Stochastic evolution equations in portfolio credit modelling

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Abstract

We consider a structural credit model for a large portfolio of credit risky assets where the correlation is due to a market factor. By considering the large portfolio limit of this system we show the existence of a density process for the asset values. This density evolves according to a stochastic partial differential equation and we establish existence and uniqueness for the solution taking values in a suitable function space. The loss function of the portfolio is then a function of the evolution of this density at the default boundary. We develop numerical methods for pricing and calibration of the model to credit indices and consider its performance pre and post credit crunch.

1 Introduction

The rapid growth of the credit derivatives market from 2000-2007 led to the development of increasingly complex credit instruments requiring new mathematical models for pricing and risk management. The subsequent contraction due to the credit crunch has placed even more emphasis on the importance of understanding the risks involved in dealing with complex credit products. Our aim in this paper is to develop the mathematical extension of standard large portfolio credit models by introducing dynamics and working with the infinite dimensional limit.

The two natural approaches to credit modelling that have been extensively developed are the structural approach and the reduced form approach, and each has been extended to the portfolio setting in a variety of ways. We consider a dynamic large portfolio model obtained by taking the large portfolio limit of a multidimensional structural model. By taking this limit we obtain a stochastic partial differential equation which models the evolution of the value of a large basket of underlying assets. The key quantities for multiname credit are then certain functions of the solution of this stochastic partial differential equation. Although this paper focuses on a simple version of this model which has shortcomings (as inherent in the underlying structural model), our aim is to provide a mathematical basis for the development of more realistic extensions.

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Our motivation for the development of our structural evolution model came originally from the lack of dynamics in the credit market's standard pricing methodology. A bottom-up approach, in which the individual entities in a credit basket are modelled, has been widely used, primarily as a result of the introduction of copulas and the subsequent conditionally independent factor (CIF) models. These models allow the problem of specifying the marginal distributions and the market co-movements to be separated and through the choice of specific copulas has led to simple, easy to implement and computationally efficient techniques for pricing credit products. However copula and CIF models have no dynamics to speak of; nowhere is it specified how their parameters or underlyings evolve. Furthermore, they only model expected defaults within one time period, which for instruments such as collateralised debt obligations (CDO) is not an issue as they are essentially one period instruments, but for those with stronger timing features this is not reasonable.

This absence of dynamics made pricing some structured credit instruments very difficult and credit market developments since mid-2007 have further exposed the limitations of the standard approaches. There is still a need for a new generation of models to enable a better fitting as well as understanding of the risks inherent in some of the more complex products. For instance the existence of 5, 7 and 10-year index and bespoke tranches requires a model that can fit the entire correlation skew term structure, not just the correlation skew for a given time horizon. Also forward starting tranches, options on tranches and STCDOs with trigger features requires information on the dynamics of spreads and information on the timings of default for their pricing.

Thus our purpose is to develop a relatively simple dynamic extension of a CIF to the large portfolio setting, extending the static work originating with [46], see [43]. This will enable us to describe the evolution of the loss function in terms of a few parameters. By investigating the behaviour of our simplified model, it is possible to gain an insight into which aspects of dynamic models are important for the pricing of more exotic structured credit products. This information can then be used to help the development of more realistic models within this framework. We note that other large portfolio analyses have been considered in for instance [45], [11].

Although many of the exotic credit instruments have traded infrequently, especially post the credit crunch, their introduction was an early indication of the need for a more sophisticated approach to portfolio credit modelling. There is a large and rapidly growing literature in this area, so we only mention a few papers [10], [45], [17], [3], [38], [11]. As an alternative to the bottom-up approach, top-down approaches such as the Markov chain model in [44] and the models of [4], [15] and [13] have been introduced. Reduced-form approaches have been extended to more than one issuer via correlated stochastic parameters. A relatively tractable example is the intensity-gamma model by [29]; another is the affine jump diffusion model of [36]. [37] provides an overview of some of the main bottom-up approaches.

1.1 Structural models

Our model falls into the class of multi-dimensional structural models and we take the approach of modelling the empirical measure of the asset prices in the basket when the underlyings have

dynamics linked through a factor model. The pricing of CDOs is then a function of the limit of the empirical measure of the large basket.

Structural models are based on the premise that when a company's asset value falls below a certain threshold barrier a default is triggered. The first model of this type was introduced by [34] and then extended by [5]. To date, there are many variants of this model but the basic type is as follows. Let A_t be the asset value of a company whose evolution is governed by

$$\frac{dA_t}{A_t} = \mu\, dt + \sigma\, dW_t,$$

where μ is the mean rate of return on the assets, σ is the asset volatility and W_t is a standard Brownian motion. If we denote the default threshold barrier by B_t we define the *distance to default*, X_t, as

$$X_t = \frac{1}{\sigma}\,(\,\log A_t - \log B_t\,)\,. \tag{1.1}$$

The event of a default by time t is now expressed as the event that X hits 0 before time t.

Structural models are appealing due to their intuitive economic interpretation and the link they provide between the equity and credit markets. They introduce spread dynamics and allow market participants to hedge spread risk with the underlying equity of the reference entity. Defaults are endogenously generated within the model and recovery rates do not need to be determined until after a default occurs.

There are however downsides that affect the practical applicability of structural models. Due to the diffusive nature of the asset process, and the assumption of perfect information regarding asset values and default thresholds, any credit event generated by the model is predictable. The immediate consequence is short term credit spreads that are near zero: a fact contradicted by empirical evidence. Extensions that try to address these issues include CreditGrades^{TM} described in [18], as well as [14], [47], [48], [22] and [8]. As structural models are extended in these ways their analytic complexity increases dramatically. Credit spread prices can then no longer be expressed in closed form and numerical methods must be employed for pricing. Another downside is that calibration of the model parameters is not a straightforward exercise.

As a result of the popularity enjoyed by CIF and copula models, multidimensional structural models have typically received less attention; as a result, the literature on this subject is relatively sparse. The first authors to incorporate default correlation into first passage models were [49] and [23]. The former extended the Black-Cox framework to include correlated asset value processes, with hitting times being calculated from a time dependent barrier in closed form for two risky assets. [23] followed Zhou's approach and moved to a higher dimensional space but had to sacrifice the analytic results. In [24] the asset value processes for a multi-dimensional structural model are correlated via a set of common factors. In this setting piecewise default barriers are calibrated to match market prices and Monte-Carlo simulation is used to value single tranche CDOs (STCDOs). Other recent papers using a structural approach include [19], [20], [10] and [9]. We aim to develop a model which can allow pricing of exotic options on CDO tranches and note that there has been some discussion of such products in [25], [27]. This application can be found in [7].

1.2 The SPDE model

The starting point for our model is similar to that used in [24]. We will develop a simple model in this paper in which all assets have the same constant volatility and are correlated via a single market factor. A more general version, in which the correlation is a function, can be found in [28]. Let $(\Omega^N, \mathcal{F}^N, \mathbb{P}^N)$ denote a probability space for a market consisting of N different companies whose asset values A_t at time t evolve under the risk neutral measure $\mathbb{P}^N$ according to a diffusion process given by

$$dA_t^i = rA_t^i\,dt + \sigma\sqrt{1-\rho}A_t^i\,dW_t^i + \sigma\sqrt{\rho}A_t^i\,dM_t, \quad i = 1, \ldots, N \tag{1.2}$$

up until the hitting time of a barrier B^i or the horizon T. We assume W_t^i and M_t are Brownian motions satisfying

$$d\left[W_t^i, M_t\right] = 0 \qquad \forall i$$

and

$$d\left[W_t^i, W_t^j\right] = \delta_{ij}\,dt,$$

where we have written $[.,.]$ for the quadratic covariation and will use $[.]$ for the quadratic variation, and $\sigma > 0$ is a constant and $\rho \in [0, 1)$ is the constant correlation. Note the co-dependence between the asset processes is provided solely by the Brownian motion M_t which can be thought of as a market wide factor influencing all of the assets.

Thus we can write (1.2) in terms of the distance to default process $X_t^i = (\ln A_t^i - \ln B^i)/\sigma$, with constant barrier B^i, as

$$\begin{aligned} dX_t^i &= \mu dt + \sqrt{1-\rho}dW_t^i + \sqrt{\rho}dM_t, \quad t < T_0^i, \\ X_t^i &= 0, \quad t \geq T_0^i, \\ X_0^i &= x^i > 0, \\ T_0^i &= \inf\{t : X_t^i = 0\}, \end{aligned} \tag{1.3}$$

for $i = 1, 2, ..., N$, where $\mu = (r - \frac{1}{2}\sigma^2)/\sigma$.

It does not matter how we label our assets so make the following assumptions. We will assume that $\{X_0^1, ..., X_0^N\}$ is a family of exchangeable, $[C_B, \infty)$-valued random variables with $\mathbb{E}(X_0^i) < \infty$, where the constant $C_B > 0$. We assume that this initial distribution is independent of $\{W^i\}$ and M.

By construction we see that our system extends to an infinite system as $N \to \infty$ and we will show that there is a limit empirical measure whose density satisfies an SPDE. We will write $(\Omega, \mathcal{F}, \mathbb{P})$ with associated expectation operator $\mathbb{E}$ for the limit probability space containing the full infinite asset value model.

In order to state our main mathematical result we need some further notation. Let $(\Omega^M, \mathcal{F}^M, \mathbb{P}^M)$ be a probability space supporting a one-dimensional Brownian motion $(M_t, \mathcal{F}_t)$. Let $\mathcal{G}^M$ denote the σ-algebra of predictable sets on $\Omega^M \times (0, \infty)$ associated with the filtration $\mathcal{F}_t^M$ and $H^1((0, \infty)) = \{f : f \in L^2((0, \infty)), f' \in L^2((0, \infty))\}$, where $L^2((0, \infty)) = \{f : \int_0^\infty f^2 dx < \infty\}$. We

write $L^2(\Omega^M \times (0,T), \mathcal{G}^M, H^1((0,\infty))) = \{f(\omega,t,.) : f(\omega,t,.) \in H^1((0,\infty)), f(\omega,t,.)$ is $\mathcal{F}_t^M$-measurable, $\mathbb{E}^M \int_0^T \|f(\omega,t)\|_{H^1}^2 dt < \infty\}$. We also write δ_x for a Dirac measure at the point x.

Let $\bar{\nu}_{N,t}$ denote the equally weighted empirical measure for the entire portfolio given by

$$\bar{\nu}_{N,t} = \frac{1}{N} \sum_{i=1}^{N} \delta_{X_t^i}. \tag{1.4}$$

Theorem 1.1. *The limit empirical measure $\bar{\nu}_t = \lim_{N\to\infty} \nu_{N,t}$ exists and is a probability measure with a natural decomposition into two components, $\bar{\nu}_t = L_t\delta_0 + \nu_t$. The measure ν_t is a measure on $(0,\infty)$ with density $v(t,x)$, which is the unique solution in $L^2(\Omega^M \times (0,T), \mathcal{G}^M, H^1((0,\infty)))$ of the SPDE*

$$\begin{cases} dv = -\frac{1}{\sigma}\left(r - \frac{1}{2}\sigma^2\right) v_x\, dt + \frac{1}{2} v_{xx}\, dt - \sqrt{\rho} v_x\, dM(t), \\ v(0,x) = v_0(x), \quad v(t,0) = 0. \end{cases} \tag{1.5}$$

The weight of the Dirac mass at 0 is

$$L_t = 1 - \int_0^\infty v(t,x)dx.$$

The prices of typical large portfolio credit products are functions of this proportionate loss function L_t. There is no analytic solution for this SPDE, though it can be viewed as the Zakai equation for a filtering problem, and thus we require numerical techniques for its solution. One natural approach is just to use a Monte Carlo technique to simulate the whole basket, and for small sizes of basket this would be a reasonable approach. However, as the basket size increases, the numerical solution of the limit SPDE becomes more computationally efficient and we discuss this in our simplified setting. It is also natural to ask about the quality of the SPDE approximation to the original large basket and, provided ρ is not too small, it is a good approximation for $N \geq 50$ [6]. However we regard this as a model in its own right and we will see in Section 5 that it can fit the traded index spreads of the iTraxx well.

Our focus here is on the mathematical development of this simple model and we note that the model has clear drawbacks for pricing which would need to be addressed before it could be used in practice. In particular, the fact that it is based on a structural model with diffusion will mean that it will not fit well for short timescales. As it is based on a single diffusive market factor it will not fit the prices of the super senior tranches well. The calibration uses only a single parameter and it would be more difficult to calibrate to multiple tranches. There is also an underlying assumption of homogeneity, in that all assets have the same volatility and correlation. Some of these problems are straightforward to address, for instance the extension of the approach to a model with multiple sectors and, with more work, a more general functional dependence of the correlation, see [28]. Others, such as addressing the low default probability at short times, could be tackled using lévy process or stochastic volatility models as drivers, see for example [29], [17] for their use in other settings to tackle this problem. Despite the issues raised here we believe that the model has good properties and is a basis for extensions which incorporate more realistic features.

An outline of the paper is as follows. We begin with a description of the mechanics and basic valuation methods of synthetic collateralised debt obligations in Section 2 in order to provide the necessary background for later sections. The mathematical core of the work is in Section 3 where we develop our infinite dimensional model for portfolio credit starting from a multidimensional structural model and prove Theorem 1.1. We make strong assumptions with the aim of delivering a relatively simple, tractable model that encapsulates the information required to calculate the loss distribution for a portfolio of risky assets. The aim in Section 4 is to develop a suitable numerical scheme for solving the SPDE. Section 5 discusses the calibration and performance of the model when pricing tranches of the iTraxx before and after the credit crunch.

2 Collateralised debt obligations

Collateralised Debt Obligations (CDOs) are securitized interests in pools of credit risky assets. These assets can include mortgages, bonds, loans and credit derivatives. The CDO repackages the credit risk of the reference portfolio into multiple tranches that are then passed on to investors. Prior to the 'credit crunch' the synthetic CDO, credit indices and single name Credit Default Swap (CDS) market together made up the majority of the total traded notional in the credit derivative market. However the index tranche market is currently the only area that is still active. The bespoke CDO business has yet to return although there are a few signs of activity.

Although there are many different types of CDO, here we will focus on what is known as a synthetic CDO i.e. one whose collateral pool consists entirely of credit default swaps. It is possible to trade single tranches within a synthetic CDO without the entire structure being constructed. In this case the two parties of the transaction, the protection buyer and protection seller, exchange payments as if the CDO had been set-up. The performance of this single tranche CDO (STCDO) is dependent on the number of defaults that occur in the reference portfolio during the lifetime of the contract.

Each tranche is defined by two points that determine its place within the capital structure: the attachment point and the higher valued detachment point. These are usually expressed as a percentage of the total portfolio notional. The tranche notional is defined as the difference between the attachment and detachment points. When losses are incurred (the loss is the notional of the defaulted entity corrected for recovery), and the cumulative loss in the collateral pool is between the attachment and detachment point, the seller pays the buyer an amount equal to the loss incurred within the tranche. The tranche notional is then reduced by this amount. This means that when the cumulative loss exceeds the detachment point the tranche notional is zero. In return for this protection, the buyer pays a quarterly premium based off a fixed spread and the outstanding tranche notional.

