \Delta z_i^\epsilon + \frac{1}{\epsilon}\arctan(((z_i^\epsilon + v) \wedge 0)^2).$$

Then we know that $z_i^\epsilon \uparrow z_i$ for $i = 1,2$. We define $w^\epsilon(t,x) := z_1^\epsilon - z_2^\epsilon - x\phi(t)$. Then w^ϵ is a weak solution to the equation

$$\frac{\partial w^\epsilon}{\partial t} = \Delta w^\epsilon + g_\epsilon(z_1^\epsilon + v_1) - g_\epsilon(z_2^\epsilon + v_2) - x\frac{\mathrm{d}\phi}{\mathrm{d}t},$$

with boundary conditions $w(t,0) = 0$, $w(t,1) = -\phi(t)$. Note that, since ϕ is increasing in t, the weak derivative of ϕ is negative. Testing this equation with $(w^\epsilon)^+$ and integrating over space and time, we obtain that

$$\begin{aligned}\frac{1}{2}\|(w_T^\epsilon)^+\|_{L^2}^2 = &-\int_0^T \int_0^1 \left|\frac{\partial (w_t^\epsilon)^+}{\partial x}\right|^2 \mathrm{d}x\mathrm{d}t + \int_0^T \int_0^1 g_\epsilon(z_1^\epsilon(t,x) + v_1(t,x))(w^\epsilon(t,x))^+\mathrm{d}x\mathrm{d}t \\ &- \int_0^T \int_0^1 g_\epsilon(z_1^\epsilon(t,x) + v_1(t,x))(w^\epsilon(t,x))^+\mathrm{d}x\mathrm{d}t - \int_0^T \int_0^1 x(w^\epsilon(t,x))^+\mathrm{d}x\;\mathrm{d}\phi_t.\end{aligned} \tag{3.1}$$

Notice that g_ϵ is a decreasing function. Also, on the set $w^\epsilon \geq 0$, $z_1^\epsilon + v_1 \geq z_2^\epsilon + v_2$. It then follows that the right hand side of (3.1) is negative. Therefore, $(w^\epsilon)^+ = 0$, and so

$$z_1^\epsilon - z_2^\epsilon \leq x\phi(t).$$

Interchanging z_1^ϵ and z_2^ϵ, we obtain that, for every $t \in [0,T]$ and every $x \in [0,1]$,

$$\sup_{t\in[0,T]} \sup_{x\in(0,1]} \left| \frac{z_1^\epsilon(t,x)}{x} - \frac{z_2^\epsilon(t,x)}{x} \right| \leq \phi(t).$$

Letting $\epsilon \downarrow 0$, we deduce that

$$\sup_{t\in[0,T]} \sup_{x\in(0,1]} \left| \frac{z_1(t,x)}{x} - \frac{z_2(t,x)}{x} \right| \leq \phi(t). \tag{3.2}$$

We now argue that the functions $z_i \in C([0,T];\mathscr{H})$. Note that we have $C([0,T];\mathscr{H})$ regularity for solutions to the obstacle problem provided that the obstacle is smooth (see, for example, Theorem 4.1 in [1]). Let v_n be a sequence of smooth obstacles such that $v_n \to v$ in $C([0,T];\mathscr{H})$, and denote by z_n the solution to the obstacle problem with obstacle v_n. Then, by (3.2), the sequence z_n is Cauchy in $C([0,T];\mathscr{H})$, and so converges in $C([0,T];\mathscr{H})$ to some $\tilde{z}$. We also have that

$$\sup_{t\in[0,T]} \sup_{x\in(0,1]} \left| \frac{z_n(t,x)}{x} - \frac{z(t,x)}{x} \right| \to 0.$$

Therefore, we have that $\tilde{z} = z$, and so $z \in C([0,T];\mathscr{H})$. This concludes the proof. □

4 Key Estimates

In order to prove existence for our reflected SPDE, we will perform a Picard iteration in the space $L^p(\Omega; C([0,T];\mathscr{H}))$. Having defined $(u^{1,n}, \eta^{1,n}, u^{2,n}, \eta^{2,n}, p^n)$, we will define our $(n+1)^{\text{th}}$ approximation of a solution to be given by the solutions of

$$\begin{aligned}
\frac{\partial u^{1,n+1}}{\partial t} =& \Delta u^{1,n+1} + f(x,t,u^{1,n}(t,\cdot)) + \frac{\partial u^{1,n}}{\partial x} h\left(\frac{\partial u^{1,n}}{\partial x}(t,p(t)^-), \frac{\partial u^{2,n}}{\partial x}(t,p(t)^+)\right) \\
&+ \sigma(x,t,u^{i,n}(t,\cdot))\dot{W} + \eta^{1,n+1}, \\
\frac{\partial u^{2,n+1}}{\partial t} =& \Delta u^{2,n+1} + f(x,t,u^{2,n}(t,\cdot)) - \frac{\partial u^{2,n}}{\partial x} h\left(\frac{\partial u^{1,n}}{\partial x}(t,p(t)^-), \frac{\partial u^{2,n}}{\partial x}(t,p(t)^+)\right) \\
&+ \sigma(x,t,u^{i,n}(t,\cdot))\dot{W}^- + \eta^{2,n+1}, \\
p^{n+1}(t) =& \int_0^t h\left(\frac{\partial u^{1,n}}{\partial x}(s,p(s)^-), \frac{\partial u^{2,n}}{\partial x}(s,p(s)^+)\right) \mathrm{d}s.
\end{aligned}$$

In order to control the difference $\mathbb{E}\left[\|u^{i,n+1} - u^{i,n}\|^p_{\mathscr{H},T}\right]$, we will use the result from Section 3. This gives

$$\mathbb{E}\left[\|u^{i,n+1} - u^{i,n}\|^p_{\mathscr{H},T}\right] \leq C_p \mathbb{E}\left[\|v^{i,n+1} - v^{i,n}\|^p_{\mathscr{H},T}\right],$$

where v_i^{n+1} solves

$$\frac{\partial v^{i,n+1}}{\partial t} = \Delta v^{i,n+1} \pm \frac{\partial u^{i,n}}{\partial x} p'(t) + f(x,u^{i,n}) + \sigma(x,u^{i,n}) \frac{\partial^2 W}{\partial x \partial t}.$$

By writing this in mild form, we see that we will require estimates on the terms

(i)
$$\int_0^t \int_0^1 \frac{1}{x} G(t-s,x,y) \left[f(x,u^{i,n}) - f(x,u^{i-1,n}) \right] \mathrm{d}y \mathrm{d}s. \tag{4.1}$$

(ii)
$$\int_0^t \int_0^1 \frac{y}{x} G(t-s,x,y) \left[\frac{\sigma(x,u^{i,n}) - \sigma(x,u^{i-1,n})}{y} \right] W(\mathrm{d}y,\mathrm{d}s). \tag{4.2}$$

(iii)
$$\int_0^t \int_0^1 \frac{y}{x} \frac{\partial G}{\partial y}(t-s,x,y) \left[u^{i,n}(s,y)(p^n)'(s) - u^{i,n-1}(s,y)(p^{n-1})'(s) \right] \mathrm{d}y \mathrm{d}s. \tag{4.3}$$

Obtaining such estimates will be the focus in this section.

Heat Kernel Estimates

In this section we will state and prove the heat kernel estimates which will be crucial in proving existence and uniqueness for our equations. In particular, they allow us to show that solutions to certain SPDEs lie in the space $C([0,T];\mathscr{H})$, and also enable us to obtain estimates in $L^p(\Omega; C([0,T];\mathscr{H}))$ for these solutions. We define G to be the Dirichlet heat kernel on $[0,1]$, so

$$G(t,x,y) := \frac{1}{\sqrt{4\pi t}} \sum_{n=-\infty}^{\infty} \left[\exp\left(-\frac{(x-y+2n)^2}{4t} \right) - \exp\left(-\frac{(x+y+2n)^2}{4t} \right) \right].$$

We then define for $t>0$, $y \in [0,1]$ and $x \in (0,1]$

$$\tilde{G}(t,x,y) := \frac{y}{x} G(t,x,y).$$

For $t>0$, $y \in [0,1]$ and $x=0$, we set

$$\tilde{G}(t,x,y) := y \frac{\partial G}{\partial x}(t,0,y).$$

We also define

$$\tilde{H}(t,x,y) := \frac{y}{x} \frac{\partial G}{\partial y}(t,x,y)$$

for $t \in [0,T]$, $x \in (0,1]$ and $y \in [0,1]$, and set

$$\tilde{H}(t,0,y) := y \frac{\partial^2 G}{\partial y \partial x}(t,0,y)$$

for $t \in [0,T]$ and $y \in [0,1]$.

Remark 4.1. *Note that*

$$\begin{aligned} G(t,x,y) = &\frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y)^2}{4t} \right) - \exp\left(-\frac{(x+y)^2}{4t} \right) \right] \\ &+ \frac{1}{\sqrt{4\pi t}} \left[\exp\left(-\frac{(x-y-2)^2}{4t} \right) - \exp\left(-\frac{(x+y-2)^2}{4t} \right) \right] + L(t,x,y), \end{aligned}$$

where L is a smooth function on $[0,T]\times[0,1]\times[0,1]$, vanishing on $t=0$ and $x=0,1$. Consequently, when proving estimates in this section, we will focus on the first of these three terms,

$$G_1(t,x,y) := \frac{1}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right].$$

Proposition 4.2. *The following estimates hold for G, $\tilde{G}$ and $\tilde{H}$.*

1. *$\exists\, C_T > 0$ such that*
$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\int_0^1 \left|\frac{1}{x}G(t,x,y)\right| dy \leq \frac{C_T}{\sqrt{t}}.$$

2. *$\exists\, C > 0$ such that for every $s > 0$,*
$$\sup_{x\in[0,1]}\int_0^1 \tilde{G}(s,x,y)^2\, dy \leq \frac{C}{\sqrt{s}}.$$

3. *For $T > 0$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that*
$$\sup_{t\in[0,T]}\int_0^t \left[\int_0^1 \left(\tilde{G}(s,x,z) - \tilde{G}(t,y,z)\right)^2 dy\right]^q ds \leq C_{T,q}|x-y|^{(2-q)/3}.$$

4. *For $T > 0$, $0 \leq s \leq t \leq T$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that*
$$\sup_{x\in[0,1]}\int_0^s \left[\int_0^1 (\tilde{G}(t-r,x,y) - \tilde{G}(s-r,x,y))^2 dy\right]^q dr \leq C_{T,q}|t-s|^{(2-q)/2}.$$

5. $$\sup_{x\in[0,1]}\int_0^1 \left|\tilde{H}(t,x,y)\right| dy \leq \frac{C}{\sqrt{t}}.$$

Proof. The proof for inequality (2) can be found in [10]. The other inequalities can be shown by adapting some of the arguments from [10], and their proofs are deferred to the appendix. □

Mild form Estimates and Continuity

We would now like to translate the estimates on the heat kernel into estimates on the terms (4.1), (4.2) and (4.3) appearing in the mild form for the problem. We would also like to ensure continuity of these terms. In the case of the drift and moving boundary terms in the mild form, this is essentially an immediate consequence of the heat kernel estimates. For the stochastic term, more work is required, as we need to control the L^p-norm of the supremum of the term over the space-time interval $[0,T]\times[0,1]$. We can achieve this by a standard application of the Garisa-Rodemich-Rumsey Lemma.

