

Self-similarity and spectral asymptotics for the continuum random tree

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Abstract

We use the random self-similarity of the continuum random tree to show that it is homeomorphic to a post-critically finite self-similar fractal equipped with a random self-similar metric. As an application we determine the mean and almost-sure leading order behaviour of the high frequency asymptotics of the eigenvalue counting function associated with the natural Dirichlet form on the continuum random tree. We also obtain short time asymptotics for the trace of the heat semigroup and the annealed on-diagonal heat kernel associated with this Dirichlet form.

1 Introduction

One of the reasons the continuum random tree of Aldous has attracted such great interest is that it connects together a number of diverse areas of probability theory. On one hand, it appears from discrete probability as the scaling limit of combinatorial graph trees and probabilistic branching processes; and on the other hand, it is intimately related with a continuous time process, namely the normalised Brownian excursion, [2]. However, with both of these representations of the continuum random tree, there does not appear to be an obvious description of the structure of the set itself. In this paper we demonstrate that the continuum random tree has a recursive description as a random self-similar fractal and show that the set is always homeomorphic to a deterministic subset of the Euclidean plane. As an application of this precise description of the random self-similarity of the continuum random tree, we deduce results about the spectrum and on-diagonal heat kernel of the natural Dirichlet form on the set using techniques developed for random recursive self-similar fractals.

From its graph tree scaling limit description, Aldous showed how the continuum random tree has a certain random self-similarity, [3]. In this article, we use this result iteratively to label the continuum random tree, \mathcal{T} , using a shift space over a three letter alphabet. This enables us to show that there is an isometry from \mathcal{T} , with its natural

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metric $d_{\mathcal{T}}$ (see Section 2 for a precise definition of \mathcal{T} and $d_{\mathcal{T}}$, and Section 3 for the decomposition of \mathcal{T} that we apply), to a deterministic subset of \mathbb{R}^2 , T say, equipped with a random metric R , \mathbf{P} -a.s., where \mathbf{P} is the probability measure on the probability space upon which all the random variables of the discussion are defined. This metric is constructed using random scaling factors in an adaptation of the now well-established techniques of [11] for building a resistance metric on a post-critically finite self-similar fractal. We note that on a tree the resistance and geodesic metrics are the same. Furthermore, we show that the isometry in question also links the natural Borel probability measures on the spaces $(\mathcal{T}, d_{\mathcal{T}})$ and (T, R) . The relevant measures will be denoted by μ and μ^T respectively, with μ arising as the scaling limit of the uniform measures on the graph approximations of \mathcal{T} (see [2], for example), and μ^T being the random self-similar measure that is associated with the construction of R . The result that we prove is the following; full descriptions of (T, R, μ^T) are given in Section 4, and the isometry is defined in Section 5.

Theorem 1.1 *There exists a deterministic post-critically finite self-similar dendrite, T , equipped with a (random) self-similar metric, R , and Borel probability measure, μ^T , such that (T, R, μ^T) is equivalent to $(\mathcal{T}, d_{\mathcal{T}}, \mu)$ as a measure-metric space, \mathbf{P} -a.s.*

Previous analytic work on the continuum random tree in [6] obtained estimates on the quenched and the annealed heat kernel for the tree. We can now adapt techniques of [8] to consider the spectral asymptotics of the tree. As a byproduct we are also able to refine the results on the annealed heat kernel to show the existence of a short time limit for $t^{2/3}\mathbf{E}p_t(\rho, \rho)$ at the root of the tree ρ , where the notation \mathbf{E} is used to represent expectation under the probability measure \mathbf{P} .

The natural Dirichlet form on $L^2(\mathcal{T}, \mu)$ may be thought of simply as the electrical energy when we consider $(\mathcal{T}, d_{\mathcal{T}})$ as a resistance network. We shall denote this form by $\mathcal{E}_{\mathcal{T}}$, and its domain $\mathcal{F}_{\mathcal{T}}$, and explain in Section 2 how it may be constructed using results of [10]. The eigenvalues of the triple $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$ are defined to be the numbers λ which satisfy

$$\mathcal{E}(u, v) = \lambda \int_{\mathcal{T}} u v d\mu, \quad \forall v \in \mathcal{F}_{\mathcal{T}} \quad (1)$$

for some eigenfunction $u \in \mathcal{F}_{\mathcal{T}}$. The corresponding eigenvalue counting function, N , is obtained by setting

$$N(\lambda) := \#\{\text{eigenvalues of } (\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu) \leq \lambda\}, \quad (2)$$

and we prove in Section 6 that this is well-defined and finite for any $\lambda \in \mathbb{R}$, \mathbf{P} -a.s. In Section 6, we also prove the following result, which shows that asymptotically the mean and \mathbf{P} -a.s. behaviour of N are identical.