Say we have N entities in our reference credit portfolio each with notional N_0. We define the total loss L_t on the portfolio as

$$L_t = \sum_{i=1}^{N} L_i 1_{\{\tau_i \leq t\}}, \tag{2.1}$$

where $L_i = N_0(1 - R_i)$, R_i and τ_i are the recovery rate and default time of the i-th entity respectively. If we assume the recovery rate is the same across all credit entities and equal to a value R then we can write

$$L_t = N_0(1 - R) \sum_{i=1}^{N} 1_{\{\tau_i \leq t\}}. \tag{2.2}$$

The outstanding tranche notional, Z_t, of a single tranche within a synthetic CDO is given by

$$Z_t = [d - L_t]^+ - [a - L_t]^+, \tag{2.3}$$

and the tranche loss Y_t as

$$Y_t = [L_t - a]^+ - [L_t - d]^+, \tag{2.4}$$

where a is the tranche attachment point and d is the tranche detachment point.

As for a Credit Default Swap (CDS) the value of a STCDO is given by the difference between the fee leg and the protection leg. The protection buyer pays a regular fixed spread on the outstanding notional of the tranche. We denote the payment dates by T_i, $1 \leq i \leq n$, the intervals by $\delta_i = T_i - T_{i-1}$ and the value of a bank account at time t by $b(t)$. Then the value of the fee leg is given by

$$sV^{fee} = s \sum_{i=1}^{n} \frac{\delta_i}{b(T_i)} \mathbb{E}[Z_{T_i}], \tag{2.5}$$

where the expectation is with respect to a suitable pricing measure. The protection seller only makes payments to the buyer when the tranche incurs losses, and the value of this payment is equal to the change in the tranche loss Y_t. However, we can express the value of the protection leg in terms of the outstanding tranche notional X_t as follows

$$V^{prot} = \sum_{i=1}^{n} \frac{1}{b(T_i)} \mathbb{E}[Z_{T_{i-1}} - Z_{T_i}], \tag{2.6}$$

assuming that the losses are paid at the coupon dates. As in a CDS contract the par spread s of the tranche is chosen to make the initial value zero hence is calculated as

$$s = \frac{V^{prot}}{V^{fee}}. \tag{2.7}$$

From (2.5) and (2.6) we see that the key to finding the par spread is obtaining the distribution of the outstanding tranche notional; from (2.3), this is equivalent to finding the distribution of the loss L_t. As all portfolio credit derivatives are essentially options on this loss variable the heart of every multiname credit model is determining its distribution.

3 An infinite dimensional structural model

Our aim in this section is to establish Theorem 1.1. We will begin by describing the system (1.3) by a measure valued process and showing that there is a limit empirical measure for the

infinite system. We then proceed to establish its behaviour near 0 before proving that its evolution can be captured by an SPDE.

3.1 The limit empirical density

Recall the equally weighted empirical measure for the entire portfolio is given by

$$\bar{\nu}_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}.$$

We can write this as

$$\bar{\nu}_{N,t} = L_{N,t}\delta_0 + \nu_{N,t},$$

where

$$\nu_{N,t} = \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i} 1_{\{t<T_i^0\}}, \quad L_{N,t} = \frac{1}{N}\sum_{i=1}^{N} 1_{\{t\geq T_i^0\}}.$$

Note that $L_{N,t}$ is a loss function in that it is the proportion of companies that have defaulted by time t.

Let $\mathbb{R}_+ = [0,\infty)$. We write $\mathcal{P}(\mathbb{R}_+)$ for the set of probability measures on $\mathbb{R}_+$ and $\mathcal{P}(C_{\mathbb{R}_+}[0,\infty))$ for the set of probability measures on $C_{\mathbb{R}_+}[0,\infty)$ where the topology is always that of weak convergence. We write $C_{\mathcal{P}(\mathbb{R}_+)}[0,\infty)$ for the continuous $\mathcal{P}(\mathcal{R}_+)$-valued functions on $[0,\infty)$.

Theorem 3.1. *There exists a $C_{\mathcal{P}(\mathbb{R}_+)}[0,\infty)$-valued random variable $\bar{\nu}$ such that*

$$\bar{\nu}_t = \lim_{N\to\infty} \bar{\nu}_{N,t} = \lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{X_t^i}, \ \mathbb{P}\text{- } a.s. \ .$$

We also have a decomposition for the limit into two subprobability measures

$$\bar{\nu}_t = L_t\delta_0 + \nu_t.$$

Proof. Let us denote the system with the same dynamics but without default by $\{\tilde{X}_t^i\}$. Then

$$X_t^i = \tilde{X}_t^i 1_{\{\min_{0\leq s\leq t} \tilde{X}_s^i > 0\}} := F\left(\tilde{X}_s^i, 0\leq s\leq t\right).$$

Since F is independent of i, exchangeability of $\{X^i\}$ in $C_{\mathbb{R}_+}[0,\infty)$ follows from exchangeability of $\{\tilde{X}^i\}$ in $C_{\mathbb{R}}[0,\infty)$.

Firstly note that $\{X_0^i\}$ is an exchangeable family and as $\tilde{X}_t^i = X_0^i + \mu t + \sqrt{1-\rho}W_t^i + \sqrt{\rho}M_t$ for all t, we have $\{\tilde{X}_t^1,\ldots,\tilde{X}_t^N\}$ is exchangeable for any t. It is then easy to see that for any N, $\{\tilde{X}_\cdot^1,...,\tilde{X}_\cdot^N\}$ is exchangeable in $C_{\mathbb{R}}[0,\infty)$ using this and exchangeability of the increments. As a consequence we have $\{X_\cdot^1,...,X_\cdot^N\}$ is exchangeable in $C_{\mathbb{R}_+}[0,\infty)$ and for a fixed t, $\{X_t^1,...,X_t^N\}$ is exchangeable in $\mathbb{R}_+$. As the system (1.3) is easily extended to an infinite particle system, by

de Finetti's theorem, see for example, [1],

$$\bar{\nu}_{\cdot} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_{\cdot}}$$

exists almost surely in $\mathcal{P}(C_{\mathbb{R}_+}[0,\infty))$.

We now need to show that the $\{\nu_t, t \in [0,\infty)\}$ is a continuous process in the space of probability measures. We define a projection mapping

$$P_t : C_{\mathbb{R}}[0,\infty) \to \mathbb{R}$$

by setting, $P_t(Y_{\cdot}) = Y_t$, for any $Y_{\cdot} \in C_{\mathbb{R}}[0,\infty)$. Then define $\bar{\nu}_t := \bar{\nu} \circ P_t^{-1} \in \mathcal{P}(\mathbb{R})$. We first show that

$$\bar{\nu}_t = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t}.$$

To establish this we denote

$$\theta_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t},\ \theta = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t},$$

where θ_N converges weakly to θ and θ exists in $\mathcal{P}(\mathbb{R})$ almost surely by the exchangeability of $\{X^i_t\}$ at any time t. For any $h \in C_b(\mathbb{R})$, the collection of all the bounded and continuous functions on $\mathbb{R}$, we have

$$\int h(x)\theta(dx) = \lim_{N\to\infty} \int h(x)\theta_N(dx).$$

Define

$$\alpha_N = \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_{\cdot}} \in \mathcal{P}(C_{\mathbb{R}}[0,\infty)),\ k = h \circ P_t \in C_b(C_{\mathbb{R}}[0,\infty)),$$

then

$$\theta_N = \alpha_N \circ P_t^{-1},$$

and α_N converges weakly to $\bar{\nu}$ in $\mathcal{P}(C_{\mathbb{R}}[0,\infty))$. Thus

$$\begin{aligned}
\int h(x)\theta(dx) &= \lim_{N\to\infty} \int k \circ P_t^{-1}(x)(\alpha_N \circ P_t^{-1})(dx) \\
&= \int k \circ P_t^{-1}(x)\bar{\nu} \circ P_t^{-1}(dx) \\
&= \int h(x)\bar{\nu}_t(dx).
\end{aligned}$$

Therefore $\bar{\nu}_t = \theta = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \delta_{X^i_t} \in \mathcal{P}(\mathbb{R})$. To show that $\bar{\nu}_t \in C_{\mathcal{P}(\mathbb{R})}[0,\infty)$ it suffices prove that when $t_n \to t_0$, we have $\bar{\nu}_{t_n} \to \bar{\nu}_{t_0}$ weakly in $\mathcal{P}(\mathbb{R})$, i.e., we want to show that for any open set $U \in \mathcal{B}(\mathbb{R})$, $\liminf_{n\to\infty} \bar{\nu}_{t_n}(U) \geq \bar{\nu}_{t_0}(U)$ [[16], Theorem 3.3.1]. By continuity of Y and Fatou's

Lemma for sets, we see that

$$\begin{aligned}\bar{\nu}_{t_0}(U) =& \bar{\nu} \circ P_{t_0}^{-1}(U) = \bar{\nu}\left(\{Y_\cdot | Y_{t_0} \in U\}\right) \\ =& \bar{\nu}\left(\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{Y_\cdot | Y_{t_k} \in U\}\right) \\ \leq& \lim_{n\to\infty} \inf_{k\geq n} \bar{\nu}\left(\{Y_\cdot | Y_{t_k} \in U\}\right) = \liminf_{n\to\infty} \bar{\nu}_{t_n}(U).\end{aligned}$$

Therefore, the process $\{\bar{\nu}_t : t \in [0,\infty)\}$ exists almost surely in $C_{\mathcal{P}(\mathbb{R})}[0,\infty)$.

The decomposition follows from the decomposition for N companies. We then define $L_t = \bar{\nu}_t(\{0\})$ and ν_t to be $\bar{\nu}_t$ restricted to $(0,\infty)$. □

For a measure ζ_t and integrable function ϕ we write

$$\langle \phi, \zeta_t \rangle = \int \phi(x)\zeta_t(dx). \tag{3.1}$$

Let $C_K^\infty(0,\infty)$ be the set of infinitely differentiable functions with compact support on $(0,\infty)$. Using the empirical measure (1.4) we define a family of processes $F_t^{N,\phi}$ for $\phi \in C_K^\infty(0,\infty)$ by

$$F_t^{N,\phi} = \langle \phi, \bar{\nu}_{N,t} \rangle = \frac{1}{N}\sum_{i=1}^{N} \phi(X_t^i) = \langle \phi, \nu_{N,t} \rangle, \tag{3.2}$$

using the fact that $\phi(0) = 0, \forall \phi \in C_K^\infty(0,\infty)$.

As $X_t^i = 0$ for $t > T_0^i$, and hence $\phi(X_t^i) = 0$ for $t > T_0^i$, in order to apply Itô's formula to $F_t^{N,\phi}$ we write $F_t^{N,\phi} = \frac{1}{N}\sum_{i=1}^{N} \phi(X_t^i) 1_{\{t<T_0^i\}}$. Thus we have

$$\begin{aligned}F_t^{N,\phi} - F_0^{N,\phi} &= \frac{1}{N}\sum_{i=1}^{N}\int_0^t 1_{\{s\leq T_0^i\}}\left(\phi'(X_s^i)dX_s^i + \frac{1}{2}\phi''(X_s^i)d[X_s^i]\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\int_0^t 1_{\{s\leq T_0^i\}}\left[\phi'(X_s^i)\mu ds + \phi'(X_s^i)\sqrt{1-\rho}dW_s^i + \phi'(X_s^i)\sqrt{\rho}dM_s + \frac{1}{2}\phi''(X_s^i)ds\right] \\ &= \int_0^t \frac{1}{N}\sum_{i=1}^{N}(\mu\phi'(X_s^i) + \frac{1}{2}\phi''(X_s^i))1_{\{s<T_0^i\}}ds + \int_0^t \frac{1}{N}\sum_{i=1}^{N}\sqrt{1-\rho}\phi'(X_s^i)1_{\{s<T_0^i\}}dW_s^i \\ &\quad + \int_0^t \frac{1}{N}\sum_{i=1}^{N}\sqrt{\rho}\phi'(X_s^i)1_{\{s<T_0^i\}}dM_s\end{aligned}$$

If we define the second order linear operator $\mathcal{A}$ by

$$\mathcal{A} = \mu\frac{\partial}{\partial x} + \frac{1}{2}\frac{\partial^2}{\partial x^2},$$

we have

$$\begin{aligned} F_t^{N,\phi} &= F_0^{N,\phi} + \int_0^t \langle \mathcal{A}\phi, \nu_{N,s} \rangle \, ds + \int_0^t \langle \sqrt{\rho}\phi', \nu_{N,s} \rangle \, dM_s \\ &\quad + \int_0^t \frac{1}{N} \sum_{i=1}^N \phi'(X_s^i) \sqrt{1-\rho} \, dW_s^i. \end{aligned} \tag{3.3}$$

In order to pass to the limit as $N \to \infty$ we first focus on the idiosyncratic term in (3.3)

$$I_{t,N}^{\phi} = \int_0^t \frac{1}{N} \sum_{i=1}^N \sqrt{1-\rho}\phi'(X_s^i) 1_{\{s<T_0^i\}} \, dW_s^i. \tag{3.4}$$

As ϕ' is bounded and $I_{t,N}^{\phi}$ is a martingale, by the independence of the W_t^i, it has quadratic variation

$$[I_N^{\phi}]_t = \int_0^t \frac{1}{N^2} \sum_{i=1}^N (1-\rho) \left(\phi'(X_s^i)\right)^2 1_{\{s<T_0^i\}} \, ds.$$

Writing K_ϕ for the constant such that $|\phi'| \leq K_\phi$, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^N \int_0^t (1-\rho) \mid \phi'(X_s^i) \mid^2 1_{\{s<T_0^i\}} \, ds \leq K_\phi^2 t,$$

and hence we have for any such ϕ

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^N \int_0^t (1-\rho) \mid \phi'(X_s^i) \mid^2 1_{\{s<T_0^i\}} \, ds \leq \lim_{N\to\infty} \frac{1}{N} K_\phi^2 t = 0, \quad \forall t \in [0,T], \quad a.s.$$

As a martingale with 0 quadratic variation must be the constant process the idiosyncratic term must vanish almost surely in the limit.