Proposition 4.3. *Let $f \in L^p(\Omega; L^\infty([0,T]\times[0,1]))$. Define $F(t,x)$ such that, for $t\in[0,T]$ and $x \in (0,1]$,*

$$F(t,x) := \int_0^t\int_0^1 G(t-s,x,y)f(s,y)\,dy\,dy.$$

Then we have that, for $p > 2$,

1. $F \in C([0,T];\mathscr{H})$ *almost surely.*
2.
$$\mathbb{E}\left[\|F\|_{\mathscr{H},T}^p\right] \leq C_{p,T}\int_0^t \mathbb{E}\left[\|f\|_{\infty,s}^p\right]ds.$$

Proof. $F \in C([0,T];\mathscr{H})$ is clear, since F is simply the solution to the heat equation with Dirichlet boundary conditions at $x=0,1$, initial data 0 and source term f. For the second part, we note that

$$\begin{aligned}\left|\int_0^t\int_0^1 \frac{1}{x}G(t-s,x,y)f(s,y)\mathrm{d}y\mathrm{d}s\right|^p \leq& \left|\int_0^t\left(\int_0^1 \frac{1}{x}G(t-s,x,y)\mathrm{d}y\right)\|f\|_{\infty,s}\mathrm{d}s\right|^p\\ \leq& C_T\left[\int_0^t\left(\int_0^1 \frac{1}{x}G(t-s,x,y)\mathrm{d}y\right)^q\mathrm{d}s\right]\times\int_0^t\mathbb{E}\left[\|f\|_{\infty,s}^p\right]\mathrm{d}s,\end{aligned}$$

where $q=p/(p-1)$. By estimate (1) in Proposition 4.2, we have that

$$\left[\int_0^t\left(\int_0^1 \frac{1}{x}G(t-s,x,y)\mathrm{d}y\right)^q\mathrm{d}s\right] \leq C_{T,p}\int_0^t\left(\frac{1}{\sqrt{t-s}}\right)^q\mathrm{d}s = C_{T,p}.$$

We therefore have the result. □

Proposition 4.4. *Let $f \in L^p(\Omega;L^\infty([0,T];\mathscr{H}))$. Define $J(t,x)$ so that*

$$J(t,x) := \int_0^t\int_0^1 \frac{\partial G}{\partial y}(t-s,x,y)f(s,y)\,dy\,ds.$$

Then we have that

1. $J \in C([0,T];\mathscr{H})$ *almost surely.*
2.
$$\mathbb{E}\left[\|J\|_{\mathscr{H},T}^p\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|_{\mathscr{H},s}^p\right]ds.$$

Proof. We have that, for $x \in (0,1]$,

$$\begin{aligned}\frac{J(t,x)}{x} :=& \int_0^t\int_0^1 \frac{y}{x}\frac{\partial G}{\partial y}(t-s,x,y)\frac{f(s,y)}{y}\mathrm{d}y\mathrm{d}s\\ =&\int_0^t\int_0^1 \frac{y}{x}\frac{\partial G}{\partial y}(s,x,y)\frac{f(t-s,y)}{y}\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Note that, by the estimates in Proposition A.6 in [4], we know that $\frac{J(t,x)}{x}$ is well defined by our integral expression and continuous on $[0,T]\times(0,1]$. So it is left to show that $\frac{J(t,x)}{x}$ can be extended to a continuous function on $[0,T]\times[0,1]$. In the following arguments, we focus on the G_1 component of G (see Remark 4.1). Suppose that (t_n,x_n) is a sequence in $[0,T]\times[0,1]$ converging to (t,x). Note that

$$\begin{aligned}&\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\frac{y}{x_n}\left|\frac{\partial G_1}{\partial y}(s,x_n,y)\frac{f(t_n-s,y)}{y}\right|\mathrm{d}y\mathrm{d}s\\ \leq& 2\|f\|_{\mathscr{H},T}\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\left|\frac{\partial G_1}{\partial y}(s,x_n,y)\right|\mathrm{d}y\mathrm{d}s.\end{aligned}$$

For $p \in (1,2)$, we have that this is at most

$$2\|f\|_{\mathscr{H},T}\left(\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y<2x_n\}}\mathrm{d}y\mathrm{d}s\right)^{1/q} \times \left(\int_0^{t_n}\int_0^1 \left|\frac{\partial G_1}{\partial y}(s,x_n,y)\right|^p \mathrm{d}y\mathrm{d}s\right)^{1/p}. \tag{4.4}$$

Since

$$\sup_{t\in[0,T]}\sup_{x\in[0,1]}\int_0^t\int_0^1 \left|\frac{\partial G_1}{\partial y}(s,x,y)\right|^p \mathrm{d}y\mathrm{d}s < \infty,$$

we have that (4.4) converges to zero as $n \to \infty$. Now note that

$$\mathbb{1}_{\{y\geq 2x_n, s\in[0,t_n]\}}\frac{y}{x_n}\frac{\partial G_1}{\partial y}(s,x_n,y)\frac{f(t_n-s,y)}{y} \to \mathbb{1}_{\{s\in[0,t]\}}y\frac{\partial^2 G}{\partial y\partial x}(s,0,y)\frac{f(t-s,y)}{y}$$

almost everywhere in $[0,T]\times[0,1]$. In addition, we have that for $y \geq 2x_n$,

$$\begin{aligned}
\sqrt{4\pi}\left|\frac{y}{x_n}\frac{\partial G_1}{\partial y}(s,x_n,y)\right| &= \left|\frac{y}{x_n s\sqrt{s}}\left((x_n-y)e^{-(x_n-y)^2/4s} + (x_n+y)e^{-(x_n+y)^2/4s}\right)\right| \\
&\leq \left|\frac{y}{s\sqrt{s}}\left(e^{-(x_n-y)^2/4s}+e^{-(x_n+y)^2/4s}\right)\right| + \left|\frac{y^2}{x_n s\sqrt{s}}\left(e^{-(x_n-y)^2/4s}-e^{-(x_n+y)/4s}\right)\right| \\
&\leq \left|\frac{y}{s\sqrt{s}}\left(e^{-(x_n-y)^2/4s}+e^{-(x_n+y)^2/4s}\right)\right| + \left|\frac{y^2}{x_n s\sqrt{s}}e^{-(x_n-y)^2/4s}\left(1-e^{-x_n y/s}\right)\right| \\
&\leq \left|\frac{y}{s\sqrt{s}}\left(e^{-(x_n-y)^2/4s}+e^{-(x_n+y)^2/4s}\right)\right| + \left|\frac{y^2}{x_n s\sqrt{s}}e^{-(x_n-y)^2/4s}\left(1-e^{-x_n y/s}\right)\right|.
\end{aligned} \tag{4.5}$$

Since $y \geq 2x_n \geq 0$, we have that

$$e^{-(x_n-y)^2/4s} \leq e^{-y^2/16s}.$$

Note also that, for $x \geq 0$, $1-e^{-x} \leq x$. Therefore, we can bound (4.5) by

$$\left|\frac{y}{s\sqrt{s}}\left(e^{-y^2/16s}+e^{-y^2/4s}\right)\right| + \left|\frac{y^2}{x_n s\sqrt{s}}e^{-y^2/16s}\times\frac{x_n y}{s}\right| \leq C\left(\frac{y}{s\sqrt{s}}e^{-y^2/16s}+\frac{y^3}{s^2\sqrt{s}}e^{-y^2/16s}\right).$$

We can similarly bound the other components of G, to obtain that, for every n,

$$\sqrt{4\pi}\left|\frac{y}{x_n}\frac{\partial G}{\partial y}(s,x_n,y)\right| \leq C\left(\frac{y}{s\sqrt{s}}e^{-y^2/16s}+\frac{y^3}{s^2\sqrt{s}}e^{-y^2/16s}\right). \tag{4.6}$$

It follows that, for every n,

$$\left|\mathbb{1}_{\{y\geq 2x_n, s\in[0,t_n]\}}\frac{y}{x_n}\frac{\partial G}{\partial y}(s,x_n,y)\frac{f(t_n-s,y)}{y}\right| \leq C\left(\frac{y}{s\sqrt{s}}e^{-y^2/16s}+\frac{y^3}{s^2\sqrt{s}}e^{-y^2/16s}\right)\|f\|_{\mathscr{H},T}$$

This function is integrable, and so we can apply the DCT to obtain that

$$\begin{aligned}
&\int_0^{t_n}\int_0^1 \mathbb{1}_{\{y\geq 2x_n\}}\frac{y}{x_n}\frac{\partial G}{\partial y}(t_n-s,x_n,y)\frac{f(s,y)}{y}\mathrm{d}y\mathrm{d}s \\
&= \int_0^{t_n}\int_0^1 \mathbb{1}_{\{y\geq 2x_n\}}\frac{y}{x_n}\frac{\partial G}{\partial y}(s,x_n,y)\frac{f(t_n-s,y)}{y}\mathrm{d}y\mathrm{d}s \to \int_0^t\int_0^1 \frac{\partial^2 G}{\partial y\partial x}(s,0,y)f(t-s,y)\mathrm{d}y\mathrm{d}s \\
&= \int_0^t\int_0^1 \frac{\partial^2 G}{\partial y\partial x}(t-s,0,y)f(s,y)\mathrm{d}y\mathrm{d}s.
\end{aligned}$$

Another application of the DCT (note that we can use the same dominating function as in (4.6)) gives that, if t_n is a sequence in $[0,T]$ and $t_n \to t$, then

$$\int_0^{t_n}\int_0^1 \frac{\partial^2 G}{\partial y \partial x}(t_n - s, 0, y) f(s,y) \mathrm{d}y \mathrm{d}s \to \int_0^{t}\int_0^1 \frac{\partial^2 G}{\partial y \partial x}(t - s, 0, y) f(s,y) \mathrm{d}y \mathrm{d}s.$$

So we have shown that $J \in C([0,T];\mathscr{H})$ almost surely.

We now prove the bound (2). For $p > 2$, we have that

$$\left|\frac{J(t,x)}{x}\right| \leq \left(\int_0^t \left[\int_0^1 \left|\frac{y}{x}\frac{\partial G}{\partial y}(s,x,y)\right| \mathrm{d}y\right]^q \mathrm{d}s\right)^{1/q} \times \left(\int_0^t \|f\|^p_{\mathscr{H},s} \mathrm{d}s\right)^{1/p}.$$

Arguing as before, considering the cases $y < 2x$ and $y \geq 2x$ separately, we have that

$$\left|\frac{y}{x}\frac{\partial G}{\partial y}(s,x,y)\right| \leq C\left(\left|\frac{\partial G}{\partial y}(s,x,y)\right| + \frac{y}{s\sqrt{s}}e^{-y^2/16s} + \frac{y^3}{s^2\sqrt{s}}e^{-y^2/16s}\right).$$

For $p > 2$ and $q = p/(p-1) \in (1,2)$, and we have that

$$\int_0^t \left[\int_0^1 \left(\left|\frac{\partial G}{\partial y}(s,x,y)\right| + \frac{y}{s\sqrt{s}}e^{-y^2/16s} + \frac{y^3}{s^2\sqrt{s}}e^{-y^2/16s}\right) \mathrm{d}y\right]^q \mathrm{d}s \leq C\int_0^t \left(\frac{1}{\sqrt{s}}\right)^q \mathrm{d}s = C_{T,p} < \infty.$$

It follows that, for $p > 2$,

$$\left|\frac{J(t,x)}{x}\right| \leq C_{T,p}\left(\int_0^t \|f\|^p_{\mathscr{H},s} \mathrm{d}s\right)^{1/p}.$$

Therefore,

$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\left|\frac{J(t,x)}{x}\right|^p \leq C_{T,p}\int_0^t \|f\|^p_{\mathscr{H},s} \mathrm{d}s.$$

The result then follows by taking expectations. □

Proposition 4.5. *Suppose that $p > 8$ and let $f \in L^p(\Omega; L^\infty([0,T];\mathscr{H}))$. Define $K(t,x)$ such that, for $t \in [0,T]$ and $x \in [0,1]$,*

$$K(t,x) := \int_0^t \int_0^1 G(t-s,x,y) f(s,y) W(dy,ds).$$

Then we have that

1. *$K \in C([0,T];\mathscr{H})$ almost surely.*
2. $$\mathbb{E}\left[\|K\|^p_{\mathscr{H},t}\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},s}\right] ds.$$

Proof. In this case, we will need to apply the Garisa-Rodemich-Rumsey Lemma to obtain continuity of $\frac{K(t,x)}{x}$, and to suitably control the supremum of this process. Define for $t \in [0,T]$ and $x \in [0,1]$

$$L(t,x) := \int_0^t \int_0^1 \tilde{G}(t-s,x,y)\frac{f(s,y)}{y} W(\mathrm{d}y,\mathrm{d}s).$$