Theorem 1.2 *There exists a deterministic constant $C_0 \in (0, \infty)$ such that*
(a) $\lambda^{-2/3}\mathbf{E}N(\lambda) \rightarrow C_0$, as $\lambda \rightarrow \infty$.
(b) $\lambda^{-2/3}N(\lambda) \rightarrow C_0$, as $\lambda \rightarrow \infty$, \mathbf{P} -a.s.

To provide some context for this result, we will now briefly discuss some related work. For the purposes of brevity, during the remainder of the introduction, we shall use the

notation $N(\lambda)$ to denote the eigenvalue counting function of whichever problem is being considered. Classically, for the usual Laplacian on a bounded domain $\Omega \subseteq \mathbb{R}^n$, Weyl's famous theorem tells us that the eigenvalue counting function satisfies

$$N(\lambda) = C_n |\Omega| \lambda^{n/2} + o(\lambda^{n/2}), \quad \text{as } \lambda \rightarrow \infty, \quad (3)$$

where C_n is a constant depending only on n , and $|\Omega|$ is the Lebesgue measure of Ω , see [14]. As a consequence, in this setting, there exists a limit for $\lambda^{-n/2} N(\lambda)$ as $\lambda \rightarrow \infty$. In the case of deterministic p.c.f. self-similar fractals it is known that

$$N(\lambda) = \lambda^{d_S/2} (G(\ln \lambda) + o(1)), \quad \text{as } \lambda \rightarrow \infty,$$

where G is a periodic function, see [11], Theorem 4.1.5. The generic case has G constant but for fractals with a high degree of symmetry, such as the class of nested fractals, (an example is the Sierpinski gasket), the function G can be proved to be non-constant, and so no limit actually exists for $\lambda^{-d_S/2} N(\lambda)$ as $\lambda \rightarrow \infty$ for these fractals. In the case of random recursive Sierpinski gaskets, as studied in [8], there are similar results, however the function G must be multiplied by a random weight variable, which can be thought of as a measure of the volume of the fractal, and roughly corresponds to the factor $|\Omega|$ in (3). Again the generic case is that the limit of the rescaled counting function exists and, in this setting, there are no known examples of periodic behaviour. For the continuum random tree, no periodic fluctuations or random weight factors appear; this is due to the non-lattice distribution of the Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ random variables that are used in the self-similar construction, and also the fact that summing the three elements of the triple gives exactly one, **P**-a.s.

It is also worth commenting upon the values of the exponent of λ in the leading order behaviour of $N(\lambda)$ in the classical and fractal setting. From Weyl's result for bounded domains in \mathbb{R}^n , we see that the limit

$$d_S := 2 \lim_{\lambda \rightarrow \infty} \frac{\ln N(\lambda)}{\ln \lambda}$$

is precisely n , matching the Hausdorff dimension of Ω . However, for deterministic and random self-similar fractals, this agreement is not generally the case. For a large class of finitely ramified fractals it has been proved that

$$d_S = \frac{2d_H}{1 + d_H}, \quad (4)$$

where d_H is the Hausdorff dimension of the fractal in the resistance metric (see [11], Theorem 4.2.1, and [8], Theorem 1.1). Due to its definition from the spectral asymptotics, the quantity d_S has become known as the spectral dimension of a (Laplacian on a) set. Clearly, from the previous theorem, we see that for the continuum random tree $d_S = 4/3$. This result could have been predicted from the self-similar fractal picture of the set given in Theorem 1.1, and (4), noting that $d_H = 2$ for the continuum random tree (see [7]). Observe that to be able to apply the result of [7], the equivalence of the resistance and geodesic metrics on trees must be used.

Finally, let X be the Markov process corresponding to $(\mathcal{E}_T, \mathcal{F}_T, \mu)$ and denote by $(p_t(x, y))_{x, y \in T, t > 0}$ its transition density; alternatively this is the heat kernel of the Laplacian associated with the Dirichlet form. The existence of p_t for $t > 0$ was proved in [6],

where it was also shown that $t^{2/3}p_t(x, x)$ exhibits logarithmic fluctuations globally, and log-logarithmic fluctuations for μ -a.e. $x \in \mathcal{T}$, as $t \rightarrow 0$. These fluctuations are caused by variations in the “thickness” of the measure μ over the space, which result in turn from the randomness of the construction. However, the result of Theorem 1.2(b) implies that these fluctuations must even out, when averaged over the entire space. In particular, applying an Abelian theorem in the way discussed in Remark 5.11 of [8], we obtain the following limit result for the trace of the heat semigroup, which we state without proof.