We also note that as ϕ', ϕ'' are bounded and $\nu_{N,s}$ is a probability measure, we can apply the dominated convergence theorem to take the limit under the integrals in the other terms in (3.3). We summarize in the following

Theorem 3.2. *The sequence of empirical measures $\nu_{N,t}$ on $(0,\infty)$ satisfies for all $\phi \in C_K^\infty(0,\infty)$,*

$$F_t^{N,\phi} \to F_t^{\phi} = \langle \phi, \nu_t \rangle \quad \text{as } N \to \infty, \quad a.s.$$

The evolution of the limit empirical measure in the weak sense is given by

$$\langle \phi, \nu_t \rangle = \langle \phi, \nu_0 \rangle + \int_0^t \langle \mathcal{A}\phi, \nu_s \rangle \, ds + \int_0^t \langle \sqrt{\rho}\phi', \nu_s \rangle \, dM_s, \quad \forall \phi \in C_K^\infty(0,\infty). \tag{3.5}$$

3.2 The boundary condition

The behaviour of ν_t, the limit empirical measure on $(0,\infty)$, at the boundary zero is given in the following theorem:

Theorem 3.3. *We have*

$$\lim_{\varepsilon\downarrow 0}\frac{\nu_t((0,\varepsilon))}{\varepsilon}=0,\quad a.s..$$

Proof. By the definition of ν_t, properties of weak convergence and an application of Fatou's Lemma, we have

$$\begin{aligned}\mathbb{E}[\nu_t((0,\varepsilon))]\leq&\mathbb{E}\left[\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}1_{\{0<X_t^i<\varepsilon\}}\right]\\ \leq&\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}\mathbb{P}\left\{X_t^i<\varepsilon,\inf_{0\leq s\leq t}X_s^i>0\right\}.\end{aligned}\tag{3.6}$$

For $t<T_0^i$, integrating the system (1.3) from time 0 to t, we have:

$$X_t^i=x^i+\mu t+\sqrt{1-\rho}W_t^i+\sqrt{\rho}M_t\overset{d}{=}x^i+\mu_t+B_t,$$

where B_t is a standard Brownian motion on the same probability space. Thus we have

$$\begin{aligned}\mathbb{P}\left\{X_t^i<\varepsilon,\inf_{0\leq s\leq t}X_s^i>0\right\}=&\mathbb{P}\left\{x^i+\mu t+B_t<\varepsilon,\inf_{0\leq s\leq t}(x^i+\mu s+B_s)>0\right\}\\ =&\mathbb{P}^{x^i}\left\{\mu t+B_t<\varepsilon\right\}-\mathbb{P}^{x^i}\left\{\mu t+B_t<\varepsilon,\inf_{0\leq s\leq t}(\mu s+B_s)\leq 0\right\}\\ =&\int_{-\infty}^{\varepsilon}\frac{1}{\sqrt{2\pi t}}e^{-(z-\mu t-x^i)^2/2t}dz-\int_{-\infty}^{\varepsilon}\frac{1}{\sqrt{2\pi t}}e^{\mu(z-x^i)-\mu^2t/2-(|z|+x^i)^2/2t}dz\\ =&\frac{1}{\sqrt{2\pi t}}\int_0^{\varepsilon}\left(e^{-(z-\mu t-x^i)^2/2t}-e^{\mu(z-x^i)-\mu^2t/2-(z+x^i)^2/2t}\right)dz\\ =&\frac{1}{\sqrt{2\pi t}}\int_0^{\varepsilon}e^{-(z-\mu t-x^i)^2/2t}\left(1-e^{-\frac{2zx^i}{t}}\right)dz\\ \leq&\frac{1}{\sqrt{2\pi t}}\left(1-e^{-\frac{2\varepsilon x^i}{t}}\right)\int_0^{\varepsilon}e^{-(z-\mu t-x^i)^2/2t}dz\\ \leq&\frac{1}{\sqrt{2\pi t}}\frac{2\varepsilon x^i}{t}\int_0^{\varepsilon}e^{-(z-\mu t-x^i)^2/2t}dz.\end{aligned}\tag{3.7}$$

Assume $\varepsilon<\frac{1}{2}C_B$. Since we have $x^i\geq C_B$, if $t<\frac{C_B-\varepsilon}{|\mu|}$, then $|z-\mu t-x^i|>0,\ \forall 0<z<\varepsilon$ and there exists $C_T^1>0$ only depending on T such that

$$\frac{1}{t^{\frac{3}{2}}}e^{-(z-\mu t-x^i)^2/2t}\leq C_T^1,\ \ \forall t<\frac{C_B-\varepsilon}{|\mu|}.$$

If $t\geq\frac{C_B-\varepsilon}{|\mu|}$, then

$$\frac{1}{t^{\frac{3}{2}}}e^{-(z-\mu t-x^i)^2/2t}\leq\frac{1}{\left(\frac{C_B-\varepsilon}{|\mu|}\right)^{\frac{3}{2}}}\leq\frac{1}{\left(\frac{C_B}{2|\mu|}\right)^{\frac{3}{2}}}.$$

Letting $C'_T := \max\left\{C^1_T, \frac{1}{\left(\frac{C_B}{2|\mu|}\right)^{\frac{3}{2}}}\right\}$, (3.7) becomes

$$\mathbb{P}\left\{X^i_t < \varepsilon, \inf_{0\le s\le t} X^i_s > 0\right\} \le \frac{2}{\sqrt{2\pi}}\varepsilon x^i \varepsilon C'_T := x^i C_T \varepsilon^2, \tag{3.8}$$

where C_T is a positive constant only depending on T. Thus by (3.6) and (3.8) we have

$$\frac{\mathbb{E}[\nu_t((0,\varepsilon))]}{\varepsilon} \le C_T\varepsilon(\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N x^i) \le KC_T\varepsilon, \tag{3.9}$$

since $\{X^1_0, ..., X^N_0\}$ is an exchangeable family of integrable random variables. By Markov's inequality, for the subsequence $\varepsilon = \frac{1}{n^2}$,

$$\mathbb{P}\left\{\frac{\nu_t((0,\frac{1}{n^2}))}{\frac{1}{n^2}} > \lambda\right\} \le \frac{KC_T}{\lambda n^2}.$$

Thus by the first Borel-Cantelli Lemma, as $\lambda > 0$ is arbitrary and also $\frac{\nu_t((0,\frac{1}{n^2}))}{\frac{1}{n^2}} \ge 0$, we must have

$$\limsup_{n\to\infty}\frac{\nu_t((0,\frac{1}{n^2}))}{\frac{1}{n^2}} = 0, \quad \text{a.s..}$$

Now for any $\varepsilon > 0$, there exists a n such that $\frac{1}{(n+1)^2} \le \varepsilon \le \frac{1}{n^2}$ and hence

$$\limsup_{\varepsilon\downarrow 0}\frac{\nu_t((0,\varepsilon))}{\varepsilon} \le \limsup_{n\to\infty}\frac{\nu_t((0,\frac{1}{n^2}))}{\frac{1}{(n+1)^2}} = \limsup_{n\to\infty}\frac{\nu_t((0,\frac{1}{n^2}))}{\frac{1}{n^2}}\frac{\frac{1}{n^2}}{\frac{1}{(n+1)^2}} = 0, \quad \text{a.s..}$$

Since $\frac{\nu_t((0,\varepsilon))}{\varepsilon} \ge 0$, therefore $v(t,0) := \lim_{\varepsilon\downarrow 0}\frac{\nu_t((0,\varepsilon))}{\varepsilon} = 0$, a.s. □

Therefore, if there is a density for the empirical measure, it will satisfy a Dirichlet boundary condition.

Next we give an estimate on $\mathbb{E}[(\nu_t((0,\varepsilon)))^2]$ which will be needed later. In order to do this we require an estimate for the distribution of the first passage times of two correlated Brownian motions, and the Brownian motions themselves.

Lemma 3.4. *Let B^1_t and B^2_t be two correlated Brownian motions with constant correlation $|\varrho| < 1$, $B^1_0 = a_1 > 0$, $B^2_0 = a_2 > 0$ and law $\mathbb{P}_B$. Then there exists $\varepsilon_0 = \frac{1}{3}\sqrt{\frac{1-\varrho}{2}}\sqrt{\frac{a_1^2+a_2^2-2\varrho a_1 a_2}{1-\varrho^2}}$ such that for all $\varepsilon < \varepsilon_0$,*

$$\mathbb{P}_B\left\{0 < B^1_t < \varepsilon, \inf_{0\le s\le t} B^1_s > 0,\ 0 < B^2_t < \varepsilon, \inf_{0\le s\le t} B^2_s > 0\right\} \le C_T\varepsilon^{2+\frac{\pi}{\alpha}},$$

where $C_T = 2^{1-\frac{\pi}{\alpha}}\left(\sqrt{\frac{a_1^2+a_2^2-2\varrho a_1 a_2}{1-\varrho^2}}\right)^{\frac{\pi}{\alpha}} K_T\left(\sqrt{\frac{2}{1-\varrho}}\right)^{2+\frac{\pi}{\alpha}}$ and K_T is a constant only depending on

T; and

$$\alpha = \begin{cases} \pi + \tan^{-1}\left(-\frac{\sqrt{1-\varrho^2}}{\varrho}\right), & \varrho > 0, \\ \frac{\pi}{2}, & \varrho = 0, \\ \tan^{-1}\left(-\frac{\sqrt{1-\varrho^2}}{\varrho}\right), & \varrho < 0. \end{cases} \tag{3.10}$$

Therefore, if $\varrho \geq 0$*, we have* $\frac{\pi}{2} \leq \alpha < \pi$ *and* $3 < 2 + \frac{\pi}{\alpha} \leq 4$*.*

Proof. We begin by making a transformation to obtain a two-dimensional Brownian motion with independent components. Although initially derived in [26], we follow the setup and statements in [35]. Let $B_t = (B_t^1, B_t^2)$ and consider the process $Z = \sigma^{-1}B$, where

$$\sigma = \begin{bmatrix} \sqrt{1-\varrho^2} & \varrho \\ 0 & 1 \end{bmatrix}.$$

We know that Z has independent components. It is easily seen that the horizontal axis is invariant under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\mathbf{x}) = \sigma^{-1}\mathbf{x}$, while the vertical axis is mapped to the line $z_1 = -\frac{\varrho}{\sqrt{1-\varrho^2}} z_2$.

Now the time that the first Brownian motion B^1 hits zero is transformed to the time τ_1 which is the first passage time of Z_t to the horizontal axis; and the time that the second Brownian motion B^2 hits zero is transformed to the time τ_2 which is the first passage time of Z_t to the line $z_2 = z_1 \tan\alpha$, where $0 < \alpha < \pi$ is given in (3.10). Moreover, in polar coordinates $Z_t = (R_t, \Theta_t)$ starts at the point z_0 given by

$$r_0 = \sqrt{\frac{a_1^2 + a_2^2 - 2\varrho a_1 a_2}{1-\varrho^2}};$$

and

$$\theta_0 = \begin{cases} \pi + \tan^{-1}\left(\frac{a_2\sqrt{1-\varrho^2}}{a_1 - \varrho a_2}\right), & a_1 < \varrho a_2, \\ \frac{\pi}{2}, & a_1 = \varrho a_2, \\ \tan^{-1}\left(\frac{a_2\sqrt{1-\varrho^2}}{a_1 - \varrho a_2}\right), & a_1 > \varrho a_2. \end{cases}$$

It is easily verified that $0 < \theta_0 < \alpha$. We denote by $\tau = \min(\tau_1, \tau_2)$ the first exit time of Z from the wedge

$$C_\alpha = \{(r\cos\theta, r\sin\theta) : r > 0, 0 < \theta < \alpha\} \subset \mathbb{R}^2.$$

If $z = (r\cos\theta, r\sin\theta)$ is a point in C_α we have, by [26],

$$\mathbb{P}_B^{z_0}\{\tau > t, Z_t \in dz\} = \frac{2r}{t\alpha} e^{-(r^2+r_0^2)/2t} \sum_{n=1}^{\infty} \sin\frac{n\pi\theta}{\alpha} \sin\frac{n\pi\theta_0}{\alpha} I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) dr d\theta, \tag{3.11}$$

where I_v denotes the modified Bessel function of the first kind of order v

Using this transformation and the formula (3.11) we have

$$\begin{aligned}
&\mathbb{P}_B\left\{0<B_t^1<\varepsilon,\ \inf_{0\le s\le t}B_s^1>0,\ 0<B_t^2<\varepsilon,\ \inf_{0\le s\le t}B_s^2>0\right\}\\
&\le\ \mathbb{P}_B\left\{\tau>t, 0<\Theta_t<\alpha, 0<R_t<\sqrt{\frac{2}{1-\varrho}}\varepsilon\right\}\\
&=\ \int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}\int_0^{\alpha}\frac{2r}{t\alpha}e^{-(r^2+r_0^2)/2t}\sum_{n=1}^{\infty}\sin\frac{n\pi\theta}{\alpha}\sin\frac{n\pi\theta_0}{\alpha}I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right)drd\theta\\
&\le\ \int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}\frac{2r}{t\alpha}e^{-(r^2+r_0^2)/2t}\int_0^{\alpha}\sum_{n=1}^{\infty}I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right)drd\theta. \qquad (3.12)
\end{aligned}$$

By the definition of the modified Bessel function, we have

$$\begin{aligned}
I_{n\pi/\alpha}\left(\frac{rr_0}{t}\right) &= \sum_{m=0}^{\infty}\frac{1}{m!\Gamma(m+\frac{n\pi}{\alpha}+1)}\left(\frac{rr_0}{2t}\right)^{2m+\frac{n\pi}{\alpha}}\\
&\le \frac{1}{\left[\frac{n\pi}{\alpha}\right]!}\left(\frac{rr_0}{2t}\right)^{\frac{n\pi}{\alpha}}\sum_{m=0}^{\infty}\frac{1}{(m!)^2}\left(\frac{rr_0}{2t}\right)^{2m}\\
&\le \frac{1}{\left[\frac{n\pi}{\alpha}\right]!}\left(\frac{rr_0}{2t}\right)^{\frac{n\pi}{\alpha}}\left[\sum_{m=0}^{\infty}\frac{1}{m!}\left(\frac{rr_0}{2t}\right)^{m}\right]^2\\
&= e^{rr_0/t}\frac{1}{\left[\frac{n\pi}{\alpha}\right]!}\left(\frac{rr_0}{2t}\right)^{\frac{n\pi}{\alpha}},
\end{aligned}$$

where $[x]$ denotes the integer part of x. Using this in (3.12) we have

$$\begin{aligned}
&\mathbb{P}_B\left\{0<B_t^1<\varepsilon,\ \inf_{0\le s\le t}B_s^1>0,\ 0<B_t^2<\varepsilon,\ \inf_{0\le s\le t}B_s^2>0\right\}\\
&\le\ \int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}\frac{2r}{t\alpha}e^{-(r^2+r_0^2)/2t}\int_0^{\alpha}e^{rr_0/t}\sum_{n=1}^{\infty}\frac{1}{\left[\frac{n\pi}{\alpha}\right]!}\left(\frac{rr_0}{2t}\right)^{\frac{n\pi}{\alpha}}drd\theta\\
&\le\ \int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}\frac{2r}{t}e^{-(r^2+r_0^2)/2t}e^{rr_0/t}(\frac{rr_0}{2t})^{\frac{\pi}{\alpha}}e^{rr_0/2t}dr\\
&=\ 2^{1-\frac{\pi}{\alpha}}r_0^{\frac{\pi}{\alpha}}\int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}r^{1+\frac{\pi}{\alpha}}\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{r^2+r_0^2-3rr_0}{2t}}dr.
\end{aligned}$$

If we choose $\varepsilon_0=\frac{r_0\sqrt{\frac{1-\varrho}{2}}}{3}$, then for any $\varepsilon<\varepsilon_0$ we have $r^2+r_0^2-3rr_0>0$. Therefore we can find a constant K_T only depending on T such that

$$\frac{1}{t^{1+\frac{\pi}{\alpha}}}e^{-\frac{r^2+r_0^2-3rr_0}{2t}}\le K_T.$$

Thus

$$\begin{aligned}
&\mathbb{P}_B\left\{0<B_t^1<\varepsilon,\ \inf_{0\le s\le t}B_s^1>0,\ 0<B_t^2<\varepsilon,\ \inf_{0\le s\le t}B_s^2>0\right\}\\
&\le\ 2^{1-\frac{\pi}{\alpha}}r_0^{\frac{\pi}{\alpha}}\int_0^{\sqrt{\frac{2}{1-\varrho}}\varepsilon}r^{1+\frac{\pi}{\alpha}}K_T dr\\
&\le\ 2^{1-\frac{\pi}{\alpha}}r_0^{\frac{\pi}{\alpha}}K_T\left(\sqrt{\frac{2}{1-\varrho}}\varepsilon\right)^{1+\frac{\pi}{\alpha}}\sqrt{\frac{2}{1-\varrho}}\varepsilon=C_T\varepsilon^{2+\frac{\pi}{\alpha}},
\end{aligned}$$

where $C_T=2^{1-\frac{\pi}{\alpha}}r_0^{\frac{\pi}{\alpha}}K_T\left(\sqrt{\frac{2}{1-\varrho}}\right)^{2+\frac{\pi}{\alpha}}$ is a constant only depending on ϱ, a_1, a_2 and T.