Note that, for $t \in [0,T]$ and $x \in (0,1]$, we then have that $L(t,x) = K(t,x)/x$. For $x, z \in (0,1]$ and $s \leq t \in [0,T]$, we have

$$\begin{aligned}\mathbb{E}\left[|L(t,x) - L(s,z)|^p\right] \leq & \mathbb{E}\left[\left|\int_s^t \int_0^1 \tilde{G}(t-r,x,y)\frac{f(r,y)}{y} W(\mathrm{d}y,\mathrm{d}r)\right|^p\right] \\ & + \mathbb{E}\left[\left|\int_0^s \int_0^1 \left[\tilde{G}(t-r,x,y) - \tilde{G}(s-r,z,y)\right]\frac{f(r,y)}{y} W(\mathrm{d}y,\mathrm{d}r)\right|^p\right].\end{aligned} \tag{4.7}$$

Applying Burkholder's inequality gives that the first term on the right hand side is at most

$$\mathbb{E}\left[\left|\int_s^t \int_0^1 \left(\tilde{G}(t-r,x,y)\frac{f(r,y)}{y}\right)^2 \mathrm{d}y\mathrm{d}r\right|^{p/2}\right] \leq \mathbb{E}\left[\left|\int_s^t \left(\int_0^1 \tilde{G}(t-r,x,y)^2 \mathrm{d}y\right) \|f\|^2_{\mathscr{H},r}\mathrm{d}r\right|^{p/2}\right].$$

An application of Hölder's inequality then bounds this by

$$\left[\int_s^t \left(\int_0^1 \tilde{G}(t-r,x,y)^2\mathrm{d}y\right)^{p/(p-2)} \mathrm{d}r\right]^{(p-2)/2} \times \int_s^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\mathrm{d}r\right].$$

By estimate (2) in Proposition 4.2, this can be bounded by

$$C|t-s|^{(p-4)/4} \times \int_s^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\mathrm{d}r\right].$$

Arguing similarly for the second term in (4.7), making use of the other estimates (3) and (4) in Proposition 4.2, we see that

$$\mathbb{E}\left[|L(t,x) - L(s,z)|^p\right] \leq C_{T,p}\left(|t-s|^{1/4} + |x-z|^{1/6}\right)^{p-4} \times \int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r.$$

Let d be the metric on $[0,T]\times[0,1]$ given by

$$d((t,x),(s,y)) := |t-s|^{1/4} + |x-y|^{1/6}.$$

Then, applying Corollary A.3 from [2] with respect to this metric, we see that there exists a random variable $X \geq 0$ such that

1. $\mathbb{E}\left[X^p\right] \leq C_{T,p}\int_0^t \mathbb{E}\left[\|f\|^p_{\mathscr{H},r}\right]\mathrm{d}r.$

2. $$|L(t,x) - L(s,y)| \leq X\left(|t-s|^{1/4} + |x-y|^{1/6}\right)^{(p-8)/p}$$

In particular, we see that L is continuous on $[0,T]\times[0,1]$ almost surely, from which it follows that $K \in C([0,T];\mathscr{H})$ almost surely. In addition, we have that for $t \in [0,T]$ and $x \in [0,1]$

$$|L(t,x)| = |L(t,x) - L(0,x)| \leq C_{T,p} \times X.$$

It follows that

$$\mathbb{E}\left[\|K\|_{\mathscr{H},T}\right] = \mathbb{E}\left[\|L\|^p_{\infty,T}\right] \leq C_{T,p}\int_0^T \mathbb{E}\left[\|f\|^p_{\mathscr{H},t}\right]\mathrm{d}t.$$

□

5 Existence and Uniqueness

We are now in position to prove existence and uniqueness for our Stefan problem. As mentioned in the introduction, we prove existence via a Picard argument for a truncated version of the problem. Existence for the main problem is then deduced by concatenation. Before doing so, the following truncation map is introduced.

Definition 5.1. *For $M > 0$, we define the map $F_M : \mathscr{H} \to \mathscr{H}$ such that*

1. *For $u \in \mathscr{H}$, $x \in (0,1]$,*
$$F_M(u)(x) := x\left[\frac{u(x)}{x} \wedge M\right].$$
2. *For $u \in \mathscr{H}$, $x = 0$,*
$$F_M(u)(x) := 0.$$

Remark 5.2. *$F_M(u)$ is a well-defined map from $\mathscr{H}$ to $\mathscr{H}$ i.e. $F_M(u) \in \mathscr{H}$ for $u \in \mathscr{H}$. Continuity on $(0,1]$ is clear, and we have defined $F_M(u)(0) = 0$. We therefore only need to argue the existence of a derivative for $F_M(u)$ at $x = 0$. Note that, for $x > 0$,*
$$\frac{F_M(u)(x)}{x} = \left[\frac{u(x)}{x} \wedge M\right].$$
This converges as $x \downarrow 0$, since $\frac{u(x)}{x}$ converges.

Theorem 5.3. *Fix some $M > 0$. Then there exists a unique solution $(v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ to the system of coupled SPDEs*
$$\begin{aligned}\frac{\partial v_M^1}{\partial t} &= \Delta v_M^1 - p'(t)\frac{\partial(F_M(v_M^1))}{\partial x} + f_{1,M}(x,t,v_M^1(t,\cdot)) + \sigma_{1,M}(x,t,v_M^1(t,\cdot))\dot{W} + \eta_M^1,\\ \frac{\partial v_M^2}{\partial t} &= \Delta v_M^2 + p'(t)\frac{\partial(F_M(v_M^2))}{\partial x} + f_{2,M}(x,t,v_M^2(t,\cdot)) + \sigma_{2,M}(x,t,v_M^2(t,\cdot))\dot{W}^- + \eta_M^2,\\ p'(t) &= h_M\left(\frac{\partial u^1}{\partial x}(t,p(t)^-), \frac{\partial u^2}{\partial x}(t,p(t)^+)\right),\end{aligned}$$
where $\dot{W}$ is space-time white noise with respect to the complete probability space $(\Omega, \mathscr{F}, \mathscr{F}_t, \mathbb{P})$, $v_M^1(t,0) = v_M^1(t,1) = v_M^2(t,0) = v_M^2(t,1) = 0$ and initial data $v_M^1(0,x) = u^1(x)$, $v_M^2(0,x) = u^2(x)$. The functions $f_{i,M}$, $\sigma_{i,M}$ and h_M here are given by
$$\begin{aligned}f_{i,M}(x,t,u) &:= f_i(x,t,F_M(u)),\\ \sigma_{i,M}(x,t,u) &:= \sigma_i(x,t,F_M(u)),\\ h_M(u,v) &:= h(F_M(u),F_M(v)).\end{aligned}$$

Proof. Note that existence and uniqueness for the problem on the infinite time interval $[0,\infty)$ follow from existence and uniqueness on the finite time intervals $[0,T]$, for $T > 0$. Fix some $T > 0$ and some $p > 8$. We perform a Picard iteration in the space $L^p(\Omega, C([0,T];\mathscr{H})) \times L^p(\Omega, C([0,T];\mathscr{H}))$. Let $v^{i,0}(t,x) = u^i(x)$ for $i = 1,2$. Given $(v^{1,n}, v^{2,n}) \in L^p(\Omega; C([0,T];\mathscr{H})) \times L^p(\Omega; C([0,T];\mathscr{H}))$, we define $z^{i,n+1}$ to be the solution to
$$\begin{aligned}\frac{\partial z^{1,n+1}}{\partial t} =& \Delta z^{1,n+1} - h_M\left(\frac{\partial v^{1,n}}{\partial x}(t,0^-), \frac{\partial v^{2,n}}{\partial x}(t,0^+)\right)\frac{\partial(F_M(v^{1,n}))}{\partial x}\\ &+ f_{1,M}(x,t,v^{1,n}(t,\cdot)) + \sigma_{1,M}(x,t,v^{1,n}(t,\cdot))\dot{W}(\mathrm{d}x,\mathrm{d}t),\end{aligned} \tag{5.1}$$

for $i = 1, 2$. We then define, for $i = 1, 2$, $w^{i,n+1}$ to be given by the solution to the obstacle problem

$$\frac{\partial w^i_{n+1}}{\partial t} = \Delta w^i_{n+1} + \eta^i_{n+1}, \qquad w^i_{n+1} \geq -z^i_{n+1},$$
$$\int_0^T \int_0^1 \left[w^i_{n+1}(t,x) + z^i_{n+1}(t,x) \right] \eta^i_{n+1}(\mathrm{d}x, \mathrm{d}t) = 0,$$

and set our $(n+1)^{\text{th}}$ approximate solution to be given by $(v^{1,n+1}, v^{2,n+1}) := (z^{1,n+1} + w^{1,n+1}, z^{2,n+1} + w^{2,n+1})$. For ease of notation, we define here

$$g_n(t,x) := h_M \left(\frac{\partial v^{1,n}}{\partial x}(t, 0^-), \frac{\partial v^{2,n}}{\partial x}(t, 0^+) \right) F_M(v^{1,n}(t,x)).$$

Writing $z^{1,n+1}$ in mild form, we have that

$$\begin{aligned} z^{1,n+1}(t,x) = &\int_0^1 G(t,x,y) u^1_0(y) \mathrm{d}y \\ &+ \int_0^t \int_0^1 \frac{\partial G}{\partial y}(t-s,x,y) g_n(s,y) \mathrm{d}y \mathrm{d}s \\ &+ \int_0^t \int_0^1 G(t-s,x,y) f_{1,M}(y,s,v^{1,n}(s,\cdot)) \mathrm{d}x \mathrm{d}s \\ &+ \int_0^t \int_0^1 G(t-s,x,y) \sigma_{1,M}(y,s,v^{1,n}(s,\cdot)) \mathrm{W}(\mathrm{d}y, \mathrm{d}s). \end{aligned}$$

Note that, by Propositions 4.3, 4.4 and 4.5, we can see from this expression that $z^{1,n+1} \in C([0,T]; \mathscr{H})$ almost surely. Using the analogous expression for $z^{1,n}$ and taking the difference, we see that

$$\begin{aligned} &\mathbb{E}\left[\| z^{1,n+1} - z^{1,n} \|^p_{\mathscr{H},T} \right] \\ \leq &\mathbb{E}\left[\left\| \int_0^t \int_0^1 \frac{\partial G}{\partial y}(t-s,x,y) \left[g_n(s,y) - g_{n-1}(s,y) \right] \mathrm{d}y \mathrm{d}s \right\|^p_{\mathscr{H},T} \right] \\ &+ \mathbb{E}\left[\left\| \int_0^t \int_0^1 G(t-s,x,y) \left[f_{1,M}(y,s,v^{1,n}(s,\cdot)) - f_{1,M}(y,s,v^{1,n-1}(s,\cdot)) \right] \mathrm{d}x \mathrm{d}s \right\|^p_{\mathscr{H},T} \right] \\ &+ \mathbb{E}\left[\left\| \int_0^t \int_0^1 G(t-s,x,y) \left[\sigma_{1,M}(y,s,v^{1,n}(s,\cdot)) - \sigma_{1,M}(y,s,v^{1,n-1}(s,\cdot)) \right] \mathrm{W}(\mathrm{d}y, \mathrm{d}s) \right\|^p_{\mathscr{H},T} \right]. \end{aligned} \tag{5.2}$$

Note that

$$\| g_n - g_{n-1} \|_{\mathscr{H},s} \leq C_{M,h} \left(\| v^{1,n} - v^{1,n-1} \|_{\mathscr{H},s} + \| v^{2,n} - v^{2,n-1} \|_{\mathscr{H},s} \right). \tag{5.3}$$