Corollary 1.3 *Let C_0 be the constant of Theorem 1.2, and Γ be the standard gamma function, then*

(a)

$$t^{2/3}\mathbf{E} \int_{\mathcal{T}} p_t(x, x)\mu(dx) \rightarrow C_0\Gamma(5/3), \quad \text{as } t \rightarrow 0,$$

(b) \mathbf{P} -a.s.,

$$t^{2/3} \int_{\mathcal{T}} p_t(x, x)\mu(dx) \rightarrow C_0\Gamma(5/3), \quad \text{as } t \rightarrow 0.$$

Another corollary, which follows from the invariance under random re-rooting of the continuum random tree, [1], allows us to deduce from part (a) of this Corollary the following limit for the annealed heat kernel at ρ , the root of \mathcal{T} (see Section 2 for a definition). This tightens the result obtained in [6], Proposition 1.7 for the annealed heat kernel.

Corollary 1.4 *Let C_0 be the constant of Theorem 1.2, and Γ be the standard gamma function, then*

$$t^{2/3}\mathbf{E}p_t(\rho, \rho) \rightarrow C_0\Gamma(5/3) \text{ as } t \rightarrow 0,$$

An outline of the paper is as follows. In Section 2 we introduce the continuum random tree and give the natural Dirichlet form associated with the tree. In Section 3 we use the decomposition of Aldous to give a description of the tree via a sequence space. Once we have established this we can map the continuum random tree into a post-critically finite self-similar tree with a random metric. Finally we show that the map ensures that the two sets are equivalent as metric measure spaces. Once we have the picture as a self-similar set with a random metric it is straightforward to deduce a decomposition of the Dirichlet form and from this a natural scaling in the eigenvalues. This leads to our results on the spectrum, and via an Abelian theorem, to results on the trace of the heat semigroup.

2 Continuum random tree

The connection between trees and excursions is an area that has been of much recent interest. In this section, we provide a brief introduction to this link, a definition of the continuum random tree, and also describe how to construct the natural Dirichlet form on this set.

We begin by defining the space of excursions, U , to be the set of continuous functions $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ for which there exists a $\tau(f) \in (0, \infty)$ such that $f(t) > 0$ if and only if $t \in (0, \tau(f))$. Given a function $f \in U$, we define a distance on $[0, \tau(f)]$ by setting

$$d_f(s, t) := f(s) + f(t) - 2m_f(s, t), \quad (5)$$

where $m_f(s, t) := \inf\{f(r) : r \in [s \wedge t, s \vee t]\}$. We then use the equivalence

$$s \sim t \quad \Leftrightarrow \quad d_f(s, t) = 0, \quad (6)$$

to define $\mathcal{T}_f := [0, \tau(f)] / \sim$. Denoting by $[s]$ the equivalence class containing s , it is elementary (see [7], Section 2) to check that $d_{\mathcal{T}_f}([s], [t]) := d_f(s, t)$ defines a metric on \mathcal{T}_f , and also that \mathcal{T}_f is a *dendrite*, which is taken to mean a path-wise connected Hausdorff space containing no subset homeomorphic to the circle. Furthermore, the metric $d_{\mathcal{T}_f}$ is a shortest path metric on \mathcal{T}_f , which means that it is additive along the paths of \mathcal{T}_f . The *root* of the tree \mathcal{T}_f is defined to be the equivalence class $[0]$, and is denoted by ρ_f . A natural volume measure to impose upon \mathcal{T}_f is the projection of Lebesgue measure on $[0, \tau(f)]$. In particular, for open $A \subseteq \mathcal{T}_f$, let

$$\mu_f(A) := \ell(\{t \in [0, \tau(f)] : [t] \in A\}),$$

where, throughout this article, ℓ is the usual 1-dimensional Lebesgue measure. This defines a Borel measure on $(\mathcal{T}_f, d_{\mathcal{T}_f})$, with total mass equal to $\tau(f)$.

We are now able to define the *continuum random tree* as the random dendrite that we get when the function f is chosen according to the law of a suitably scaled Brownian excursion. More precisely, we shall assume that there exists an underlying probability space, with probability measure \mathbf{P} , upon which is defined a process $W = (W_t)_{t=0}^1$ which has the law of the normalised Brownian excursion, where, throughout this article “normalised” is taken to mean “scaled to return to the origin for the first time at time 1”. In keeping with the notation used so far in this section, the measure-metric space of interest should be written $(\mathcal{T}_W, d_{\mathcal{T}_W}, \mu_W)$, the distance on $[0, \tau(W)]$, defined at (5), d_W , and the root, ρ_W . However, we shall omit the subscripts W with the understanding that we are discussing the continuum random tree in this case. We note that $\tau(W) = 1$, \mathbf{P} -a.s., and so $[0, \tau(W)] = [0, 1]$ and μ is a probability measure on \mathcal{T} , \mathbf{P} -a.s. Moreover, that μ is non-atomic is readily checked using simple path properties of W . Note that our definition differs slightly from the Aldous continuum random tree, which is based on the random function $2W$. Since this extra factor only has the effect of increasing distances by a factor of 2, our results are readily adapted to apply to Aldous’ tree.