Moreover, it is obvious that $0<\alpha<\pi$ and $\frac{\pi}{2}\le\alpha<\pi$ if $\varrho\ge 0$. In the latter case we have $3<2+\frac{\pi}{\alpha}\le 4$. □

Lemma 3.5. *There exists $\tilde{\varepsilon}_0>0$ only depending on ρ and the lower bound C_B for the $\{X_0^i\}$, such that for any $\eta>0$, for all $\varepsilon<\tilde{\varepsilon}_0$ we have*

$$\mathbb{E}[(\nu_t((0,\varepsilon)))^2]\le K_T\varepsilon^{2+\frac{\pi}{\alpha}-\eta},$$

where K_T is a positive constant depending on T and α is given in (3.2).

Proof. By definition of ν_t, properties of weak convergence and Fatou's Lemma

$$\begin{aligned}
\mathbb{E}[(\nu_t((0,\varepsilon)))^2]\le&\mathbb{E}\left[\liminf_{N\to\infty}\frac{1}{N}\sum_{i=1}^N 1_{\{0<X_t^i<\varepsilon,\inf_{0\le s\le t}X_s^i>0\}}\liminf_{M\to\infty}\frac{1}{M}\sum_{j=1}^M 1_{\{0<X_t^j<\varepsilon,\inf_{0\le s\le t}X_s^j>0\}}\right]\\
\le&\liminf_{N\to\infty,M\to\infty}\frac{1}{NM}\sum_{i=1}^N\sum_{j=1}^M\mathbb{E}\left[1_{\{0<X_t^i<\varepsilon,\ \inf_{0\le s\le t}X_s^i>0,\ 0<X_t^j<\varepsilon,\inf_{0\le s\le t}X_s^j>0\}}\right]\\
=&\liminf_{N\to\infty,M\to\infty}\frac{1}{NM}\sum_{i=1}^N\sum_{j\ne i,j=1}^M\mathbb{E}\left[1_{\{0<X_t^i<\varepsilon,\ \inf_{0\le s\le t}X_s^i>0,\ 0<X_t^j<\varepsilon,\inf_{0\le s\le t}X_s^j>0\}}\right].
\end{aligned}\tag{3.13}$$

Since neither of the firms i or j has defaulted by time t, we have

$$X_t^i=x^i+\mu t+\sqrt{1-\rho}W_t^i+\sqrt{\rho}M_t\overset{d}{=}x^i+\mu t+B_t^1;$$

$$X_t^j=x^j+\mu t+\sqrt{1-\rho}W_t^j+\sqrt{\rho}M_t\overset{d}{=}x^j+\mu t+B_t^2,$$

where B_t^1 and B_t^2 are correlated Brownian motions with correlation ρ.

We use the Girsanov theorem (e.g. [42]) to change the measure and set

$$Z_t(\mu)=\exp\left(-\frac{\mu}{1+\rho}\left(B_t^1+B_t^2+\mu t\right)\right),$$

which is easily seen to be a true martingale by Novikov's condition. We write $\tilde{\mathbb{P}}$ for the probability

measure on $\mathcal{F}_T$ given by

$$\tilde{\mathbb{P}}(A) := \mathbb{E}[1_A Z_T(\mu)];\ A \in \mathcal{F}_T, \tag{3.14}$$

and $\tilde{\mathbb{E}}$ for expectation with respect to $\tilde{\mathbb{P}}$. Thus for each fixed $T \in [0, \infty)$, the process

$$\{(\tilde{B}_t^1, \tilde{B}_t^2) := (B_t^1 + \mu t, B_t^2 + \mu t), \mathcal{F}_t, 0 \le t \le T\}$$

is a two-dimensional Brownian motion on $(\Omega, \mathcal{F}_T, \tilde{\mathbb{P}})$, where $\tilde{B}_t^1$ and $\tilde{B}_t^2$ have correlation ρ.

We now calculate the term $\mathbb{E}\left[1_{\{0<X_t^i<\varepsilon, \inf_{0\le s\le t} X_s^i>0,\ 0<X_t^j<\varepsilon, \inf_{0\le s\le t} X_s^j>0\}}\right]$ in (3.13). We have

$$\begin{aligned}
&\mathbb{E}\left[1_{\{0<X_t^i<\varepsilon, \inf_{0\le s\le t} X_s^i>0,\ 0<X_t^j<\varepsilon, \inf_{0\le s\le t} X_s^j>0\}}\right] \\
&= \tilde{\mathbb{E}}\left[1_{\{0<X_t^i<\varepsilon, \inf_{0\le s\le t} X_s^i>0,\ 0<X_t^j<\varepsilon, \inf_{0\le s\le t} X_s^j>0\}} \frac{1}{Z_T(\mu)}\right] \\
&\le \left\{\tilde{\mathbb{E}}\left[1_{\{0<X_t^i<\varepsilon, \inf_{0\le s\le t} X_s^i>0,\ 0<X_t^j<\varepsilon, \inf_{0\le s\le t} X_s^j>0\}}\right]\right\}^{1/a} \cdot \left\{\tilde{\mathbb{E}}\left[\left(\frac{1}{Z_T(\mu)}\right)^b\right]\right\}^{1/b} \\
&= J_1 \cdot J_2,
\end{aligned}$$

by Hölder's inequality, with $1/a + 1/b = 1, a > 1, b > 1$, and

$$J_1 = \left\{\tilde{\mathbb{E}}\left[1_{\{0<X_t^i<\varepsilon, \inf_{0\le s\le t} X_s^i>0,\ 0<X_t^j<\varepsilon, \inf_{0\le s\le t} X_s^j>0\}}\right]\right\}^{1/a},$$

$$J_2 = \left\{\tilde{\mathbb{E}}\left[\left(\frac{1}{Z_T(\mu)}\right)^b\right]\right\}^{1/b}.$$

For J_1, we have

$$\begin{aligned}
J_1 &= \left(\tilde{\mathbb{P}}\left\{0 < X_t^i < \varepsilon, \inf_{0\le s\le t} X_s^i > 0,\ 0 < X_t^j < \varepsilon, \inf_{0\le s\le t} X_s^j > 0\right\}\right)^{1/a} \\
&= \left(\tilde{\mathbb{P}}\left\{0 < x^i + \tilde{B}_t^1 < \varepsilon, \inf_{0\le s\le t} x^i + \tilde{B}_s^1 > 0,\ 0 < x^j + \tilde{B}_t^2 < \varepsilon, \inf_{0\le s\le t} x^j + \tilde{B}_s^2 > 0\right\}\right)^{1/a}.
\end{aligned}$$

By Lemma 3.4 with $\varrho = \rho, a_1 = x^i, a_2 = x^j$ we know that there exists $\varepsilon_0 = \frac{1}{3}\sqrt{\frac{1-\rho}{2}}\frac{\sqrt{(x^i)^2+(x^j)^2-2\rho x^i x^j}}{\sqrt{1-\rho^2}}$ and α as in (3.2), such that for all $\varepsilon < \varepsilon_0$ we have

$$\tilde{\mathbb{P}}\left\{0 < x^i + \tilde{B}_t^1 < \varepsilon, \inf_{0\le s\le t} x^i + \tilde{B}_s^1 > 0,\ 0 < x^j + \tilde{B}_t^2 \le <, \inf_{0\le s\le t} x^j + \tilde{B}_s^2 > 0\right\} \le C_T \varepsilon^{2+\frac{\pi}{\alpha}}.$$

As $x^i \ge C_B$ and $x^j \ge C_B$, we have

$$\sqrt{(x^i)^2 + (x^j)^2 - 2\rho x^i x^j} \ge \sqrt{2(1-\rho)} C_B.$$

Thus we can choose a new $\tilde{\varepsilon}_0 := \frac{1}{3}\sqrt{\frac{1-\rho}{1+\rho}} C_B \le C_B$, such that for all $\varepsilon < \tilde{\varepsilon}_0$ we have, for all i, j,

$$J_1 \le C_T \varepsilon^{\frac{2+\frac{\pi}{\alpha}}{a}}.$$

For J_2 we have

$$\begin{aligned}
J_2 &= \left\{ \tilde{\mathbb{E}} \left[\left(\frac{1}{Z_T(\mu)} \right)^b \right] \right\}^{1/b} \\
&= \left\{ \tilde{\mathbb{E}} \left[\exp\left(\frac{b\mu}{1+\rho} \left(\tilde{B}_T^1 + \tilde{B}_T^2 - \mu T \right) \right) \right] \right\}^{1/b} \\
&= \exp\left(-\frac{\mu^2 T}{1+\rho} \right) \left\{ \tilde{\mathbb{E}} \left[\exp\left(\frac{b\mu}{1+\rho} \left(\tilde{B}_T^1 + \tilde{B}_T^2 \right) \right) \right] \right\}^{1/b} \\
&= \exp\left(\frac{(b-1)\mu^2 T}{1+\rho} \right) := J_T < \infty, \ \ \forall b > 1.
\end{aligned}$$

Thus we have

$$\mathbb{E}[\nu_t((0,\varepsilon))^2] \leq J_1 \cdot J_2 \leq C_T J_T \varepsilon^{\frac{2+\frac{\pi}{\alpha}}{a}}, \ \forall \varepsilon < \tilde{\varepsilon}_0.$$

To conclude we note that for any $0 < \eta < \frac{\pi}{\alpha} - 1$ we can choose $1 < a = (2 + \frac{\pi}{\alpha})/(2 + \frac{\pi}{\alpha} - \eta) < (2 + \frac{\pi}{\alpha})/3$ and hence

$$\mathbb{E}[\nu_t((0,\varepsilon))^2] \leq K_T \varepsilon^{2+\frac{\pi}{\alpha}-\eta}, \ \forall \varepsilon < \tilde{\varepsilon}_0,$$

where K_T is a positive constant only depending on T. □

We will write $\beta = \frac{\pi}{\alpha} - \eta - 1 > 0$ so that $2 + \frac{\pi}{\alpha} - \eta = 3 + \beta$.

3.3 The existence and uniqueness of the density

In order to prove our main Theorem we need to recharacterise the evolution obtain in (3.5) as the stochastic PDE. Thus we need the measure ν_t to be absolutely continuous with respect to Lebesgue measure to write $\nu_t(dx) = v(t,x)dx$ for some density v.

We introduce some notation first. Let $H^0 = L^2((0,\infty))$ be the usual Hilbert space with L^2-norm $||\cdot||_0$ and inner product $\langle \cdot, \cdot \rangle_0$ given by $||\phi||_0^2 = \int_0^\infty |\phi(x)|^2 dx$ and $\langle \phi, \psi \rangle_0 = \int_0^\infty \phi(x)\psi(x)dx$. In the following we adapt the approach in [32] to our setting. The idea to prove the existence of an $L^2((0,\infty))$-density is to transform our $\mathcal{M}((0,\infty))$-valued process to an H^0-valued process, by convolving the measure with the absorbing heat kernel, where $\mathcal{M}((0,\infty))$ denotes the set of finite Borel measures on $(0,\infty)$.

For any $\varrho \in \mathcal{M}((0,\infty))$ and $\delta > 0$, we write

$$(T_\delta \varrho)(x) = \int_0^\infty G_\delta(x,y)\varrho(dy), \tag{3.15}$$

where G_δ is the absorbing heat kernel in $\mathbb{R}^+$ given by

$$G_\delta(x,y) = \frac{1}{\sqrt{2\pi\delta}} \left(e^{-\frac{(x-y)^2}{2\delta}} - e^{-\frac{(x+y)^2}{2\delta}} \right), \ \forall x, y > 0.$$

We use the same notation for the Brownian semigroup on $C_b(\mathbb{R}_+)$, the bounded and continuous

functions on $\mathbb{R}_+$, i.e.,

$$T_t\phi(x) = \int_0^\infty G_t(x,y)\phi(y)dy, \quad \forall \phi \in C_b(\mathbb{R}_+).$$

We will also need to consider the reflecting heat kernel $G^r_\delta(x,y)$, defined by

$$G^r_\delta(x,y) = \frac{1}{\sqrt{2\pi\delta}}\left(e^{-\frac{(x-y)^2}{2\delta}} + e^{-\frac{(x+y)^2}{2\delta}}\right), \ \forall x,y > 0.$$

We write the associated semigroup as

$$T^r_\delta \nu_t(x) = \int_0^\infty G^r_\delta(x,y)\nu_t(dy).$$

It is an easy calculation to see that

$$\partial_x G_\delta(x,y) = -\partial_y G^r_\delta(x,y). \tag{3.16}$$

It is not difficult to prove the following lemma.

Lemma 3.6. *If $\varrho \in \mathcal{M}((0,\infty))$ and $\delta > 0$, then $T_\delta \varrho \in H^0$.*

We will write $\nu_t \in H^0$ if the measure ν_t has a density which is in H^0. Let $Z_\delta(s) = T_\delta \nu_s$, where ν is an $\mathcal{M}((0,\infty))$-valued solution to (3.5). Our aim is to obtain an estimate for the H^0-norm of the process Z_δ.