By making use of Propositions 4.3, 4.4 and 4.5, we are able to bound (5.2). We obtain that

$$\begin{aligned} \mathbb{E}\left[\| z^{1,n+1} - z^{1,n} \|^p_{\mathscr{H},T} \right] \leq &C_{T,p} \int_0^T \mathbb{E}\left[\| f_{1,M}(\cdot,s,v^{1,n}(s,\cdot)) - f_{1,M}(\cdot,s,v^{1,n-1}(s,\cdot)) \|^p_\infty \right] \mathrm{d}s \\ &+ \int_0^T \mathbb{E}\left[\| g_n - g_{n-1} \|^p_{\mathscr{H},s} \right] \mathrm{d}s \\ &+ \int_0^T \mathbb{E}\left[\| \sigma_{1,M}(\cdot,s,v^{1,n}(s,\cdot)) - \sigma_{1,M}(\cdot,s,v^{1,n-1}(s,\cdot)) \|^p_{\mathscr{H}} \right] \mathrm{d}s. \end{aligned}$$

By the Lipschitz-type conditions on f, σ and the inequality (5.3), we see that this is at most

$$C_{M,h,T,p}\int_0^T \mathbb{E}\left[\|v^{1,n}-v^{1,n-1}\|^p_{\mathscr{H},s}+\|v^{2,n}-v^{2,n-1}\|^p_{\mathscr{H},s}\right]\mathrm{d}s.$$

By arguing in the same way, we obtain the same bound for $\mathbb{E}\left[\|z^{2,n+1}-z^{2,n}\|^p_{\mathscr{H},T}\right]$. Adding these together gives that

$$\begin{aligned}&\mathbb{E}\left[\|z^{1,n+1}-z^{1,n}\|^p_{\mathscr{H},T}\right]+\mathbb{E}\left[\|z^{2,n+1}-z^{2,n}\|^p_{\mathscr{H},T}\right]\\&\quad\le C_{M,h,T,p}\int_0^T \mathbb{E}\left[\|v^{1,n}-v^{1,n-1}\|^p_{\mathscr{H},s}\right]+\mathbb{E}\left[\|v^{2,n}-v^{2,n-1}\|^p_{\mathscr{H},s}\right]\mathrm{d}s.\end{aligned}\tag{5.4}$$

By Theorem 3.2, we have that for $i=1,2$,

$$\|w^{i,n+1}-w^{i,n}\|_{\mathscr{H},T}\le\|z^{i,n+1}-z^{i,n}\|_{\mathscr{H},T}$$

almost surely. Therefore, for $i=1,2$,

$$\|v^{i,n+1}-v^{i,n}\|_{\mathscr{H},T}\le\|z^{i,n+1}-z^{i,n}\|_{\mathscr{H},T}\tag{5.5}$$

almost surely. It follows from (5.4) and (5.5) that

$$\begin{aligned}&\mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},T}\right]+\mathbb{E}\left[\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},T}\right]\\&\quad\le C_{M,h,T,p}\int_0^T \mathbb{E}\left[\|v^{1,n}-v^{1,n-1}\|^p_{\mathscr{H},s}\right]+\mathbb{E}\left[\|v^{2,n}-v^{2,n-1}\|^p_{\mathscr{H},s}\right]\mathrm{d}s.\end{aligned}$$

By iterating this inequality, we obtain that

$$\begin{aligned}&\mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},T}+\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},T}\right]\\
\le&C_{M,p,T}\int_0^T \mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},s}+\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},s}\right]\mathrm{d}s\\
\le&C^2_{M,p,T}\int_0^T\int_0^s \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right]\mathrm{d}u\,\mathrm{d}s\\
=&C^2_{M,p,T}\int_0^T\int_u^T \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right]\mathrm{d}s\,\mathrm{d}u\\
=&C^2_{M,p,T}\int_0^T \mathbb{E}\left[\|v^{1,n-1}-v^{1,n-2}\|^p_{\mathscr{H},u}+\|v^{2,n-1}-v^{2,n-2}\|^p_{\mathscr{H},u}\right](T-u)\,\mathrm{d}u.\end{aligned}$$

Therefore,

$$\begin{aligned}&\mathbb{E}\left[\|v^{1,n+1}-v^{1,n}\|^p_{\mathscr{H},T}+\|v^{2,n+1}-v^{2,n}\|^p_{\mathscr{H},T}\right]\\
&\quad\le C^n_{M,p,T}\int_0^T \mathbb{E}\left[\|v^{1,1}-v^{1,0}\|^p_{\mathscr{H},s}+\|v^{2,1}-v^{2,0}\|^p_{\mathscr{H},s}\right]\frac{(T-s)^{n-1}}{(n-1)!}\mathrm{d}s\\
&\quad\le C^n_{M,p,T}\times\mathbb{E}\left[\|v^{1,1}-v^{1,0}\|^p_{\mathscr{H},s}+\|v^{2,1}-v^{2,0}\|^p_{\mathscr{H},s}\right]\frac{T^n}{n!}.\end{aligned}$$

Hence, for $m > n \geq 1$, we have that

$$\begin{aligned}&\mathbb{E}\left[\|v^{1,n+1} - v^{1,n}\|^p_{\mathscr{H},T} + \|v^{2,n+1} - v^{2,n}\|^p_{\mathscr{H},T}\right]\\&\quad\leq \sum_{k=n}^{m-1}\left[\frac{\tilde{C}^k_{M,p,T}T^k}{k!}\right]\mathbb{E}\left[\|v^1_1 - v^1_0\|^p_{\mathscr{H},T} + \|v^2_1 - v^2_0\|^p_{\mathscr{H},T}\right] \to 0.\end{aligned}$$

as $m, n \to \infty$. Therefore, the sequence $(v^{1,n}, v^{2,n})$ is Cauchy in the space $L^p(\Omega; C([0,T]; \mathscr{H})))^2$ and so converges to some pair (v^1_M, v^2_M). We now verify that this is indeed a solution to our evolution equation. Let (z^1_M, z^2_M) solve the SPDEs

$$\frac{\partial z^1_M}{\partial t} = \Delta z^1_M - p'(t)\frac{\partial(F_M(v^1_M))}{\partial x} + f_{1,M}(x,t,v^1_M) + \sigma_{1,M}(x,t,v^1_M)\dot{W},$$

$$\frac{\partial z^2_M}{\partial t} = \Delta z^2_M + p'(t)\frac{\partial(F_M(v^2_M))}{\partial x} + f_{2,M}(x,t,v^2_M) + \sigma_{2,M}(x,t,v^2_M)\dot{W}^-,$$

where

$$p'(t) := h_M\left(\frac{\partial v^1_M}{\partial x}(t,0^-), \frac{\partial v^2_M}{\partial x}(t,0^+)\right).$$

We then define w^i_M as solutions to the obstacle problem, with obstacles z^i_M, and set $\tilde{v}^i_M := z^i_M + w^i_M$. Therefore, we have that

$$\frac{\partial \tilde{v}^1_M}{\partial t} = \Delta \tilde{v}^1_M - p'(t)\frac{\partial(F_M(v^1_M))}{\partial x} + f_{1,M}(x,t,v^1_M) + \sigma_{1,M}(x,t,v^1_M)\dot{W} + \eta^1_M,$$

$$\frac{\partial \tilde{v}^2_M}{\partial t} = \Delta \tilde{v}^2_M + p'(t)\frac{\partial(F_M(v^2_M))}{\partial x} + f_{2,M}(x,t,v^2_M) + \sigma_{2,M}(x,t,v^2_M)\dot{W} + \eta^2_M.$$

By reproducing the same argument as that used to achieve the estimate (5.5), we are able to show that

$$\begin{aligned}&\mathbb{E}\left[\|\tilde{v}^1_M - v^{1,n}\|^p_{\mathscr{H},T} + \|\tilde{v}^2_M - v^{2,n}\|^p_{\mathscr{H},T}\right]\\&\quad\leq C_{M,h,T,p}\int_0^T \mathbb{E}\left[\|v^1_M - v^{1,n}\|^p_{\mathscr{H},s} + \|v^2_M - v^{2,n}\|^p_{\mathscr{H},s}\right]\mathrm{d}s \to 0.\end{aligned}$$

It follows that $\tilde{v}^1 = v^1$ in $L^p(\Omega; C([0,T]; \mathscr{H}))$. The same applies to v^2, so it follows that the pair (v^1, v^2), together with the reflection measures $(\tilde{\eta}^1_M, \tilde{\eta}^2_M)$, do indeed satisfy our problem.

Uniqueness follows by essentially the same argument. Given two solutions with the same initial data, (v^1_1, v^2_1) and (v^1_2, v^2_2) (together with their reflection measures), we argue as before to obtain that, for $t \in [0,T]$,

$$\mathbb{E}\left[\|v^1_1 - v^1_2\|^p_{\mathscr{H},t} + \|v^2_1 - v^2_2\|^p_{\mathscr{H},t}\right] \leq \int_0^t \mathbb{E}\left[\|v^1_1 - v^1_2\|^p_{\mathscr{H},s} + \|v^2_1 - v^2_2\|^p_{\mathscr{H},s}\right]\mathrm{d}s.$$

The equivalence then follows by Gronwall's inequality. □

Before stating and proving our main theorem, we first prove the following technical result which gives us a universal representation for terminal times of maximal solutions, and a canonical localising sequence.

Proposition 5.4. *Let $(v^1, \eta^1, v^2, \eta^2)$ be a maximal solution to the moving boundary problem in the relative frame. Then we have that the terminal time, τ, is given by*

$$\tau := \sup_{M>0} \tau_M,$$

where

$$\tau_M := \inf \left\{ t \geq 0 \mid \|v^1\|_{\mathscr{H},t} + \|v^2\|_{\mathscr{H},s} \geq M \right\}.$$

Furthermore, τ_M can be taken as a localising sequence for the solution.

Proof. Let σ_n be a localising sequence for the solution. Define $\sigma := \sup_{n\geq 1} \sigma_n$. Fix some $M > 0$ and consider the localising sequence $(\sigma_n \wedge \tau_M)_{n\geq 1}$ for the solution, with this solution on a potentially smaller stochastic interval. Taking the limit as $n \to \infty$ for each $M > 0$, we obtain a local solution which agrees with $(v^1, \eta^1, v^2, \eta^2)$ until $\tau \wedge \sigma$ (its own terminal time), and has the localising sequence $(\sigma \wedge \tau_M)_{M>0}$. Arguing as in Theorem 5.3, for every $M > 0$, this must agree with the M-truncated problem until the time $\sigma \wedge \tau_M$. Therefore, for every $M > 0$, we have that $\sigma \geq \tau_M$, as otherwise we could use the solution to the M-truncated problem to propagate the solution to a later time. Hence, the localising sequence $\sigma \wedge \tau_M$ is simply just τ_M. In addition, we have that $\sigma \geq \tau_M$ for every M almost surely, which implies that $\sigma \geq \tau$ almost surely. Clearly, by agreement with the M-truncated problems until the times τ_M, we have that $\tau \geq \sigma$, since the solution cannot be propagated beyond a blow-up in the $\mathscr{H}$-norm. We therefore have the result. □

We are now in position to prove our main result, Theorem 2.11.

Proof of Theorem 2.11. We prove existence of a maximal solution by concatenating solutions to the truncated problems. For $M > 0$, let $\mathscr{V}_M := (v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ solve the M-truncated problem as in Theorem 5.3. Note that for $M_1 \geq M_2 > 0$, $\mathscr{V}_{M_1}$ solves (2.2) until the random stopping time

$$\tau := \inf \left\{ t \in [0, \infty) \mid \max \left\{ \|v_{M_1}^1\|_{\mathscr{H},t}, \|v_{M_1}^2\|_{\mathscr{H},t} \right\} \geq M_2 \right\}.$$

Let $z_{M_j}^i$ solve the unreflected SPDEs associated with $v_{M_j}^i$ for $i, j = 1, 2$, as in the equation (5.1). By considering the difference between $z_{M_1}^i(t,x)\mathbb{1}_{\{t\leq\tau\}}$ and $z_{M_2}^i(t,x)\mathbb{1}_{\{t\leq\tau\}}$ for $i = 1, 2$, we are able to argue as in the proof of Theorem 5.3 to obtain that, for $T > 0$, $p > 8$ and $t \in [0, T]$

$$\begin{aligned}
&\mathbb{E}\left[\|z_{M_1}^1 - z_{M_2}^1\|_{\mathscr{H},\tau\wedge t}^p + \|z_{M_1}^2 - z_{M_2}^2\|_{\mathscr{H},\tau\wedge t}^p\right] \\
&\quad \leq C_{T,p} \int_0^t \mathbb{E}\left[\|z_{M_1}^1 - z_{M_2}^1\|_{\mathscr{H},\tau\wedge s}^p + \|z_{M_1}^2 - z_{M_2}^2\|_{\mathscr{H},\tau\wedge s}^p\right] \mathrm{d}s.
\end{aligned}$$

Gronwall's lemma then applies to give that $z_{M_1}^i = z_{M_2}^i$ almost surely for $i = 1, 2$, until the random time $\tau \wedge T$. Since $(v_{M_j}^i, \eta_{M_j}^i)$ are simply the solutions to the obstacle problem with obstacles $z_{M_j}^i$, it follows that $(v_{M_1}^1, \eta_{M_1}^1, v_{M_1}^2, \eta_{M_1}^2) = (v_{M_2}^1, \eta_{M_2}^1, v_{M_2}^2, \eta_{M_2}^2)$ until time $\tau \wedge T$. This holds for every $T > 0$, so we have that they almost surely agree until time τ We have therefore shown that the solutions to our M-truncated problems are consistent. This allows us to concatenate them. We define the process $(u^1, \eta^1, u^2, \eta^2)$ such that, for all $M > 0$, it agrees with $(v_M^1, \eta_M^1, v_M^2, \eta_M^2)$ until the time

$$\tau_M := \inf \left\{ t \in [0, \infty) \mid \max \left\{ \|v_M^1\|_{\mathscr{H},t}, \|v_M^2\|_{\mathscr{H},t} \right\} \geq M \right\}.$$

Note that

$$\tau_M = \inf\left\{t \in [0,\infty) \mid \max\left\{\|u^1\|_{\mathscr{H},t}, \|u^2\|_{\mathscr{H},t}\right\} \geq M\right\}.$$

Since $\mathscr{V}_M$ solves (2.2) until time τ_M, we have that $(u^1, \eta^1, u^2, \eta^2)$ solves (2.2) on the random interval $[0, \tilde{\tau})$, where

$$\tilde{\tau} = \sup_{M>0} \tau_M,$$

with localising sequence τ_M. This solution is maximal since, on the set $\{\tilde{\tau} < \infty\}$, we have that

$$\max\left\{\|u^1_M\|_{\mathscr{H},t}, \|u^2_M\|_{\mathscr{H},t}\right\} \to \infty$$

as $t \uparrow \tilde{\tau}$ almost surely. Therefore, we know that a maximal solution exists. Uniqueness follows by the same arguments as those made for consistency among solutions to the truncated. Fix $T > 0$ and let $(\tilde{u}^1, \tilde{\eta}^1, \tilde{u}^2, \tilde{\eta}^2)$ be a maximal solution to (2.2). Note that the localising sequence can be taken as in Proposition 5.4. We then have that for every $M > 0$, it must agree with the solution to the M-truncated problem, and therefore with $(u^1, \eta^1, u^2, \eta^2)$, until

$$\inf\left\{t \in [0,T] \mid \max\left\{\|\tilde{u}^1_M\|_{\mathscr{H},t}, \|\tilde{u}^2_M\|_{\mathscr{H},t}\right\} \geq M\right\}.$$

Uniqueness then follows. □

We can now show that, as one might expect, when the boundary term depends only on the derivatives of the two SPDEs at the boundary, τ corresponds to when one of these derivatives blow up.

Proposition 5.5. *Let $(u^1, \eta^1, u^2, \eta^2)$ be a maximal solution to the moving boundary problem on the maximal interval $[0, \tau)$. Suppose that the function h in the formulation is of the form $h(u, v) = h(u'(0), v'(0))$. Then*

$$\tau = \sup_{M>0} \sigma_M,$$

where

$$\sigma_M = \inf\left\{t \in [0,\infty) \mid \max\left\{\frac{\partial u^1}{\partial x}(t, 0^-), \frac{\partial u^2}{\partial x}(t, 0^+)\right\} \geq M\right\}.$$

Proof. By Proposition 5.4, it is sufficient to show that for every $M > 0$, $\sigma_M \leq \tau$ almost surely. Suppose that, for every $T > 0$,

$$\mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))}\left[\|u^1(t,\cdot)\|^p_{\mathscr{H}} + \|u^2(t,\cdot)\|^p_{\mathscr{H}}\right]\right] < \infty. \tag{5.6}$$

Then, since

$$\sup_{t\in[0,\tau\wedge T)}\left[\|u^1(t,\cdot)\|_{\mathscr{H}} + \|u^2(t,\cdot)\|_{\mathscr{H}}\right] = \infty$$

almost surely on the set that $\{\tau \leq T\}$ we must have that $\sigma_M < \tau \wedge T$ almost surely on the set $\{\tau \leq T\}$. Therefore, for every $T > 0$, we'd have that

$$\sigma_M \mathbb{1}_{\{\tau\leq T\}} < \tau \wedge T$$

almost surely. Letting $T \uparrow \infty$ (via a countable sequence), we obtain that

$$\sigma_M \mathbb{1}_{\{\tau<\infty\}} \leq \tau$$

almost surely, so that $\sigma_M \leq \tau$ almost surely, giving the result. So it is sufficient to prove that (5.6) holds for every $M, T > 0$. Fix some $T > 0$ and some $N > 0$. By Theorem 3.2,

$$\mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))} \|u^1(t,\cdot)\|^p_{\mathscr{H}}\right] \leq 2^p \mathbb{E}\left[\sup_{t\in[0,(\sigma_M\wedge\tau\wedge T))} \|z^1(t,\cdot)\|^p_{\mathscr{H}}\right],$$

where z^1 solves the unreflected problem corresponding to u^1 on the random interval $[0, \tau\wedge T)$. That is, z^1 solves

$$\frac{\partial z^1}{\partial t} = \Delta z^1 - h\left(\frac{\partial u^1}{\partial x}(t,0^-), \frac{\partial u^2}{\partial x}(t,0^+)\right)\frac{\partial u^1}{\partial x} + f(x,u^1) + \sigma(x,u^1)\dot{W}(\mathrm{d}x,\mathrm{d}t),$$

on the interval $[0, \tau \wedge T)$, where we take the localising sequence

$$\tau_N := \inf\left\{t \in [0,\infty) \mid \max\left\{\|u^1\|_{\mathscr{H},t} + \|u^2\|_{\mathscr{H},t}\right\} \geq N\right\}.$$

Writing z^1 in mild form, we have that

$$\begin{aligned} &z^1(t,x)\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}} = \\ &\quad \mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^1 G(t,x,y)u_0^1(y)\mathrm{d}y\right] + \\ &\quad \mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 \frac{\partial G}{\partial y}(t-s,x,y)h\left(\frac{\partial u^1}{\partial x}(t,0^-), \frac{\partial u^2}{\partial x}(t,0^+)\right)u^1(s,y)\mathrm{d}y\mathrm{d}s\right] + \\ &\quad \mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 G(t-s,x,y)f(y,u^1(s,y))\mathrm{d}x\mathrm{d}s\right] + \\ &\quad \mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 G(t-s,x,y)\sigma(y,u^1(s,y))\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right], \end{aligned} \tag{5.7}$$

We aim to bound the $L^p(\Omega; L^\infty([0,T];\mathscr{H}))$ norms of the four terms on the right hand side uniformly over N. The first term is simple, so we omit the proof for this. For the second term, note that

$$\begin{aligned} &\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\left[\int_0^t\int_0^1 \frac{\partial G}{\partial y}(t-s,x,y)h\left(\frac{\partial u^1}{\partial x}(s,0^-), \frac{\partial u^2}{\partial x}(s,0^+)\right)u^1(s,y)\mathrm{d}y\mathrm{d}s\right] \\ &\qquad \leq \left[\int_0^t\int_0^1 \frac{\partial G}{\partial y}(t-s,x,y)h\left(\frac{\partial u^1}{\partial x}(s,0^-), \frac{\partial u^2}{\partial x}(s,0^+)\right)u^1(s,y)\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{d}y\mathrm{d}s\right]. \end{aligned}$$

By the linear growth condition on h, this is at most

$$J(t,x) := C_M\left[\int_0^t\int_0^1 \frac{\partial G}{\partial y}(t-s,x,y)u^1(s,y)\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{d}y\mathrm{d}s\right].$$

An application of Proposition 4.4 then gives that

$$\mathbb{E}\left[\|J\|^p_{\mathscr{H},t}\right] \leq \int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} \|u^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right] \mathrm{d}s.$$

Arguing in the same way, applying Proposition 4.3 in place of Proposition 4.4, we are able to control the third term of (5.7). For the fourth term of (5.7), we note that

$$\begin{aligned}\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}} &\left|\int_0^t\int_0^1 G(t-s,x,y)\sigma(y,u^1(s,y))\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right| \\ &\leq \left|\int_0^t\int_0^1 G(t-s,x,y)\sigma(y,u^1(s,y))\mathbb{1}_{\{s<\sigma_M\wedge\tau_N\wedge T\}}\mathrm{W}(\mathrm{d}y,\mathrm{d}s)\right| =: K(t,x).\end{aligned}$$

Applying Proposition 4.5 and noting the linear growth condition on σ then gives that, for $t\in[0,T]$

$$\mathbb{E}\left[\|K\|^p_{\mathscr{H},t}\right] \leq C_{T,p}\left[1+\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} \|u^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right] \mathrm{d}s\right].$$

Putting the four terms together, we have that, for $t\in[0,T]$,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\left\|z^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\right\|^p_{\mathscr{H}}\right] \leq C_{T,p,M}\left[1+\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} \|u^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right] \mathrm{d}s\right].$$

Therefore,

$$\mathbb{E}\left[\sup_{r\in[0,t]}\left\|u^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\right\|^p_{\mathscr{H}}\right] \leq C_{T,p,M}\left[1+\int_0^t \mathbb{E}\left[\sup_{r\in[0,s]} \|u^1(r,\cdot)\mathbb{1}_{\{r<\sigma_M\wedge\tau_N\wedge T\}}\|^p_{\mathscr{H}}\right] \mathrm{d}s\right].$$

It follows by Gronwall's inequality that

$$\sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau_N\wedge T)}\left\|u^1(t,\cdot)\right\|^p_{\mathscr{H}}\right] = \sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,T]}\left\|u^1(t,\cdot)\mathbb{1}_{\{t<\sigma_M\wedge\tau_N\wedge T\}}\right\|^p_{\mathscr{H}}\right] \leq K_{T,p,M}<\infty.$$

Note that, by the MCT

$$\sup_{N>0}\mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau_N\wedge T)}\left\|u^1(t,\cdot)\right\|^p_{\mathscr{H}}\right] = \mathbb{E}\left[\sup_{t\in[0,\sigma_M\wedge\tau\wedge T)}\left\|u^1(t,\cdot)\right\|^p_{\mathscr{H}}\right]$$

The same estimates hold for u^2. This concludes the proof. □

Proposition 5.6. *Suppose that the function h is globally bounded. Then the unique maximal solution to (2.2) is global i.e. $\tau=\infty$ almost surely.*

Proof. The proof is essentially the same as the proof of Proposition 5.5. By following the same steps as in that proof, we are able to show that, for every $T>0$

$$\mathbb{E}\left[\sup_{t\in[0,\tau\wedge T)}\left[\|u^1(t,\cdot)\|^p_{\mathscr{H}}+\|u^2(t,\cdot)\|^p_{\mathscr{H}}\right]\right]<\infty. \tag{5.8}$$

The key difference here is that in Proposition 5.5, the non-Lipschitz term from the moving boundary was controlled by noting that $h\left(\frac{\partial u^1}{\partial x}(t,0^-),\frac{\partial u^2}{\partial x}(t,0^+)\right)$ is bounded at times before the derivatives first hit a finite threshold M. We need not concern ourselves with that detail here, as we assume that h is globally bounded from the outset. It follows from (5.8) that the solution must be global. □