It is easy to see that $T_\delta\phi \in C^\infty_K(0,\infty)$ for any $\phi \in C^\infty_K(0,\infty)$. Thus, replacing $\phi \in C^\infty_K(0,\infty)$ by $T_\delta\phi$ in (3.5) and using Fubini, we have

$$\begin{aligned}\langle Z_\delta(t), \phi\rangle_0 =& \langle T_\delta\phi, \nu_t\rangle \\ =& \langle T_\delta\phi, \nu_0\rangle + \int_0^t \langle \mu(T_\delta\phi)'(x) + \frac{1}{2}(T_\delta\phi)''(x), \nu_s\rangle ds + \int_0^t \langle \sqrt{\rho}(T_\delta\phi)'(x), \nu_s\rangle dM_s. \end{aligned} \tag{3.17}$$

The integrands can be rewritten as

$$\begin{aligned}\langle \mu(T_\delta\phi)'(x), \nu_s\rangle =& \mu\int_0^\infty (T_\delta\phi)'(x)\nu_s(dx) \\ =& \mu\int_0^\infty \partial_x\left(\int_0^\infty G_\delta(x,y)\phi(y)dy\right)\nu_s(dx) \\ =& \mu\int_0^\infty \left(\int_0^\infty (\partial_x G_\delta(x,y))\phi(y)dy\right)\nu_s(dx)\end{aligned}$$

Applying (3.16) and Fubini we have

$$\begin{aligned}\langle \mu(T_\delta\phi)'(x), \nu_s\rangle =& \mu\int_0^\infty \left(\int_0^\infty (-\partial_y G^r_\delta(x,y))\phi(y)dy\right)\nu_s(dx) \\ =& \mu\int_0^\infty \left(\int_0^\infty G^r_\delta(x,y)\phi'(y)dy\right)\nu_s(dx)\end{aligned}$$

$$
\begin{aligned}
&=\mu \int_0^\infty (T_\delta^r \nu_s)(y)\phi'(y)dy \\
&= -\mu \int_0^\infty \phi(y)\partial_y(T_\delta^r \nu_s(y))dy \\
&= -\mu\langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0.
\end{aligned}
$$

Similarly, for the term $\langle \sqrt{\rho}(T_\delta\phi)'(x), \nu_s\rangle$ we have

$$
\langle \sqrt{\rho}(T_\delta\phi)'(x), \nu_s\rangle = -\sqrt{\rho}\langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0.
$$

For the term $\langle \frac{1}{2}(T_\delta\phi)''(x), \nu_s\rangle$ we can perform the same type of calculation to see

$$
\langle \frac{1}{2}(T_\delta\phi)''(x), \nu_s\rangle = \frac{1}{2}\langle \phi, \partial_x^2 T_\delta(\nu_s)\rangle_0.
$$

Therefore (3.17) becomes

$$
\langle Z_\delta(t), \phi\rangle_0 = \langle T_\delta\nu_0, \phi\rangle_0 - \mu\int_0^t \langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0 ds + \frac{1}{2}\int_0^t \langle \phi, \partial_x^2 T_\delta(\nu_s)\rangle_0 ds - \sqrt{\rho}\int_0^t \langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0 dM_s. \tag{3.18}
$$

By using Itô's formula on $\langle Z_\delta(s), \phi\rangle_0^2$ we have

$$
\begin{aligned}
\langle Z_\delta(t), \phi\rangle_0^2 =& \langle Z_\delta(0), \phi\rangle_0^2 + \int_0^t d\langle Z_\delta(s), \phi\rangle_0^2 \\
=& \langle Z_\delta(0), \phi\rangle_0^2 + \int_0^t 2\langle Z_\delta(s), \phi\rangle_0 d\langle Z_\delta(s), \phi\rangle_0 + \int_0^t d\Big\langle \langle Z_\delta(s), \phi\rangle_0, \langle Z_\delta(s), \phi\rangle_0 \Big\rangle \\
=& \langle Z_\delta(0), \phi\rangle_0^2 - 2\mu\int_0^t \langle Z_\delta(s), \phi\rangle_0 \langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0 ds + \int_0^t \langle Z_\delta(s), \phi\rangle_0 \langle \phi, \partial_x^2 T_\delta(\nu_s)\rangle_0 ds \\
&- 2\sqrt{\rho}\int_0^t \langle Z_\delta(s), \phi\rangle_0 \langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0 dM_s + \rho\int_0^t |\langle \phi, \partial_x T_\delta^r(\nu_s)\rangle_0|^2 ds.
\end{aligned}
$$

We can choose a set of $\phi \in C_K^\infty(0,\infty)$ to be a complete, orthonormal basis of H^0 and taking expectations, we have

$$
\begin{aligned}
\mathbb{E}||Z_\delta(t)||_0^2 =& ||Z_\delta(0)||_0^2 - 2\mu\mathbb{E}\int_0^t \langle Z_\delta(s), \partial_x T_\delta^r(\nu_s)\rangle_0 ds + \mathbb{E}\int_0^t \langle Z_\delta(s), \partial_x^2 T_\delta(\nu_s)\rangle_0 ds \\
&+ \rho\mathbb{E}\int_0^t ||\partial_x T_\delta^r(\nu_s)||_0^2 ds \\
=& ||Z_\delta(0)||_0^2 - 2\mu\mathbb{E}\int_0^t \langle T_\delta(\nu_s), \partial_x T_\delta^r(\nu_s)\rangle_0 ds + \mathbb{E}\int_0^t \langle T_\delta(\nu_s), \partial_x^2 T_\delta(\nu_s)\rangle_0 ds \\
&+ \rho\mathbb{E}\int_0^t ||\partial_x T_\delta^r(\nu_s)||_0^2 ds.
\end{aligned} \tag{3.19}
$$

We now control the integral terms on the right-hand side of (3.19) in terms of the integral of $\mathbb{E}||T_\delta(\nu_s)||_0^2$ plus some constant which goes to 0 as $\delta \to 0$.

Lemma 3.7. *There exist constants C_T^1, C_2 such that for $\delta < \tilde{\varepsilon}_0^2/2$ we have*

$$\mathbb{E}[|-2\mu\langle T_\delta(\nu_s), \partial_x T_\delta^r(\nu_s)\rangle_0|] \le |\mu| \cdot \mathbb{E}[||T_\delta(\nu_s)||_0^2] + C_T^1 \delta^{\frac{\beta}{2}} + \frac{C_2 \tilde{\varepsilon}_0}{\delta^2} e^{-\tilde{\varepsilon}_0^2/2\delta}. \tag{3.20}$$

Proof.

$$\begin{aligned}
\langle T_\delta(\nu_s), \partial_x T_\delta^r(\nu_s)\rangle_0 &= \int_0^\infty T_\delta(\nu_s)(x)\partial_x T_\delta^r(\nu_s)(x)dx \\
&= \int_0^\infty T_\delta(\nu_s)(x)\left(\int_0^\infty \partial_x G_\delta^r(x,y)\nu_s(dy)\right)dx \\
&= \int_0^\infty T_\delta(\nu_s)(x)\left(\int_0^\infty (\partial_x G_\delta(x,y) - \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta})\nu_s(dy)\right)dx \\
&= \int_0^\infty T_\delta(\nu_s)(x)\int_0^\infty \partial_x G_\delta(x,y)\nu_s(dy)dx \\
&\quad - \int_0^\infty T_\delta(\nu_s)(x)\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)dx \\
&= \int_0^\infty T_\delta(\nu_s)(x)\partial_x T_\delta(\nu_s)(x)dx - \int_0^\infty T_\delta(\nu_s)(x)\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)dx \\
&= \frac{1}{2}\int_0^\infty \partial_x[(T_\delta(\nu_s)(x))^2]dx - \int_0^\infty T_\delta(\nu_s)(x)\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)dx \\
&= -\int_0^\infty T_\delta(\nu_s)(x)\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|-2\mu\langle T_\delta(\nu_s), \partial_x T_\delta^r(\nu_s)\rangle_0| &= \left|2\mu\int_0^\infty T_\delta(\nu_s)(x)\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)dx\right| \\
&\le \left|\mu\int_0^\infty (T_\delta(\nu_s)(x))^2 dx + \mu\int_0^\infty \left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)^2 dx\right| \\
&\le |\mu| \cdot ||T_\delta(\nu_s)||_0^2 + |\mu|\int_0^\infty \left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)^2 dx.
\end{aligned}$$

Now let us denote

$$P_1 := \int_0^\infty \left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)^2 dx$$

and derive a bound for P_1.

$$\begin{aligned}
P_1 &= \int_0^\infty \left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}} \frac{x+y}{\delta}\nu_s(dy)\right)^2 dx \\
&= \int_0^\infty \int_0^\infty \int_0^\infty (\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{(x+y_1)^2+(x+y_2)^2}{2\delta}} \frac{(x+y_1)(x+y_2)}{\delta^2}\nu_s(dy_1)\nu_s(dy_2)dx \\
&= \int_0^\infty \int_0^\infty \nu_s(dy_1)\nu_s(dy_2)\int_0^\infty (\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{(x+y_1)^2+(x+y_2)^2}{2\delta}} \frac{(x+y_1)(x+y_2)}{\delta^2}dx \\
&= \int_0^\infty \int_0^\infty \nu_s(dy_1)\nu_s(dy_2)\int_0^\infty (\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{1}{\delta}\left[(x+\frac{y_1+y_2}{2})^2+(\frac{y_1-y_2}{2})^2\right]} \frac{(x+\frac{y_1+y_2}{2})^2 - (\frac{y_1-y_2}{2})^2}{\delta^2}dx.
\end{aligned}$$

By changing variables using

$$z^2 = (x + \frac{y_1 + y_2}{2})^2 + (\frac{y_1 - y_2}{2})^2,$$

we have

$$P_1 = \int_0^\infty \int_0^\infty \nu_s(dy_1)\nu_s(dy_2) \int_{\sqrt{\frac{y_1^2+y_2^2}{2}}}^\infty (\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2 - \frac{(y_1-y_2)^2}{2}}{\delta^2} \frac{z}{\sqrt{z^2 - \left(\frac{y_1-y_2}{2}\right)^2}} dz.$$

Now, since

$$1 \le \frac{z}{\sqrt{z^2 - \left(\frac{y_1-y_2}{2}\right)^2}} \le \sqrt{2}$$

when $z \ge \sqrt{\frac{y_1^2+y_2^2}{2}}$, we have

$$\begin{aligned}
P_1 &\le \int_0^\infty \int_0^\infty \nu_s(dy_1)\nu_s(dy_2) \int_{\sqrt{\frac{y_1^2+y_2^2}{2}}}^\infty \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2 - \frac{(y_1-y_2)^2}{2}}{\delta^2} dz \\
&= \int_0^\infty \int_0^\infty \nu_s(dy_1)\nu_s(dy_2) \int_0^\infty 1_{\{y_1^2+y_2^2<2z^2\}} \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2 - \frac{(y_1-y_2)^2}{2}}{\delta^2} dz \\
&= \int_0^\infty \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} dz \int_0^\infty \int_0^\infty \frac{z^2 - \frac{(y_1-y_2)^2}{2}}{\delta^2} 1_{\{y_1^2+y_2^2<2z^2\}} \nu_s(dy_1)\nu_s(dy_2) \\
&\le \int_0^\infty \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2}{\delta^2} dz \int_0^\infty \int_0^\infty 1_{\{y_1^2+y_2^2<2z^2\}} \nu_s(dy_1)\nu_s(dy_2) \\
&\le \int_0^\infty \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2}{\delta^2} dz \int_0^\infty \int_0^\infty 1_{\{y_1<\sqrt{2}z, y_2<\sqrt{2}z\}} \nu_s(dy_1)\nu_s(dy_2) \\
&\le \int_0^\infty \sqrt{2}(\frac{2}{\sqrt{2\pi\delta}})^2 e^{-\frac{z^2}{\delta}} \frac{z^2}{\delta^2} (\nu_s((0, \sqrt{2}z)))^2 dz \\
&= \int_0^\infty \frac{2}{\pi\delta} e^{-\frac{z^2}{2\delta}} \frac{z^2}{2\delta^2} (\nu_s((0, z)))^2 dz.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}[| - 2\mu\langle T_\delta(\nu_s), \partial_x T_\delta^r(\nu_s)\rangle_0|] \\
&\quad \le \quad |\mu| \cdot \mathbb{E}[||T_\delta(\nu_s)||_0^2] + |\mu|\mathbb{E}[P_1] \\
&\quad \le \quad |\mu| \cdot \mathbb{E}[||T_\delta(\nu_s)||_0^2] + |\mu|\mathbb{E}\left[\int_0^\infty \frac{2}{\pi\delta} e^{-\frac{z^2}{2\delta}} \frac{z^2}{2\delta^2} (\nu_s((0, z)))^2 dz\right] \\
&\quad = \quad |\mu| \cdot \mathbb{E}[||T_\delta(\nu_s)||_0^2] + |\mu| \int_0^\infty \frac{2}{\pi\delta} e^{-\frac{z^2}{2\delta}} \frac{z^2}{2\delta^2} \mathbb{E}[\nu_s((0, z))^2] dz.
\end{aligned}$$

By Lemma 3.5 in the last section we know that for the measure-valued solution ν_s of (3.5), there exists $\tilde{\varepsilon}_0 > 0$ and a $\beta > 0$ such that for all $z < \tilde{\varepsilon}_0$ we have

$$\mathbb{E}[\nu_s^+((0, z))^2] \le K_T z^{3+\beta}.$$

Hence we have

$$
\begin{aligned}
&\mathbb{E}[|-2\mu\langle T_\delta(\nu_s),\partial_x T_\delta^r(\nu_s)\rangle_0|]\\
\leq&|\mu|\cdot\mathbb{E}[||T_\delta(\nu_s)||_0^2]+|\mu|\int_0^{\tilde{\varepsilon}_0}\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}\mathbb{E}[\nu_s((0,z))^2]dz+|\mu|\int_{\tilde{\varepsilon}_0}^\infty\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}\mathbb{E}[(\nu_s((0,z]))^2]dz\\
\leq&|\mu|\cdot\mathbb{E}[||T_\delta(\nu_s)||_0^2]+|\mu|\int_0^{\tilde{\varepsilon}_0}\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}K_T z^{3+\beta}dz+4|\mu|\int_{\tilde{\varepsilon}_0}^\infty\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}dz\\
\leq&|\mu|\cdot\mathbb{E}[||T_\delta(\nu_s)||_0^2]+|\mu|\int_0^{\infty}\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}K_T z^{3+\beta}dz+4|\mu|\int_{\tilde{\varepsilon}_0}^\infty\frac{2}{\pi\delta}e^{-\frac{z^2}{2\delta}}\frac{z^2}{2\delta^2}dz\\
=&|\mu|\cdot\mathbb{E}[||T_\delta(\nu_s)||_0^2]+|\mu|K_T\frac{2^{3+\frac{\beta}{2}}}{\pi}\delta^{\frac{\beta}{2}}\int_0^\infty e^{-x^2}x^{5+\beta}dx+4|\mu|\frac{2\sqrt{2}}{\pi\delta^{\frac{3}{2}}}\int_{\frac{\tilde{\varepsilon}_0}{\sqrt{2\delta}}}^\infty e^{-x^2}x^2dx
\end{aligned}
$$

Finally we observe that for $\eta>0$

$$\int_\eta^\infty x^2e^{-x^2}dx\leq\frac{1}{2}(\eta+\frac{1}{\eta})e^{-\eta^2},$$

and hence setting $\eta=\frac{\tilde{\varepsilon}_0}{\sqrt{2\delta}}$, so that by assumption $\eta>1$, we have

$$\mathbb{E}[|-2\mu\langle T_\delta(\nu_s),\partial_x T_\delta^r(\nu_s)\rangle_0|]\leq|\mu|\cdot\mathbb{E}[||T_\delta(\nu_s)||_0^2]+C_T^1\delta^{\frac{\beta}{2}}+\frac{C_2\tilde{\varepsilon}_0}{\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta}$$

where

$$C_T^1=|\mu|K_T\frac{2^{3+\frac{\beta}{2}}}{\pi}\int_0^\infty e^{-x^2}x^{5+\beta}dx\quad\text{and}\quad C_2=|\mu|\frac{8\sqrt{2}}{\pi}.$$

□

Lemma 3.8. *For $\delta<\tilde{\varepsilon}_0^2/2$ we have*

$$\mathbb{E}[\langle T_\delta(\nu_s),\partial_x^2T_\delta(\nu_s)\rangle_0+\rho||\partial_xT_\delta^r(\nu_s)||_0^2]\leq\frac{\rho}{1-\rho}\left(C_T^1\delta^{\frac{\beta}{2}}+\frac{C_2\tilde{\varepsilon}_0}{\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta}\right),\tag{3.21}$$

where C_T^1,C_2 and $\tilde{\varepsilon}_0$ are the same as in Lemma 3.7.