6 Numerical Illustrations

In this section, we implement a simple numerical scheme in order to illustrate some typical profiles for solutions to our Stefan problems and point out some of the features which were highlighted in the previous analysis. In particular, we are able to see the presence of spatial derivatives at the shared boundary when we choose the drift and volatility parameters appropriately. We will also contrast this with the behaviour at the boundary when the volatility only decays there like $\sqrt{x}$, and where the volatility is constant, providing numerical evidence that a spatial derivative does not exist at the boundary in these cases. This is of interest as we have not been able to show that the linear decay condition we required of the volatility was optimal. To begin, we briefly describe the numerical scheme used. We will consider drift and volatility functions which only depend on the position in space (in the relative frame) and the value of the solution at that particular position at that particular time. In particular, these functions will be the same for both sides of the Stefan problem. In addition, our functions h which determine the boundary behaviour will take the classical form

$$h(u,v) = \gamma\left(u'(0) - v'(0)\right).$$

We simulate the process on the interval $[0,T]$ and space interval $[p(t)-1, p(t)+1]$ at a given $t \in [0,T]$. That is, we simulate the equation in the relative frame on $[0,T]\times[-1,1]$, with v^1, v^2 supported on $[0,T]\times[0,1]$. We discretise the equation into M time steps and N space steps, and define $t_i := iT/M$, $x_i := i/N$. Given our simulated solution up to the j^{th} time step, we define the values of the approximate solution at the $(j+1)^{\text{th}}$ time step by setting

$$\begin{aligned} v^1(t_{j+1},x_i) := \Bigg| & v^1(t_j,x_i) + \frac{1}{MN^2}\Big(v^1(t_j,x_{i+1}) + v^1(t_j,x_{i-1}) - 2v^1(t_j,x_i)\Big) \\ & - \frac{1}{MN^2}\Big(v^1(t_j,x_1) - v^2(t_j,x_1)\Big)\Big(v^1(t_j,x_{i+1} - v(t_j,x_i)\Big) \\ & + \frac{1}{M} f(x_i, v^1(t_j,x_i)) + \frac{\sqrt{TN}}{\sqrt{M}}\sigma(x_i, v^1(t_j,x_i)) Z_{i,j}\Bigg|, \end{aligned}$$

where the $Z_{i,j}$ are independent unit normal random variables. $v^2(t_{j+1},x_i)$ is defined similarly. We note that taking the absolute value on the right hand side here is intended to capture the fact that the process is reflected at zero. Given our simulated processes v^1 and v^2, we are then able to put to reproduce the boundary process by noting that $p'(t) = h(v^1(t,\cdot), v^2(t,\cdot))$. We can therefore take

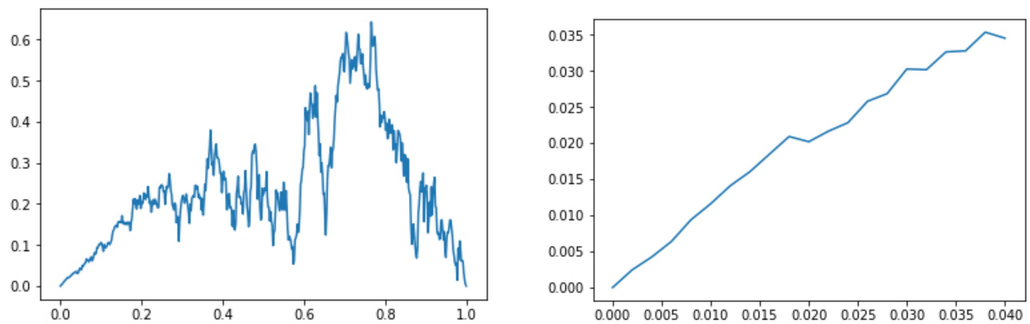
$$p(t_j) := \sum_{k=1}^{j} \frac{1}{MN}\Big(v^1(t_k,x_1) - v^2(t_k,x_1)\Big).$$

As we are interested in the effects of different volatility functions on our equations, we will fix the other parameters here. Throughout all of our simulations, we will choose the drift functions f and the boundary motion function h such that $f = 1$ and $h(u,v) = 10(u'(0) - v'(0))$. The initial data for both v^1 and v^2 will be given by the function u_0, where $u_0(0) = u_0(1) = 0$, and $u_0(0.5) = 1$, with linear interpolation in between these points. The volatility functions for which we perform our simulations are given by $\sigma_a(x), \sigma_b(u), \sigma_c(x)$ and σ_d, where

1. $\sigma_a(x)$ is linear between the points 0, 0.5 and 1, with $\sigma_a(0) = 0$, $\sigma_a(0.5) = \sigma_a(1) = 1$.
2. $\sigma_b(u) = u$.
3. $\sigma_c(x) = \sqrt{x}$.

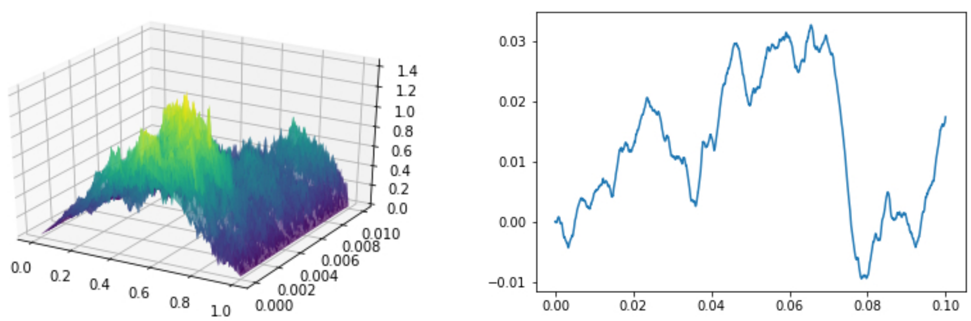
4. $\sigma_d = 1$.

The following figures depict the results of our simulations.



(a) (b)

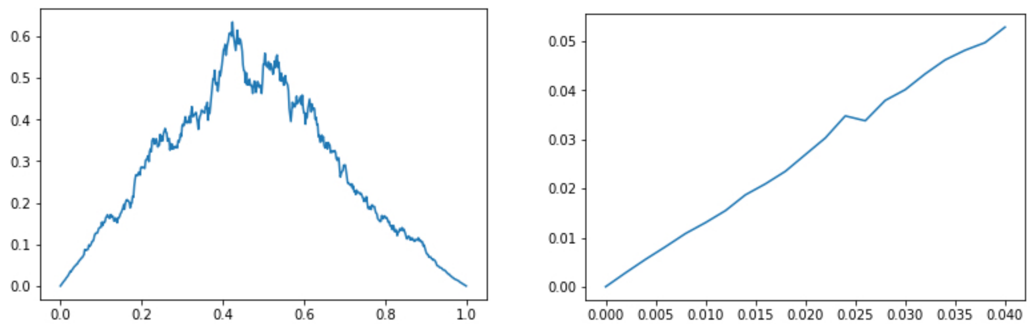
Figure 1: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_a(x)$, so that σ decays linearly at the boundary.



(a) (b)

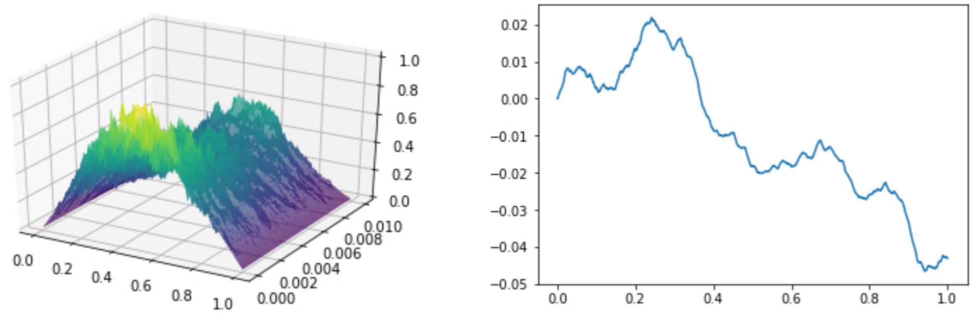
Figure 2: 3D plot of v^1 and plot of the boundary motion in the case where $\sigma = \sigma_a(x)$, so that σ decays linearly at the boundary.

We see in Figure 1 that the profile is less volatile as the boundary $x = 0$ is approached, and the derivative at the boundary is visible. The boundary function appears to be smooth to some degree in Figure 2 (b), as expected. Our second set of simulations deals with the case when the volatility is multiplicative, which also falls within the framework of our earlier analysis. Figures 3 and 4 show the results of our simulation in this case.



(a) (b)

Figure 3: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_b(u) = u$.

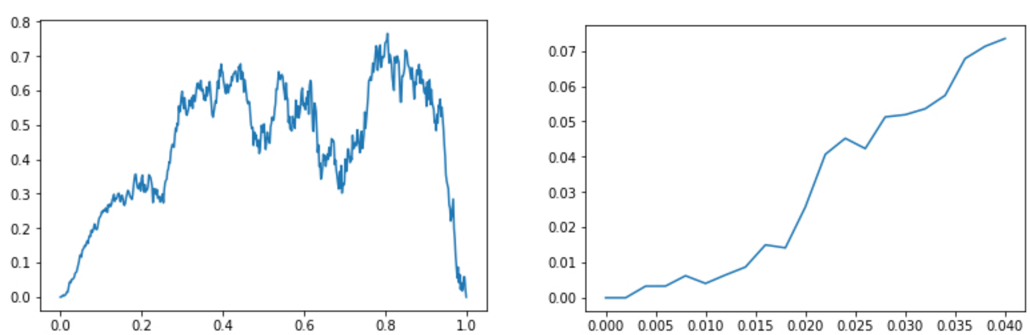


(a) (b)

Figure 4: 3D plot of v^1 and plot of the boundary motion in the case where $\sigma = \sigma_b(u) = u$.

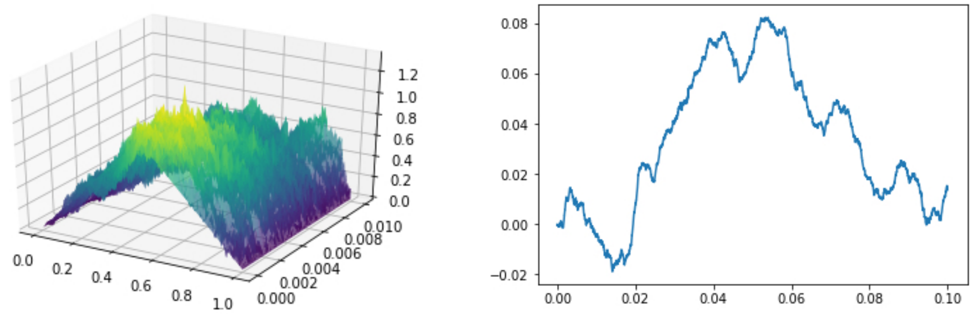
We can see the presence of spatial derivatives at both ends of the profiles. This is simply because the equation is symmetric. The Dirichlet condition is imposed at both 0 and 1, so that the volatility decays sufficiently quickly at both ends.

We now present the cases where the volatility decays like $\sqrt{x}$ at the shared boundary and where the volatility is constant, which falls outside of our proof for existence and uniqueness.



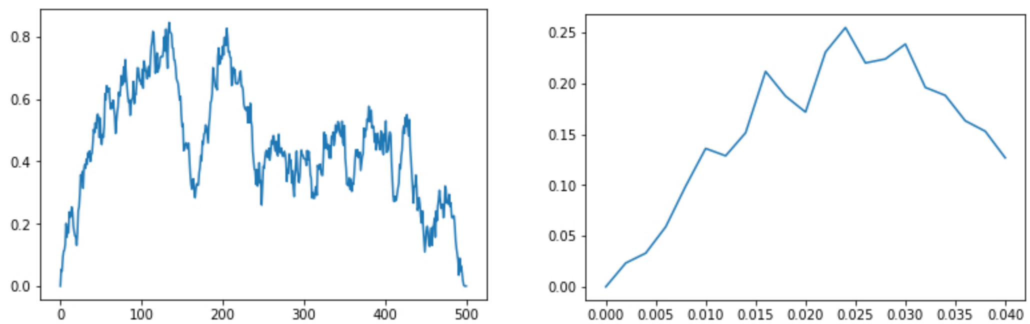
(a) (b)

Figure 5: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_c(x) = \sqrt{x}$.