Proof. Integrating by parts gives $\langle T_\delta(\nu_s),\partial_x^2T_\delta(\nu_s)\rangle_0=-||\partial_xT_\delta\nu_s||_0^2$. We also have

$$
\begin{aligned}
||\partial_xT_\delta^r(\nu_s)||_0^2=&\int_0^\infty(\partial_xT_\delta^r(\nu_s)(x))^2dx\\
=&\int_0^\infty\left(\partial_xT_\delta\nu_s(x)-\int_0^\infty\frac{2}{\sqrt{2\pi\delta}}e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2dx\\
=&||\partial_xT_\delta\nu_s||_0^2-2\int_0^\infty\partial_xT_\delta\nu_s(x)\int_0^\infty\frac{2}{\sqrt{2\pi\delta}}e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)dx\\
&+\int_0^\infty\left(\int_0^\infty\frac{2}{\sqrt{2\pi\delta}}e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2dx.
\end{aligned}
$$

Putting these together

$$
\begin{aligned}
&\langle T_\delta(\nu_s), \partial_x^2 T_\delta(\nu_s)\rangle_0 + \rho||\partial_x T_\delta^r(\nu_s)||_0^2 \\
&= -2\rho\int_0^\infty \partial_x T_\delta \nu_s(x)\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)dx \\
&\quad +\rho\int_0^\infty\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2 dx - (1-\rho)\int_0^\infty (\partial_x T_\delta(\nu_s)(x))^2 dx \\
&\le \left|2\rho\int_0^\infty \sqrt{\frac{1-\rho}{\rho}}\partial_x T_\delta \nu_s(x)\int_0^\infty \sqrt{\frac{\rho}{1-\rho}}\frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)dx\right| \\
&\quad +\rho\int_0^\infty\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2 dx - (1-\rho)\int_0^\infty (\partial_x T_\delta(\nu_s)(x))^2 dx \\
&\le \rho\int_0^\infty\left(\sqrt{\frac{1-\rho}{\rho}}\partial_x T_\delta \nu_s(x)\right)^2 dx + \rho\int_0^\infty\left(\int_0^\infty \sqrt{\frac{\rho}{1-\rho}}\frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2 dx \\
&\quad +\rho\int_0^\infty\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2 dx - (1-\rho)\int_0^\infty (\partial_x T_\delta(\nu_s)(x))^2 dx \\
&= \frac{\rho}{1-\rho}\int_0^\infty\left(\int_0^\infty \frac{2}{\sqrt{2\pi\delta}} e^{-\frac{(x+y)^2}{2\delta}}\frac{x+y}{\delta}\nu_s(dy)\right)^2 dx.
\end{aligned}
$$

By the estimate for P_1 obtained in Lemma 3.7 we have

$$
\mathbb{E}[\langle T_\delta(\nu_s), \partial_x^2 T_\delta(\nu_s)\rangle_0 + \rho||\partial_x T_\delta^r(\nu_s)||_0^2] \le \frac{\rho}{1-\rho}\mathbb{E}[P_1] \le \frac{\rho}{1-\rho}\left(C_T^1\delta^{\frac{\beta}{2}} + \frac{C_2\tilde{\varepsilon}_0}{\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta}\right),
$$

where C_T^1, C_2 and $\tilde{\varepsilon}_0$ are the same as in Lemma 3.7. □

Now, combining Lemma 3.7 and 3.8 gives the following

Theorem 3.9. *If ν_t is an $\mathcal{M}(\mathbb{R}^+)$-valued solution of (3.5) and $Z_\delta(t) = T_\delta\nu_t$, we have for $\delta < \tilde{\varepsilon}_0^2/2$,*

$$
\mathbb{E}||Z_\delta(t)||_0^2 \le ||Z_\delta(0)||_0^2 + |\mu|\int_0^t \mathbb{E}||T_\delta(\nu_s)||_0^2 ds + \frac{1}{1-\rho}C_T^1\delta^{\frac{\beta}{2}}t + \frac{C_2 t\tilde{\varepsilon}_0}{(1-\rho)\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta}. \tag{3.22}
$$

Corollary 3.10. *If ν_t is a measure-valued solution of (3.5) and $\nu_0 \in H^0$, then $\nu_t \in H^0$, a.s. and $\mathbb{E}||\nu_t||_0^2 < \infty$, $\forall t \ge 0$.*

Proof. By (3.22) we have for small δ that

$$
\begin{aligned}
\mathbb{E}||Z_\delta(t)||_0^2 &\le ||Z_\delta(0)||_0^2 + |\mu|\int_0^t \mathbb{E}||T_\delta(\nu_s)||_0^2 ds + \frac{1}{1-\rho}C_T^1\delta^{\frac{\beta}{2}}T + \frac{C_2 T\tilde{\varepsilon}_0}{(1-\rho)\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta} \\
&:= ||Z_\delta(0)||_0^2 + |\mu|\int_0^t \mathbb{E}||Z_\delta(s)||_0^2 ds + f(\delta, T),
\end{aligned}
$$

where

$$
f(\delta, T) = \frac{1}{1-\rho}C_T^1\delta^{\frac{\beta}{2}}T + \frac{C_2 T\tilde{\varepsilon}_0}{(1-\rho)\delta^2}e^{-\tilde{\varepsilon}_0^2/2\delta}.
$$

Applying Gronwall's inequality we have

$$\mathbb{E}||Z_\delta(t)||_0^2 \leq (||Z_\delta(0)||_0^2 + f(\delta,T))e^{|\mu|t}.$$

It is clear that $\lim_{\delta\to 0} f(\delta,T) = 0$. Now let $\{\phi_j\}$ be a complete, orthonormal system for H^0 such that $\phi_j \in C_b(\mathbb{R}^+)$. Then by Fatou's lemma,

$$\mathbb{E}\left[\sum_j \langle \phi_j, \nu_t\rangle^2\right] = \mathbb{E}\left[\sum_j \lim_{\delta\to 0}\langle \phi_j, T_\delta \nu_t\rangle^2\right] \leq \liminf_{\delta\to 0} \mathbb{E}||Z_\delta(t)||_0^2 \leq ||\nu_0||_0^2 e^{|\mu|t},$$

Therefore $\nu_t \in H^0$ and $\mathbb{E}||\nu_t||_0^2 < \infty$, $\forall t \geq 0$. □

Now we have proved the existence of an L^2-density for the limit empirical measure ν_t, given that ν_0 has an L^2-density.

Theorem 3.11. *Suppose that $\nu_0 \in H^0$. Then (3.5) has at most one measure-valued solution.*

Proof. Let ν_t^1 and ν_t^2 be two measure-valued solutions with the same initial value ν_0, and both of them satisfy the boundary condition stated in Lemma 3.5. By Corollary 3.10, $\nu_t^1, \nu_t^2 \in H^0$ a.s.. Let $\nu_t = \nu_t^1 - \nu_t^2$. Then $\nu_t \in H^0$ and also ν_t is a signed measure-valued solution to the equation (3.5). It is straightforward to extend all the estimates we have obtained to the case of the difference of two solutions as $|\nu_t| \leq \nu_t^1 + \nu_t^2$ and the equations are linear.

Therefore by the appropriate extension of Theorem 3.9 we have for $\delta < \tilde{\varepsilon}_0^2/2$

$$\mathbb{E}||T_\delta \nu_t||_0^2 \leq |\mu| \int_0^t \mathbb{E}||T_\delta(|\nu_s|)||_0^2 ds + \frac{2}{1-\rho} C_T^1 \delta^{\frac{\beta}{2}} T + \frac{2C_2 T\tilde{\varepsilon}_0}{(1-\rho)\delta^2} e^{-\tilde{\varepsilon}_0^2/2\delta}.$$

As before, taking $\delta \to 0$, we have

$$\mathbb{E}||\nu_t||_0^2 \leq |\mu| \int_0^t \mathbb{E}|||\nu_s|||_0^2 ds = |\mu| \int_0^t \mathbb{E}||\nu_s||_0^2 ds,$$

and by Gronwall's inequality, we have $\nu_t \equiv 0$. □

This completes the proof of the uniqueness of the L^2-valued solution to the equation (3.5).

3.4 The limit SPDE

Substituting the Lebesgue representation for the empirical measure into (3.5), integrating by parts and writing $\mathcal{A}^\dagger$ for the adjoint operator of $\mathcal{A}$, we get

$$\begin{aligned}\int \phi(x)v(t,x)\,dx &= \int \phi(x)v(0,x)\,dx + \int_0^t \int \mathcal{A}\phi(x)v(s,x)\,dx\,ds + \int_0^t \int \sqrt{\rho}\phi'(x)v(s,x)\,dx\,dM_s \\ &= \int \phi(x)\left(v(0,x) + \int_0^t \mathcal{A}^\dagger v(s,x)\,ds - \int_0^t \frac{\partial}{\partial x}\left(\sqrt{\rho}v(s,x)\right)\,dM_s\right)\,dx.\end{aligned}$$

As this holds $\forall \phi \in C_K^\infty(0,\infty)$ we have shown that we have a weak solution to the SPDE given by

$$v(t,x) = v(0,x) + \int_0^t \mathcal{A}^\dagger v(s,x)\, ds - \int_0^t \frac{\partial}{\partial x}\left(\sqrt{\rho} v(s,x)\right)\, dM_s, \tag{3.23}$$

with $v(t,0) = 0$ for all $t \in [0,T]$. Alternatively, we can write this in differential form

$$dv(t,x) = -\mu \frac{\partial v}{\partial x}(t,x)dt + \frac{1}{2}\frac{\partial^2 v}{\partial x^2}(t,x)dt - \sqrt{\rho}\frac{\partial v}{\partial x}(t,x)\mathrm{d}M_t, \tag{3.24}$$

with $v(t,0) = 0$ for all $t \in [0,T]$ and $v(0,x) = v_0(x)$. This is a stochastic PDE that describes the evolution of the distance to default of an infinite portfolio of assets whose dynamics are given by (1.2). However note that the derivatives are only defined in the weak sense.

We can now use the limiting empirical measure ν_t to approximate the loss distribution for a portfolio of fixed size N whose assets also follow (1.2). We do this by matching the initial conditions, thus setting

$$v(0,x) = \frac{1}{N}\sum_{i=1}^N \delta_{X_0^i}(x), \tag{3.25}$$

where the $X_0^i > 0$, $i = 1,\dots,N$ are the initial values for the distance to default of the assets in our fixed portfolio of size N.

3.5 Solving the SPDE

The SPDE (1.5) without the boundary condition is easily solved. A simple check with the Ito formula shows that

$$v(t,x) = u(t, x - \sqrt{\rho}M_t), \quad \forall x \in \mathbb{R}, t > 0, \tag{3.26}$$

where $u(t,x)$ is the solution to the deterministic PDE

$$u_t = \frac{1}{2}(1-\rho)u_{xx} - \frac{1}{\sigma}(r - \frac{1}{2}\sigma^2)u_x, \tag{3.27}$$

with $u(0,x) = v_0(x)$.

The SPDE with the boundary condition has been treated in [31]. This allows us to complete the proof of our existence and uniqueness theorem.

Theorem 3.12. *Let $v_0(x) \in H^1((0,\infty))$. The SPDE (1.5) has a unique solution $u \in L^2(\Omega \times (0,T), \mathcal{G}, H^1((0,\infty)))$ and is such that $xu_{xx} \in L^2(\Omega \times (0,T), \mathcal{G}, L^2((0,\infty)))$.*

Proof. The result follows from Theorem 2.1 of [31]. Thus all we have to do is ensure that the conditions of that Theorem hold in our setting. The boundary of the domain $(0,\infty)$ is the single point 0 and hence we can take the function $\psi(x) = \min(x,1)$ in the Theorem. The single point boundary trivially satisfies the Hypothesis 2.1 of [31]. The coefficients of our SPDE are constants and hence satisfy the measurability requirement of Hypothesis 2.2 and the Lipschitz condition of Hypothesis 2.4. Hypothesis 2.3 also follows as the coefficients are constants and the initial condition is in H^1. □

Proof. (of Theorem 1.1): Our previous work has shown that the empirical measure satisfies (3.5) and has a unique density in $L^2((0,\infty))$. By Theorem 3.12 the SPDE with boundary condition has a unique solution in $H^1((0,\infty))$. As this solution satisfies (3.5), by the uniqueness of solutions, it must be the density for our empirical measure. Thus our density satisfies the SPDE. □

We can derive a formal expression for L_t in terms of the density after integrating by parts.

$$\begin{aligned}
L_t =& 1 - \int_0^\infty v(t,x)dx \\
=& 1 - \int_0^\infty \left(v(0,x) - \int_0^t \mu v_x(s,x)ds + \int_0^t \frac{1}{2} v_{xx}(s,x)ds - \int_0^t \sqrt{\rho} v_x(s,x)dM_s \right) dx \\
=& 1 - \int_0^\infty v(0,x)dx + \mu \int_0^t v(s,x)|_{x=0}^{x=\infty} ds - \int_0^t \frac{1}{2} v_x(s,x)|_{x=0}^{x=\infty} ds + \sqrt{\rho} \int_0^t v(s,x)|_{x=0}^{x=\infty} dM_s.
\end{aligned}$$

Since $x^i > 0, \forall i$ and X_t^i is a continuous process, we can conclude that $T_0^i > 0, \forall i$ and $L_0 = 0$. Therefore

$$\bar{\nu}(\mathbb{R}^+ \cup \{0\}) = 1 = \int_0^\infty v(0,x)dx.$$

Moreover we have $v(s,x) \to 0, v_x(s,x) \to 0$, as $x \to \infty$ and $v(s,0) = 0$, $\forall s$. Therefore, provided that $v_x(s,0)$, the right derivative of $v(s,x)$ with respect to x at the point $x = 0$, exists we would have

$$L_t = \frac{1}{2} \int_0^t v_x(s,0)ds.$$

One issue that has not been addressed is the existence of C^2 solutions to this equation. We note that the work of Lototsky [33] shows that there is a classical C^2 solution to this SPDE over a bounded domain $(0,K)$, with Dirichlet boundary conditions at 0 and K, provided that the initial condition is smooth enough.