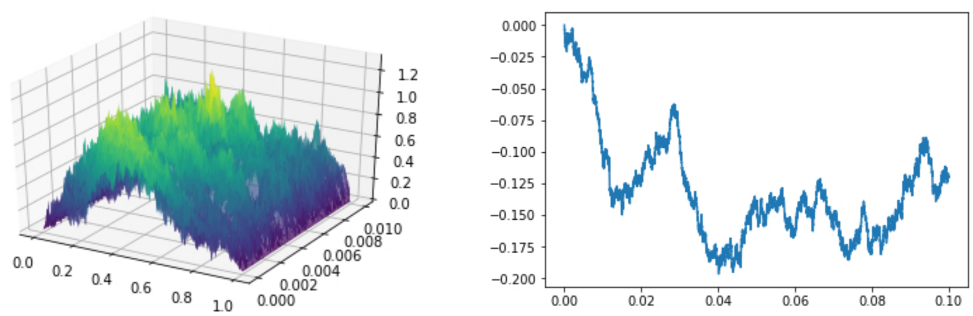
(a) (b)

Figure 6: 3D plot of v^1 and plot of the boundary motion in the case where $\sigma = \sigma_c(x) = \sqrt{x}$.



(a) (b)

Figure 7: Static Snapshot of v^1 at time t=0.1 on [0,1] and on [0,0.04], in the case where $\sigma = \sigma_d = 1$.



(a) (b)

Figure 8: 3D plot of v^1 and plot of the boundary motion in the case where $\sigma = \sigma_d = 1$.

In both of these cases, the profiles can be seen to be significantly rougher close to the boundary, and do not appear to have spatial derivatives there (although in the case when the volatility is given by $\sigma_c(x)$, this is inconclusive). Interestingly, in both instances, the boundary motion appears to be rough to some degree, with this particularly striking when the volatility is constant. We note, however, that the scheme did not blow up- we were able to obtain sensible profiles and sensible processes for the movement of the boundary. This suggests that, perhaps by considering the derivative at the boundary in a suitable weak sense, equations of this form may be solvable without the need for linear decay of the volatility, and could produce rough paths for the boundary motion. One should, however, be careful here, as the change of variables used to translate the moving boundary problem onto a relative frame required the use of the chain rule, which may no longer be applicable if we are anticipating rough paths for the boundary motion.

We perform a simple numerical test in order to give some (albeit naive) quantification of how linear the profiles are close to the boundary in the cases presented above. For a profile at a given time, we fit the values in the spatial interval $[0, 0.04]$ to a line passing through the origin by the method of least squares. Calculating the sum of the squared residuals and dividing by the square of height of the fitted line at position 0.04, we obtain values indicating how well the profile can be fitted to a line, with these values now independent of scale and so comparable across different profiles. Time averaging these values for each of the four cases, we obtain the following values, which clearly indicate that there is significant deviation from being linear close to zero when $\sigma = \sigma_c(x) = \sqrt{x}$ for $i = 1, 2$, and when $\sigma = \sigma_d = 1$.

$\sigma_a(x)$	$\sigma_b(u)$	$\sigma_c(x)$	σ_d
0.04274	0.01473	0.28378	1.19952

Table 1: Time-Averaged measure of deviation of profile from linear fit on the spatial interval $[0, 0.04]$.

A Proof of Proposition 4.2

We provide the main details for the proofs of the inequalities stated in Proposition 4.2. Recalling Remark 4.1, we focus on proving the estimates for the component J of G, where J is given by

$$J(t,x,y) = \frac{1}{\sqrt{4\pi t}}\left[\exp\left(-\frac{(x-y)^2}{4t}\right) - \exp\left(-\frac{(x+y)^2}{4t}\right)\right].$$

We note here that the proofs of the estimates in Lemma 3.2 of [10] were very helpful.

Proof of Inequality 1

Proof. The first inequality in the Proposition states that, for every $T > 0$, there exists a constant C_T such that

$$\sup_{t\in[0,T]}\sup_{x\in(0,1]}\int_0^1 \frac{1}{x}G(t,x,y)\mathrm{d}y \le \frac{C_T}{\sqrt{t}}.$$

We prove the corresponding bound for H.

$$\begin{aligned}\frac{1}{x}J(t,x,y) =& \frac{1}{x\sqrt{4\pi t}}\left[\exp\left(-(x-y)^2/4t\right) - \exp\left(-(x+y)^2/4t\right)\right]\\ =& \frac{1}{x\sqrt{4\pi t}}\exp\left(-(x-y)^2/4t\right)\left[1-\exp\left(-xy/t\right)\right].\end{aligned}$$

Note that

$$xy = x((y-x)+x) \le 2x^2\mathbb{1}_{\{y-x\le x\}} + 2x(y-x)\mathbb{1}_{\{y-x>x\}}.$$

Therefore

$$\big(1-\exp(-xy/t)\big) \le \Big(1-\exp(-2x^2/t)\Big)\,\mathbb{1}_{\{y-x\le x\}} + \big(1-\exp(-2x(y-x)/t)\big)\,\mathbb{1}_{\{y-x>x\}}.$$

Since, for $x \ge 0$

$$(1-e^{-x}) \le C\min(x,\sqrt{x}),$$

we have that

$$\big(1-\exp(-xy/t)\big) \le C\left[\frac{x}{\sqrt{t}} + \frac{x|y-x|}{t}\right].$$

Putting these bounds together, we see that

$$\begin{aligned}\int_0^1 \frac{1}{x}J(t,x,y)\,\mathrm{d}y &\le C\int_0^1 \frac{1}{\sqrt{t}}\exp\left(-\frac{(x-y)^2}{4t}\right)\left[\frac{1}{\sqrt{t}}+\frac{|y-x|}{t}\right]\mathrm{d}y \\ &= C\int_{\mathbb{R}}\left[\frac{1}{t}e^{-y^2/4t}+\frac{y}{t^{3/2}}e^{-y^2/4t}\right]\mathrm{d}y = \frac{C}{\sqrt{t}}.\end{aligned}$$

□

Proof of Inequality 3

Proof. We begin by recalling the inequality. For every $T>0$ and $q\in(0,1)$, we claim that $\exists C_{T,q}$ such that

$$\sup_{t\in[0,T]}\int_0^t\left[\int_0^1\Big(\tilde{G}(s,x,z)-\tilde{G}(s,y,z)\Big)^2\mathrm{d}z\right]^q\mathrm{d}s \le C_{T,q}|x-y|^{(2-q)/3}.$$

We will prove the corresponding bound for $\tilde{J}$, where we define

$$\tilde{J}(t,x,y) := \frac{y}{x}J(t,x,y). \tag{A.1}$$

The arguments made in the proof of [10] are followed here, and we make adjustments where necessary. Assume wlog that $x\le y$, and define $h=y-x$. By following the proof of [10], we arrive at the the inequality

$$\begin{aligned}\int_0^1\Big(\tilde{G}(s,x,z)-\tilde{G}(s,y,z)\Big)^2\mathrm{d}y \le& \frac{C}{\sqrt{s}}\Bigg[1+s\left(\frac{1-e^{-x^2/4s}}{x^2}+\frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}\right) \\ &- e^{-h^2/16s}\left(1+2s\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\Bigg] =: CR(s,x,h)\ge 0.\end{aligned}$$

So it is enough to prove that

$$\int_0^T R(s,x,h)^q\,\mathrm{d}s \le C_q h^{(2-q)/3}.$$

In the spirit of [10] once again, we control this integral by proving two separate estimates for it which we will then combine. These estimates are obtained by splitting R up into different components. Note that

$$R(s,x,h) = R_1(s,x,h)+R_2(s,x,h)\ge 0,$$

where we define

$$R_1(s,x,h) := \frac{1}{\sqrt{s}}\left(1 - e^{-h^2/16s}\right),$$

and

$$R_2(s,x,h) := \sqrt{s}\left[\frac{1-e^{-x^2/4s}}{x^2} + \frac{1-e^{-(x+h)^2/4s}}{(x+h)^2} - 2e^{-h^2/16s}\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right].$$

Therefore, if $I(s,x,h) \geq 0$ such that $R_2(s,x,h) \leq I(s,x,h)$ we have that

$$R(s,x,h)^q \leq (R_1(s,x,h) + I(s,x,h))^q \leq C_q\left(R_1(s,x,h)^q + I(s,x,h)^q\right).$$

Our approach will therefore be to bound the integral of the R_1 and I terms separately here, for two different functions I. Starting with the R_1 term, we have that for $\alpha > 0$,

$$\int_0^T R_1(s,x,h)^q \mathrm{d}s \leq \int_0^{h^\alpha}\left(\frac{1}{\sqrt{s}}\right)^q \mathrm{d}s + \int_{h^\alpha}^\infty \frac{h^2}{16s^{3/2}}\mathrm{d}s,$$

where we once again make use of the inequality $(1-e^{-x}) \leq x$. Calculating and making the choice $\alpha = q/2$, we see that this is equal to $Ch^{(2-q)/q}$. For the R_2 term, we have the bound

$$R_2(s,x,h) \leq \sqrt{s}\left[\frac{1}{2s} - 2e^{-h^2/16s}\left(\frac{1}{4s} - \frac{x(x+h)}{32s^2}\right)\right],$$

where we have made use of the fact that

$$x - \frac{x^2}{2} \leq 1 - e^x \leq x.$$

This is equal to

$$\frac{1}{2\sqrt{s}}\left[1 - e^{-h^2/16s}\right] + \frac{x(x+h)}{16s^{3/2}}e^{-h^2/16s}.$$

The first of these terms is in the same form as R_1, and so we can control it in the same way. For the second term, we have that for $\alpha > 0$,

$$\left(\frac{x(x+h)}{16s^{3/2}}e^{-h^2/16s}\right)^q = C_q x^q(x+h)^q\left(\frac{e^{-(qh^2)/16s}}{s^{3q/2}}\right) \leq C_q x^q (x+h)^q s^{(2-3q)/2}h^{-2},$$

since $e^{-x} \leq \frac{C}{x}$. This bound then gives that, for $\alpha > 0$,

$$\int_0^T \left(\frac{x(x+h)}{16s^{3/2}}e^{-h^2/16s}\right)^q \mathrm{d}s \leq C_q x^q (x+h)^q\left(\int_0^{h^\alpha} s^{(2-3q)/2}h^{-2}\,\mathrm{d}s + \int_{h^\alpha}^\infty \frac{1}{s^{3/2}}\mathrm{d}s\right).$$

Calculating and choosing $\alpha = 2$ gives that the right hand side is equal to

$$Cx^q(x+h)^q h^{2-3q}.$$

So we obtain our first bound

$$\int_0^T R(s,x,h)^q\,\mathrm{d}s \leq C_q\left[h^{(2-q)/q} + x^q(x+h)^q h^{2-3q}\right]. \tag{A.2}$$

We prove our second estimate for the integral by bounding R_2 differently. We have that

$$
\begin{aligned}
R_2(s,x,h) \leq & \sqrt{s}\left|\frac{1-e^{-x^2/4s}}{x^2}+\frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}-2\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right| \\
& \sqrt{s}\left|2\left(1-e^{-h^2/16s}\right)\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right|.
\end{aligned}
\tag{A.3}
$$

The second term on the right hand side is at most

$$\frac{C}{\sqrt{s}}\left(1-e^{-h^2/16s}\right).$$

This is once again in the form of R_1, and so can be controlled in the same way. In order to bound the first term on the right hand side of (A.3), we define here the function

$$\phi(t,x) := \frac{1-e^{-x/4s}}{x}.$$

Then we can calculate that, for $x \in [0,1]$

$$\left|\frac{\partial\phi}{\partial x}(s,x)\right| = \left|-\frac{1}{x^2}(1-e^{-x/4s})+\frac{1}{4s}e^{-x/4s}\right| \leq \frac{C}{xs}. \tag{A.4}$$