3.6 The portfolio loss

We would like to price portfolio credit derivatives whose values depend on the cumulative defaults occurring within a reference basket of risky assets. The key to pricing these instruments is determining the joint loss distribution. We have just derived an equation that describes the evolution of the empirical measure of the limiting large portfolio of assets. At any future value in time, we can determine the loss in the portfolio by calculating the total mass of the empirical measure of assets that have not defaulted. Thus the portfolio loss L_t^N can be approximated by

$$L_t^N = NL_t,$$

where N is the number of assets in the portfolio. We note that given the initial condition (3.25) we have $L_0^N = 0$. Also, due to the way in which defaults are incorporated into the model, we have

$$\begin{aligned}
0 \leq \ L_t \leq 1, & \qquad \text{for } t \geq 0 \\
P(L_s \geq K) \leq P(L_t \geq K), & \quad \text{for } s \leq t,
\end{aligned}$$

which ensures that there is no arbitrage in the loss distribution.

3.7 A connection with filtering

We note that the SPDE can be viewed as a PDE with a Brownian drift. This is easily seen through an interpretation as the Zakai equation for a filtering problem. Let $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ be a probability space. Under $\tilde{\mathbb{P}}$ we define the signal process X to be a stochastic process satisfying

$$\begin{aligned} dX &= \mu dt - \sqrt{\rho} dM + \sqrt{1-\rho} dW, \quad t \leq \tau_0 \\ X_t &= 0, \quad t > \tau_0 \end{aligned}$$

where $\tau_0 = \inf\{t : X_t = 0\}$, where μ, ρ are constants and M and W are independent Brownian motions and $X_0 = x$. The observation process Y is taken to be just the market noise,

$$dY_t = dM_t,$$

then the Zakai equation (see for example [2]) for the conditional distribution of the signal given the observations is exactly our SPDE.

Thus, by standard filtering theory, if we want to compute a functional of the signal we need to calculate

$$m_\psi(t) = \tilde{\mathbb{E}}(\psi(X_t)|\mathcal{F}_t^M) = \int \psi(y) u(t,x) dx.$$

This means that the probability distribution for the position of a company given the market noise has a density $u(t,x)$ satisfying

$$du(t,x) = (-\mu u_x(t,x) + \frac{1}{2} u_{xx}(t,x))dt - \sqrt{\rho} u_x(t,x) dM_t,$$

with $u(0,x) = u_0(x)$, that is the initial guess at X_0 is the density $u_0(x)$ and $u(t,0) = 0$. Thus for the loss function we are interested in computing the proportion of companies that have defaulted by time t and this can be found by computing $m_\psi(t)$ for $\psi(t) = I_{\{\tau_0 < t\}}$. If we start from a given fixed point so that $u_0(x)$ is a delta function at x. Then

$$L_t = m_\psi(t) = \tilde{\mathbb{P}}^x(\inf_{s \leq t} X_s < 0 | \mathcal{F}_t^M).$$

Now the process X can be written as a Brownian motion with drift

$$X_t = x + \mu t - \sqrt{\rho} M_t + \sqrt{1-\rho} W_t,$$

and if we are given M, this can be expressed as

$$X_t = \sqrt{1-\rho} \left(\frac{x + f(t)}{\sqrt{1-\rho}} + W_t \right),$$

where $f(t) = \mu t - \sqrt{\rho} M_t$ is a deterministic time dependent drift function, a fixed random path.

To compute the random loss function we set $x' = x/\sqrt{1-\rho}, g(t) = f(t)/\sqrt{1-\rho}$, giving

$$L_t = \tilde{\mathbb{P}}^x(\inf_{s\leq t} X_s < 0|\mathcal{F}_t^M) = \tilde{\mathbb{P}}(\inf_{s\leq t} g(s) + W_s < -x'|\mathcal{F}_t^M).$$

In the case where we have a general initial distribution $u_0(x)$, the loss function is then

$$L_t = \int_0^\infty u_0(x)\tilde{\mathbb{P}}(\inf_{s\leq t} g(s) + W_s < -x/\sqrt{1-\rho}|\mathcal{F}_t^M)dx.$$

Thus we can try to compute this by solving the hitting time problem for Brownian motion with time dependent drift for a fixed realization of the market noise. It is straightforward to use this to simulate a realization of the loss function.

To derive this SPDE we made some simplifying assumptions. The first of these arose when specifying the asset processes in (1.2). We had to set the drift and volatility of all the assets to some common value. For the drift this is not a problem, because under the risk neutral measure it will be transformed to a value that excludes arbitrage. The fact that there is only one yield curve means that this value will be the same for all assets. If our reference portfolio contained entities denominated in more than one currency this would not be the case and some approximation would have to be made.

This argument cannot be used for the volatility as it is not affected by a change of measure. Therefore, it would seem that giving the assets one common value of volatility is a very restrictive assumption. However, for any given value of the volatility we still have the freedom to choose the default barrier specific to any one asset. Via the distance-to-default transformation this freedom manifests itself in our particular choice of starting value for each process. The effect of changing the barrier and changing the volatility is very similar. To see this note that default risk is measured by how many standard deviations away from the barrier our process is. To increase the default risk we need to reduce this distance which can be done by either increasing the standard deviation or moving the barrier closer. Although these are clearly not equivalent transformations they have a very similar effect and so the single volatility assumption is not as restrictive as it initially appears.

Having a single volatility number also eases calibration as we do not have to estimate the volatilities of all of the entities within our portfolio. Instead, we will have to replace it by some 'average' market volatility. Not only will this help day-to-day calibration stability but it means that credit derivative prices will be a function of one volatility parameter only. This is usually a desirable property from a practitioner's point of view as it allows one to take a view on that parameter; this cannot be done if there were a single parameter for each entity within our portfolio.

The major simplification that allowed us to derive our SPDE came when we moved to an infinite dimensional limit. In this limit, the idiosyncratic noise of the individual assets is averaged out. In fact, we could have any number of idiosyncratic components, provided they are independent and uncorrelated, and they would average out to zero. It is only the correlated components between the assets that remain i.e. the market risk. Note that this means that if the limiting portfolio was fully diversified, that is had no correlation, there would be no noise in the limit and

the limit portfolio would evolve deterministically!

4 Numerical solution

We outline in the following a numerical method for approximating the solution to the SPDE, which we use in the market pricing examples in the next section. We start with the SPDE (3.5) in weak form, repeated here for convenience,

$$\langle \phi, \nu_t \rangle = \langle \phi, \nu_0 \rangle + \int_0^t \langle \mathcal{A}\phi, \nu_s \rangle \, ds + \int_0^t \langle \sqrt{\rho}\phi', \nu_s \rangle \, dM_s$$

for almost all t and all smooth test functions $\phi \in C_K^\infty(0,\infty)$. It follows from Theorem 1.1 that ν_t has as one component the density v (describing the non-absorbed element) satisfying

$$(\phi, v(t,\cdot)) = (\phi, v(0,\cdot)) + \int_0^t (\mathcal{A}\phi, v(s,\cdot)) \, ds + \sqrt{\rho} \int_0^t (\phi', v(s,\cdot)) \, dM_s, \tag{4.1}$$

where here we write $(\cdot,\cdot)$ for the L^2 inner product. Integrating by parts, noting from Theorem 3.12 that $v(t,\cdot) \in H_0^1$ with dense subspace $C_K^\infty(0,\infty)$,

$$(\phi, v(t,\cdot)) + \int_0^t a(\phi, v(s,\cdot)) ds = (\phi, v(0,\cdot)) + \sqrt{\rho} \int_0^t (\phi', v(s,\cdot)) \, dM_s$$

for all $\phi \in H_0^1$, where

$$a(\phi, v) = \frac{1}{2}(\phi', v') - \sqrt{\rho}(\phi', v).$$

4.1 Finite element approximation

Let $V_h \subset H_0^1([x_0, x_N])$ be the space of piecewise linear functions on a grid $x_1 < \ldots < x_N$, which are zero at $x_1 = 0$ and x_N a sufficiently large value (see 4.3). Denote further by $\{\phi_n : 1 \le n \le N\}$ the standard finite element basis (see e.g. [40] for standard finite element theory and approximations to PDEs). Restricting both the solution and test functions to V_h,

$$(\phi_n, v_h(t,\cdot)) + \int_0^t a(\phi_n, v_h(s,\cdot)) ds = (\phi_n, v_h(0,\cdot)) + \sqrt{\rho} \int_0^t (\phi_n', v_h(s,\cdot)) \, dM_s$$

(for all $1 \le n \le N$) defines a semi-discrete finite element approximation.

Using the stochastic θ-scheme (see [21]) for the time discretisation of the resulting SDE system,

$$\left(\phi_n, v_h^{m+1}\right) + \theta \Delta t \, a(\phi_n, v_h^{m+1}) = (\phi_n, v_h^m) - (1-\theta)\Delta t \, a(\phi_n, v_h^m) + \sqrt{\rho} \, (\phi_n', v_h^m) \sqrt{\Delta t} \Phi_m, \tag{4.2}$$

where $\Phi_m \sim N(0,1)$, $\Delta t = t_{m+1} - t_m$ is assumed constant and $v_h^m = \sum_{n=1}^N v_n^m \phi_n$. Thus one gets a linear system

$$(M + \theta \Delta t A) v^{m+1} = (M - (1-\theta)\Delta t A) v^m + \sqrt{\rho}\sqrt{\Delta t} \Phi_m D v^m, \tag{4.3}$$

where $v^m = (v_1^m, \ldots, v_N^m)$ and the standard finite element matrices are given by

$$\begin{aligned} M_{ij} &= (\phi_i, \phi_j), \quad 1 \le i, j \le N, \\ A_{ij} &= a(\phi_i, \phi_j), \quad 1 \le i, j \le N, \\ D_{ij} &= (\phi_i', \phi_j), \quad 1 \le i, j \le N. \end{aligned}$$

This gives a pathwise (in M, the market factor) approximation to the SPDE solution via timestepping from an initial density $v_h(0, \cdot)$, which is found by L^2 projection of $\bar{\nu}_{N_f,t}$ from (1.4) with N_f firms onto the finite element space (see e.g. [39], [41]).

4.2 Simulating tranche spreads

For a given (numerical) realisation of the market factor, we can approximate the loss functional L_{T_k} at time T_k by

$$L_{T_k}^h = 1 - \int_0^{x_N} v_h(T_k, x)\,\mathrm{d}x \approx 1 - h \sum_{n=1}^{N-1} v_n^m \tag{4.4}$$

where $m = T_k/\Delta t$. If we explicitly include the dependence on the Monte Carlo samples $\Phi = (\Phi_i)_{1 \le i \le I}$ in $L_{T_k}^h(\Phi)$, where Φ_i as in (4.2) are drawn independently from a standard normal distribution, then for N_{sims} simulations with samples $\Phi^l = (\Phi_i^l)_{1 \le i \le I}$, $1 \le l \le N_{sims}$, we simulate the outstanding tranche notional (2.3) as

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[Z_{T_k}] &\approx \mathbb{E}^{\mathbb{Q}}[\max(d - L_{T_k}^h, 0) - \max(a - L_{T_k}^h, 0)] \\ &\approx \frac{1}{N_{sims}} \sum_{l=1}^{N_{sims}} \left(\max(d - L_{T_k}^h(\Phi^l), 0) - \max(a - L_{T_k}^h(\Phi^l), 0)\right). \end{aligned}$$

This gives simulated tranche spreads via (2.5), (2.6) and (2.7).

4.3 Accuracy and further approximations

We now discuss the approximations made previously and further simplifications made in the numerical implementation of the examples in the next section.

It is necessary for the finite element discretisation to approximate the semi-infinite boundary value problem for the SPDE by one on a finite domain. It is expected that if the upper boundary is sufficiently large, dependent on the initial distances-to-default and model parameters, the probability of crossing this boundary can be made negligible and zero boundary conditions are appropriate. We have checked this to be the case for the following numerical simulations but do not have a theoretical justification at this point.

The derivation of the SPDE and finite element solution assume H^1 initial data, however in practice we want to use a sum of atomic measures (3.25) corresponding to the distance-to-default of individual firms, as backed out from CDS spreads. We deal with this by projecting these data onto the finite element basis (see e.g. [39], [41]).

The majority of the literature on stochastic finite element methods deals with stochastic

diffusion coefficients (see e.g. [12] and subsequent work) and we are not aware of results which cover our setting with stochastic drift. From standard finite element approximation results for PDEs (see e.g. [40]), one would expect (pathwise) convergence order two in h for solutions in H^2, but Theorem 3.12 suggests weaker regularity at the absorbing boundary, which we also observe in the numerical solutions. This does not show a measurable impact on the numerical accuracy in practice. The weak approximation order of the Euler scheme for SDEs, and that for the chosen fully implicit scheme for PDEs ($\theta = 1$ in (4.2)), is one (in Δt). In this case, the scheme is stable in the mean-square sense of [21]. This is confirmed by numerical experiments, but a rigorous numerical analysis is beyond the scope of this paper.

A common approximation to the finite element system is to 'lump' M in (4.3) in diagonal form, interpretable as application of a quadrature rule, and ultimately results in M being replaced by a multiple of the identity matrix. With this approximation, the finite element scheme becomes identical to a central finite difference approximation.

A further simplification is suggested by the solution (3.26) of the SPDE without absorbing boundary condition, which decouples the solution into the PDE solution (3.27) on a doubly-infinite domain, and a random (normal) offset. This is easy to implement if we apply boundary conditions only at a discrete set of times. In analogy to discretely sampled barrier options, this corresponds to a situation where we observe default not continuously, but only at discrete dates. The numerical results in the next section were obtained in this way with default monitoring at payment dates for computational convenience. This introduces a small shift in the calibrated parameters compared to the SPDE with continuously absorbing barrier but the reported results on tranche spreads are almost identical.

The Monte Carlo estimates of outstanding tranche notionals and subsequently tranche spreads converge per $N_{sims}^{-1/2}$. The variance relative to the spread is larger for senior tranches due to the rarity of losses in these tranches, as illustrated by Figure 1. Importance sampling could cure this

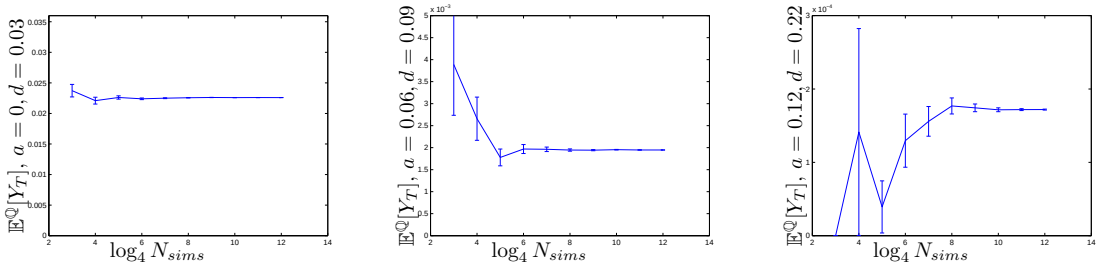


Figure 1: Monte Carlo estimators with standard error bars for expected losses (2.4) in tranches $[0, 3\%]$, $[6\%, 9\%]$, $[12\%, 22\%]$, for $N_{sims} = 16 \cdot 4^{k-1}$, $k = 1, ..., 10$, and a typical set of parameters, maturity $T = 5$.

problem but was not found necessary for the purposes of this study.