Therefore,

$$
\begin{aligned}
&\sqrt{s}\left|\frac{1-e^{-x^2/4s}}{x^2}+\frac{1-e^{-(x+h)^2/4s}}{(x+h)^2}-2\left(\frac{1-e^{-\frac{x(x+h)}{4s}}}{x(x+h)}\right)\right| \\
&= \sqrt{s}\left|\phi(t,x^2)+\phi(t,(x+h)^2)-2\phi(t,x(x+h))\right|.
\end{aligned}
$$

Using (A.4), we see that this is at most

$$\frac{C}{x^2\sqrt{s}} \times (xh+h^2) = C\left(\frac{h}{x\sqrt{s}}+\frac{h^2}{x^2\sqrt{s}}\right).$$

We have that

$$C\int_0^T \left[\left(\frac{h}{x\sqrt{s}}+\frac{h^2}{x^2\sqrt{s}}\right)\right]^q \mathrm{d}s = C_{T,q}\left(\frac{h^q}{x^q}+\frac{h^{2q}}{x^{2q}}\right).$$

Putting these parts together, we have deduced our second estimate,

$$\int_0^T R(s,x,h)^q \mathrm{d}s \leq C_{T,q}\left(h^{(2-q)/q}+\frac{h^q}{x^q}+\frac{h^{2q}}{x^{2q}}\right).$$

We can now conclude our proof by splitting into cases, and using the two inequalities in the different scenarios. For $x \leq h^{\frac{4q-2}{3q}}$, we use our first bound, (A.2), and obtain that

$$
\begin{aligned}
\int_0^T R(s,x,h)^q \,\mathrm{d}s &\leq C_q\left[h^{(2-q)/q}+x^q(x+h)^q h^{2-3q}\right] \\
&\leq C_q\left[h^{(2-q)/q}+h^{\frac{8q-4+6-9q}{3}}\right] \leq C_q h^{\frac{2-q}{3}}.
\end{aligned}
$$

On the other hand, if $x > h^{\frac{4q-2}{3q}}$, we have by our second bound that

$$\begin{aligned}\int_0^T R(s,x,h)^q \mathrm{d}s &\le C_{T,q}\left(h^{(2-q)/q} + \frac{h^q}{x^q} + \frac{h^{2q}}{x^{2q}}\right) \\ &\le C_{T,q}\left(h^{(2-q)/q} + h^{q-\frac{4q-2}{3}} + h^{2q-\frac{8q-4}{3}}\right) \\ &= C_{T,q}\left(h^{(2-q)/q} + h^{(2-q)/3} + h^{2(2-q)/3}\right) \\ &\le C_{T,q} h^{(2-q)/3},\end{aligned}$$

recalling that $h \in [0,1]$ and $q \in (1,2)$. This concludes the proof. □

Proof of Inequality 4

Proof. We begin by recalling the inequality. For $T > 0$, $0 \le s \le t \le T$ and $q \in (1,2)$, $\exists\, C_{T,q} > 0$ such that

$$\sup_{x\in[0,1]} \int_0^s \left[\int_0^1 (\tilde{G}(t-r,x,y) - \tilde{G}(s-r,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \le C_{T,q}|t-s|^{(2-q)/2}.$$

Once again, we will prove the corresponding bound for $\tilde{J}$, given by (A.1). The proof will have two steps- we will first perform a change of variables, which will allow us to bound the integral by $|t-s|^{(2-q)/2}$ multiplied by some integral which does not depend on t, s, which we denote by $I(x)$. We then show that

$$\sup_{x\ge 0} I(x) < \infty.$$

Let $k := t - s$. We have that

$$\begin{aligned}&\int_0^s \left[\int_0^1 (\tilde{J}(s-r+k,x,y) - \tilde{J}(s-r,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \\ &\quad = \int_0^s \left[\int_0^1 (\tilde{J}(s+k,x,y) - \tilde{J}(s,x,y))^2 \mathrm{d}y\right]^q \mathrm{d}r \\ &\quad = \int_0^s \left[\int_0^1 \frac{1}{k^2}\left(\tilde{J}\left(\frac{s}{k}+1, \frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right) - \tilde{J}\left(\frac{s}{k}, \frac{x}{\sqrt{k}}, \frac{y}{\sqrt{k}}\right)\right)^2 \mathrm{d}y\right]^q \mathrm{d}r.\end{aligned}$$

Performing the change of variables $z := y/\sqrt{k}$ and $u = r/k$, we obtain that this is at most $k^{(2-q)/q} \times I(\frac{x}{\sqrt{k}})$, where

$$I(x) = \int_0^\infty \left[\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u.$$

Arguing as in [10], we have that

$$\sup_{x\ge 0}\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z \le \frac{C}{\sqrt{u}}.$$

It follows that, for $q \in (0,1)$,

$$\int_0^1 \left[\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u = C_q < \infty.$$

So it is only left to control

$$\int_1^\infty \left[\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u. \tag{A.5}$$

We split the space integral here into two cases, when $\{z \geq 2x\}$, and when $\{z < 2x\}$. When $\{z < 2x\}$, we have that,

$$\begin{aligned}
\int_1^\infty \left[\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \leq C_q \int_1^\infty \left[\int_0^\infty \left(J(u+1,x,z) - J(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq & C_q \int_1^\infty \left[\int_0^\infty \frac{1}{u}\left[e^{-(x-z)^2/4(u+1)} - e^{-(x-z)^2/4u}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& + C_q \int_1^\infty \left[\int_0^\infty \frac{1}{u}\left[e^{-(x+z)^2/4(u+1)} - e^{-(x+z)^2/4u}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& + C_q \int_1^\infty \left[\int_0^\infty \left(\frac{1}{\sqrt{u}} - \frac{1}{\sqrt{u+1}}\right)^2 \left(e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq & C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{1}{u}\left(e^{-z^2/4(u+1)} - e^{-z^2/4u}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& + C_q \int_1^\infty \frac{1}{u^{3q}} \left[\int_0^\infty \left(e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
\leq & C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{1}{u} e^{-z^2/2(u+1)} \left(1 - e^{-z^2\left(\frac{1}{4u} - \frac{1}{4(u+1)}\right)}\right)^2 \mathrm{d}z\right]^q \mathrm{d}u + C_q \int_1^\infty \frac{1}{u^{5q/2}} \mathrm{d}u \\
\leq & C_q \int_1^\infty \left[\int_{\mathbb{R}} \frac{z^4}{u^5} e^{-z^2/2(u+1)} \mathrm{d}z\right]^q \mathrm{d}u + C_q \int_1^\infty \frac{1}{u^{5q/2}} \mathrm{d}u = C_q \int_1^\infty \frac{1}{u^{5q/2}} \mathrm{d}u.
\end{aligned}$$

This last term is finite and doesn't depend on x, so we have successfully bounded (A.5) in the case when $\{z < 2x\}$. When $\{z \geq 2x\}$, we have that

$$\begin{aligned}
& \int_1^\infty \left[\int_0^\infty \left(\tilde{J}(u+1,x,z) - \tilde{J}(u,x,z)\right)^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& = C \int_1^\infty \left[\int_0^\infty \frac{z^2}{x^2} \left[e^{-(x-z)^2/4(u+1)} \left(1 - e^{-z^2/4u(u+1)}\right) - e^{-(x+u)^2/4(u+1)} \left(1 - e^{-z^2/4u(u+1)}\right)\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& \leq \int_1^\infty \left[\int_0^\infty \frac{z^6}{16x^2u^2(1+u)^2} \left[e^{-(x-z)^2/4(u+1)} - e^{-(x+z)^2/4(u+1)}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& \leq \int_1^\infty \left[\int_0^\infty \frac{z^6}{16x^2u^2(1+u)^2} e^{-z^2/(u+1)} \left[1 - e^{-xz/(u+1)}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& \leq \int_1^\infty \left[\int_0^\infty \frac{z^6}{16x^2u^2(1+u)^2} e^{-z^2/(u+1)} \left[1 - e^{-xz/(u+1)}\right]^2 \mathrm{d}z\right]^q \mathrm{d}u \\
& \leq \int_1^\infty \left[\int_0^\infty \frac{z^8}{16u^2(1+u)^4} e^{-z^2/(u+1)} \mathrm{d}z\right]^q \mathrm{d}u \leq \int_1^\infty \frac{1}{u^{3q/2}} \mathrm{d}u < \infty.
\end{aligned}$$

This concludes the proof. □

Proof of Inequality 5

Proof. We want to prove that

$$\sup_{x\in[0,1]} \int_0^1 \left|\tilde{H}(t,x,y)\right| \mathrm{d}y \leq \frac{C}{\sqrt{t}}.$$

We prove the corresponding bound for K, where we define, for $x \in (0,1]$, $y \in [0,1]$ and $t \geq 0$,

$$K(t,x,y) := \frac{y}{x}\frac{\partial J}{\partial y}(t,x,y),$$

and for $y \in [0,1]$, $t \geq 0$ we define

$$K(t,0,y) := y\frac{\partial^2 J}{\partial x \partial t}(t,0,y).$$

Note that, for $x \in (0,1]$, $y \in [0,1]$ and $y \geq 2x$,

$$\begin{aligned}
\left|K(t,x,y)\right| \leq& \frac{Cy}{t^{3/2}}\left[e^{-(x-y)^2/4t} + e^{-(x+y)^2/4t}\right] + \frac{Cy^2}{t^{3/2}x}\left[e^{-(x-y)^2/4t} - e^{-(x+y)^2/4t}\right] \\
\leq& \frac{Cy}{t^{3/2}}e^{-y^2/16t} + \frac{Cy^2}{t^{3/2}x}e^{-(x-y)^2/4t}\left[1 - e^{-xy/t}\right] \\
\leq& e^{-y^2/16t}\left[\frac{Cy}{t^{3/2}} + \frac{Cy^3}{t^{5/2}}\right].
\end{aligned}$$

It follows that

$$\sup_{x\in[0,1]} \int_0^t \left|K(t,x,y)\right| \mathrm{d}y \leq C\int_0^t e^{-y^2/16t}\left[\frac{y}{t^{3/2}} + \frac{y^3}{t^{5/2}}\right] \mathrm{d}y \leq \frac{C}{\sqrt{t}}.$$

□

References

[1] T. Arnarson and J. Eriksson, *On the size of the non-coincidence set of parabolic obstacle problems with applications to American option pricing*, Mathematica Scandinavica, Vol. 101(1), pp. 148, 2007.

[2] R. C. Dalang, D. Khoshnevisan and E. Nualart, *Hitting probabilities for systems of non-linear stochastic heat equations with additive noise*, ALEA Lat. Am. J. Probab. Math. Stat., Vol. 3, pp. 231-271, 2007.

[3] C. Donati-Martin and E. Pardoux, *White noise driven SPDEs with reflection*, Probab. Theory Relat. Fields, Vol. 95, pp.1-24, 1993.

[4] B. Hambly and J. Kalsi, *A Reflected Moving Boundary Problem Driven by Space-Time White Noise*

[5] Martin Keller-Ressel and Marvin S. Müller, *Stefan-type stochastic moving boundary problem*, Stoch. Partial Differ. Equ. Anal. Comput., Vol.4, pp.746-790, 2015.

[6] Kunwoo Kim, Zhi Zheng and Richard B. Sowers, *A Stochastic Stefan Problem*, J. Theor. Probab., Vol. 25(4), pp. 1040-1080, 2012.

[7] Marvin S. Müller, *A Stochastic Stefan-Type Problem Under First Order Boundary Conditions*, forthcoming in The Annals of Applied Probability.

[8] D. Nualart and E. Pardoux, *White noise driven quasilinear SPDEs with reflection*, Probability Theory and Related Fields, Vol.93, pp. 77-89, 1992.

[9] T. Xu and T. Zhang, *White noise driven SPDEs with reflection: Existence, uniqueness and large deviation principles*, Stoch. Process. Appl., Vol 119(10), pp.3453-3470, 2009.

[10] Z. Zheng, *Stochastic Stefan Problems: Existence, Uniqueness and Modeling of Market Limit Orders*, PhD thesis, Graduate College of the University of Illinois at Urbana-Champaign, 2012.