Numerical parameters were in the following adjusted such that the (heuristically) estimated approximation error was sufficiently small compared to the effects observed by varying model parameters.

5 Market pricing examples

In this section, we analyse our model's ability to price regular index tranches for all maturities and investigate the implied correlation skew. We consider performance pre and post the onset of the credit crunch, illustrating the model's inherent ability to cope with a variety of credit environments. We refer to [7] for more extensive examples of the use of the model to price forward index tranches.

Throughout the analysis, we infer the initial condition from market spreads for the underlying index constituents, rather than allowing it to be a free parameter to be fixed by calibration to index tranches. This is to be consistent with CDS spreads for the individual constituents. We do this by backing out the distance-to-default for each constituent from its five-year CDS spread and then aggregating these. Note that as we model the distance-to-default as in (1.1), different volatilities of the underlying firms can be taken into account by rescaling. As a consequence, the initial condition is driven by both the level of constituent spreads and their dispersion.

We study the ability of our model to price index tranches on two dates: February 22, 2007 and December 5, 2008. These dates are chosen specifically to investigate the flexibility of the model to cope with different market and spread environments. February 22, 2007 was pre-crisis when spreads were tight and curves upward sloping; December 5, 2008 was at the height of market volatility, when spreads were at their widest and curves frequently inverted.

We set $R = 40\%$, the level typically assumed by the market for investment grade names, and for each date, calibrate the model to 5, 7 and 10-year index spreads using the volatility, σ. r is the risk-free rate obtained from the Euro swap curve. (N.B. the correlation parameter, ρ, does not come into this calibration since index spreads depend only on the expected losses, which are identical to the sum of default probabilities and hence correlation-independent.)

Table 1 shows the traded and model index spreads for Feb 22, 2007. Since we derive the initial condition from constituent spreads, we only have one free parameter, the volatility σ, for calibrating all three index spreads. Increasing σ to increase model spreads also causes the initial distance-to-default for each constituent to increase (since CDS spreads are fixed), so index and tranche spreads are less sensitive to changes in volatility than they would be if the initial condition was specified independently.

Maturity Date	Fixed Coupon (bp)	Traded Spread (bp)	Model Spread (bp)
20/12/2011	30	21	19.6
20/12/2013	40	30	30.7
20/12/2016	50	41	41.0

Table 1: The fixed coupons, traded spreads and model spreads for the iTraxx Main Series 6 index on February 22, 2007. Parameters used for the model spreads are $r = 0.042$, $\sigma = 0.22$, $R = 0.4$.

Table 3 shows the same results for Dec 5, 2008. In this highly distressed state, we notice that spreads are dramatically wider and the curve is inverted with 5-year > 7-year > 10-year spreads. Our simple model again does a good job of calibrating all three spreads. This is achieved by a smaller distance-to-default for the initial positions in combination with a lower volatility, triggering

more defaults in the near future. The 5-year point is a little low, which is a shortcoming of using a purely diffusive driving process: it can be hard to generate sufficient short-term losses. We refer to Section 6 for a discussion of extensions to jump and stochastic volatility driven processes.

For the parameters from the calibration in Table 1, Table 2 illustrates the correlation sensitivity of the 5, 7 and 10-year index tranches in the pre-crunch environment. We note that model spreads illustrate the behaviour we would anticipate:

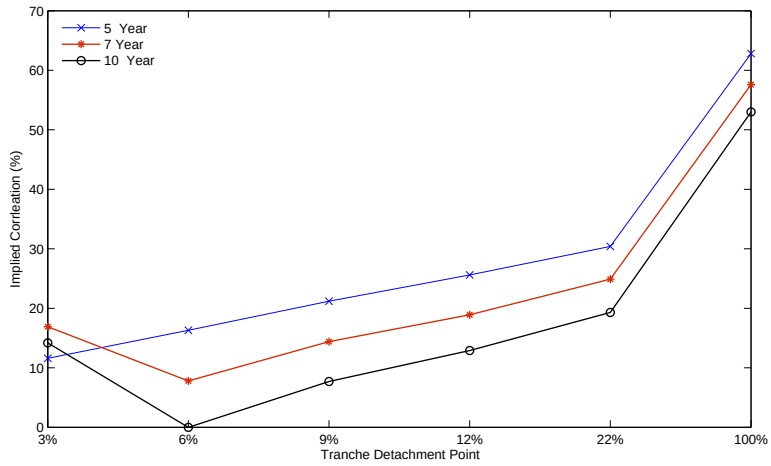
- Equity tranche spreads decline with increasing correlation whilst spreads for other tranches generally increase with correlation. As correlation increases, there are less likely to be a few defaults, and so the equity tranche becomes less risky and its spread decreases. The probability of a greater number of defaults increases with increasing correlation and so spreads on the more senior tranches increase with correlation.

- A notable exception is the 10-year junior mezzanine tranche $(3\% - 6\%)$ which behaves more like an equity tranche and has declining spreads with increasing correlation. This is because, for the parameters used, the expected index loss is between 3% and 6%. The risk of this tranche therefore decreases, along with the spread, as correlation increases, making losses in this tranche less likely.

- The 7-year junior mezzanine tranche $(3\% - 6\%)$ spreads indicate the transition, as maturity increases, from positive to negative correlation sensitivity by exhibiting a humped shape.

- For the 5 and 7-year junior mezzanine and 10-year senior mezzanine tranches, spreads decline with increasing correlation for high values of correlation.

Figure 2: Implied Correlation Skew for iTraxx Main Series 6 Tranches, Feb 22, 2007.



The implied correlation for each tranche is the value of correlation that gives a model tranche spread equal to the market tranche spread given in Table 2. Model parameters are $r = 0.042$, $\sigma = 0.22$, $R = 0.4$.

	5 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	7.19 %	7.55 %	4.99 %	2.14 %	-0.71 %	-3.48 %	-6.17 %	-8.78 %
3%-6%	41	15.6	55.6	86.4	106.1	116.2	119.5	117.4
6%-9%	10.8	0.7	9.1	25	40.3	54.5	65.2	71.7
9%-12%	5	0	2.2	8.2	18.8	28.6	37.2	45.4
12%-22%	1.8	0	0.2	1.7	4.9	9.8	16.1	22.5
22%-100%	0.9	0	0	0	0.1	0.3	0.7	1.5
	7 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	22.1 %	27.45 %	19.97 %	13.79 %	8.31 %	3.27 %	-1.47 %	-6.04 %
3%-6%	110	130.6	183.3	202.2	206	201.5	191.6	177.8
6%-9%	32.5	15.3	52.4	80.5	99.1	110.6	116.1	116.9
9%-12%	15	1.8	17.4	37.1	54.3	67.1	76.5	82.7
12%-22%	4.9	0.1	2.3	8.9	19	29.9	39.5	47.9
22%-100%	2	0	0	0.1	0.4	1.1	2.3	4.1
	10 Year							
Tranche	Market	$\rho = 0.1$	$\rho = 0.2$	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$
0%-3%	38 %	42.51 %	32.51 %	24.13 %	16.65 %	9.71 %	3.11 %	-3.33 %
3%-6%	302.5	375.8	354.9	331.9	308.1	283.5	258.3	231.9
6%-9%	83	101.4	147.3	166.2	173.6	174.4	170.8	163.8
9%-12%	37	24.3	64.1	90.9	107.7	117.8	122.9	124.1
12%-22%	12.5	2	13.5	29.1	44.4	57.5	68.2	76.5
22%-100%	3.6	0	0.1	0.6	1.5	3	5.1	7.7

Table 2: Model tranche spreads (bp) for varying values of the correlation parameter. The equity tranches are quoted as an upfront assuming a 500bp running spread. The model is calibrated to the iTraxx Main Series 6 index for Feb 22, 2007. Market levels shown are for this date; model parameters are $r = 0.042$, $\sigma = 0.22$, $R = 0.4$.

Figure 2 illustrates the 5, 7 and 10-year implied correlation skew – the value of correlation that gives a model spread equal to the market spread for each tranche and maturity.

- With the exception of the $0\% - 3\%$ tranche, we see similar behaviour and levels for all three maturities. This consistency across the term-structure suggests that the dynamics underlying the model are realistic, even in its simple form.

- 5-year implied correlations are generally high relative to the others and 10-year values relatively low. To achieve consistency of the correlation parameter across maturities, a driving process with the ability to generate more default events in the short-term would be required. This could be a more general Levy process for the market factor or one incorporating stochastic volatility.

- An anomaly is revealed by the $3\% - 6\%$ implied correlations and the corresponding row data in Table 2, where it is seen that the correlation dependence of model tranche spreads flips from increasing to hump-shaped to decreasing for maturities running from 5 to 10 years. This has the following effect: for 5 years, there is a unique implied correlation for this tranche; for 7 years, a second, higher, correlation (just under 1) also fits this tranche; for 10 years,

only a single high correlation can fit the market spread. Essentially, the implied correlation curves in Figure 2 are shifted downwards with increasing maturity. The alternative higher branches, where applicable, are not included in the Figure. When a curve crosses zero (in the case of the 10-year $3\%-6\%$ tranche), we have set the implied correlation to zero (instead of the value of around 0.42 from the higher branch which exactly reproduces the market quote). For pricing and (especially) hedging purposes, continuous dependence of implied correlations with respect to maturity and market data is clearly desirable. The lack of a calibration which is both stable *and* exact underlines the need for a richer model.

Maturity Date	Fixed Coupon (bp)	Traded Spread (bp)	Model Spread (bp)
20/12/2013	120	215	207
20/12/2015	125	195	195
20/12/2018	130	175	176

Table 3: The fixed coupons, traded spreads and model spreads for the iTraxx Main Series 10 index on December 5, 2008. Parameters used for the model spreads are $r = 0.033$, $\sigma = 0.136$, $R = 0.4$.

Table 4 shows the correlation sensitivity of the Dec 5, 2008 index tranches with parameters from the calibration in Table 3. We notice that relative to Table 2, spreads are highly distressed, the index is inverted and tranche spreads are flat to inverted across maturities. As a result, the tranches exhibit very different sensitivity to correlation than before, however there are some common themes and extensions to earlier behaviour:

- Default probabilities for the index and its constituents are very high. The index expected loss is therefore much greater than before, illustrated by the fact the first three 5-year tranches and the first four 7 and 10-year tranches have declining spreads with increasing correlation. This contrasts with just the equity and 10-year junior mezzanine tranches in Feb 2007.
- Much higher levels of ρ are needed to replicate market prices than in pre-crunch times, consistent with the fact that systematic risk is a much greater concern at this time.
- Too much of our model's portfolio loss distribution lies in the middle tranches: $6\% - 22\%$; more weight needs to be in the tail to be able to replicate $22\% - 100\%$ tranche values. The same model shortcoming holds for all maturities and reflects the need for a more sophisticated driving process.

6 Conclusions

We have illustrated the ability of our simple model to crudely calibrate to the index term-structure in wildly different market environments, and have shown that the correlation sensitivity of tranche spreads demonstrates the behaviour expected. More importantly, using just two parameters and without making them time-dependent, we have shown that our very simple structural

	5 Year							
Tranche	Market	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
0%-3%	71.5 %	81.88 %	75.9 %	69.56 %	63.02 %	56.25 %	49.16 %	41.65 %
3%-6%	1576.3	2275.2	1978.5	1743.2	1546.8	1374.6	1222.8	1090.1
6%-9%	811.5	1273.1	1168.2	1079.7	1001.4	931.3	864.6	796.3
9%-12%	506.1	775.7	765.8	748.6	724.7	695.8	663.2	629.1
12%-22%	180.3	307.8	353.3	384.7	405.5	418.1	423.4	420.5
22%-100%	77.9	9.2	16.5	25	34.3	44.5	55.7	68.1
	7 Year							
Tranche	Market	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
0%-3%	72.9 %	84.03 %	78.98 %	73.26 %	66.93 %	60 %	52.41 %	44.13 %
3%-6%	1473.2	2327.3	1985.7	1715.2	1493.4	1308	1147.8	1001.3
6%-9%	804.2	1344.2	1199	1085.2	988.2	900.7	820.9	747.9
9%-12%	512.4	855.4	808.4	765.3	725.3	684.8	643	600.4
12%-22%	182.6	375.4	401.7	417.6	425.6	427.4	423.1	411.8
22%-100%	75.8	14	22	30.6	39.6	49.3	59.7	71.2
	10 Year							
Tranche	Market	$\rho = 0.3$	$\rho = 0.4$	$\rho = 0.5$	$\rho = 0.6$	$\rho = 0.7$	$\rho = 0.8$	$\rho = 0.9$
0%-3%	73.8 %	85.13 %	80.57 %	74.99 %	68.51 %	61.31 %	53.31 %	44.22 %
3%-6%	1385.5	2270.8	1895.7	1611.1	1385.8	1195.3	1032	889.6
6%-9%	824.7	1332.2	1164.2	1033.7	925.5	833.5	749.8	669.7
9%-12%	526.1	870.8	798.8	740.7	689.3	640.5	592.1	543.1
12%-22%	174.1	406.1	414.9	417.5	415.6	409.8	400.2	385.3
22%-100%	76.3	18.3	26.1	34	42.1	50.6	59.7	69.8

Table 4: Model tranche spreads (bp) for varying values of the correlation parameter. The equity tranches are quoted as an upfront assuming a 500bp running spread. The model is calibrated to the iTraxx Main Series 10 index for Dec 5, 2008. Market levels shown are for this date; model parameters are $r = 0.033$, $\sigma = 0.136$, $R = 0.4$.

evolution model displays realistic term-structure dynamics. Using just the volatility parameter, it is able to calibrate well to all three index spreads and correlation sensitivities of the various tranches are fairly stable across maturities. This is an improvement on the majority of pricing models which lack a coherent means of incorporating dynamics.

The next stage, which has not been the focus here, is to extend the framework so that it can calibrate to all tranches with a single set of parameters. This would involve replacing the simple Brownian Motion driving the process with a more general stochastic volatility or Levy or jump-diffusion process. Jumps in the market factor are conceptually easy to include and result in a jump process driving the SPDE drift. Similarly, a single stochastic volatility factor affecting all firms will result in a stochastic term driving the SPDE diffusion. Contagion may be incorporated by making model parameters, notably the correlation, loss dependent. These extensions would allow the loss distribution process to become more skewed, allocating more weight to the tail and increasing super senior tranche spreads, as well as generally allowing more flexibility to match observed data.

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