A further observation that will be useful to us is that between any three points of a dendrite there is a unique branch-point. We shall denote the branch-point of $x, y, z \in \mathcal{T}$ by $b(x, y, z)$, which is the unique point in \mathcal{T} lying on the arcs between x and y , y and z , and z and x .

Finally, we note that it is easy to check the conditions of [10], Theorem 5.4 to deduce that it is possible to build a natural Dirichlet form on the continuum random tree.

Theorem 2.1 *\mathbf{P} -a.s. there exists a local regular Dirichlet form $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$ on $L^2(\mathcal{T}, \mu)$, which is associated with the metric $d_{\mathcal{T}}$ through, for every $x \neq y$,*

$$d_{\mathcal{T}}(x, y)^{-1} = \inf\{\mathcal{E}_{\mathcal{T}}(f, f) : f \in \mathcal{F}_{\mathcal{T}}, f(x) = 0, f(y) = 1\}. \quad (7)$$

This final property means that the metric $d_{\mathcal{T}}$ is indeed the resistance metric associated with $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}})$. It will be the eigenvalue counting function defined from $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$ as at (2) for which we deduce asymptotic results in this article.

3 Decomposition of the continuum random tree

To make precise the decomposition of the continuum random tree that we shall apply, we use the excursion description of the set introduced in the previous section. This allows us to prove rigorously the independence properties that are important to our argument. However, it may not be immediately obvious exactly what the excursion picture is telling us about the continuum random tree, and so, after Lemma 3.1, we present a more heuristic discussion of the procedure we use in terms of the related dendrites.

The initial object of consideration is a triple (W, U, V) , where W is the normalised Brownian excursion, and U and V are independent $U[0, 1]$ random variables, independent of W . From this triple it is possible to define three independent Brownian excursions. The following decomposition is rather awkward to write down, but is made clearer by Figure 1. First, suppose $U < V$. On this set, it is \mathbf{P} -a.s. possible to define $H \in [0, 1]$ by

$$\{H\} := \{t \in [U, V] : W_t = \inf_{s \in [U, V]} W_s\}. \quad (8)$$

We also define

$$H_- := \sup\{t < U : W_t = W_H\}, \quad H_+ := \inf\{t > V : W_t = W_H\}, \quad (9)$$

$$\begin{aligned} \Delta_1 &:= 1 + H_- - H_+, & \Delta_2 &:= H - H_-, & \Delta_3 &:= H_+ - H, \\ \tilde{U}_1 &:= \frac{H_-}{\Delta_1}, & U_2 &:= \frac{U - H_-}{\Delta_2}, & U_3 &:= \frac{V - H}{\Delta_3}, \end{aligned}$$

and for $t \in [0, 1]$,

$$\tilde{W}_t^1 := \Delta_1^{-1/2}(W_{t\Delta_1} \mathbf{1}_{\{t \leq \tilde{U}_1\}} + W_{H_+ + (t - \tilde{U}_1)\Delta_1} \mathbf{1}_{\{t > \tilde{U}_1\}}),$$

$$W_t^2 := \Delta_2^{-1/2}(W_{H_- + t\Delta_2} - W_H),$$

$$W_t^3 := \Delta_3^{-1/2}(W_{H + t\Delta_3} - W_H).$$

Finally, it will be convenient to shift \tilde{W}^1 by \tilde{U}_1 so that the root of the corresponding tree is chosen differently. Thus, we define W^1 by

$$W_t^1 := \begin{cases} W_{\tilde{U}_1} + W_{\tilde{U}_1 + t} - 2m(\tilde{U}_1, \tilde{U}_1 + t), & 0 \leq t \leq 1 - \tilde{U}_1 \\ W_{\tilde{U}_1} + W_{\tilde{U}_1 + t - 1} - 2m(\tilde{U}_1 + t - 1, \tilde{U}_1), & 1 - \tilde{U}_1 \leq t \leq 1, \end{cases}$$

and set $U_1 := 1 - \tilde{U}_1$. If $U > V$, the definition of these quantities is similar, with W^1 again being the rescaled, shifted excursion containing $t = 0$, W^2 being the rescaled excursion containing $t = U$, and W^3 being the rescaled excursion containing $t = V$. A minor adaptation of [3], Corollary 3, using the invariance under random re-rooting of the continuum random tree (see [1], Section 2.7), then gives us the following result, which we state without proof.

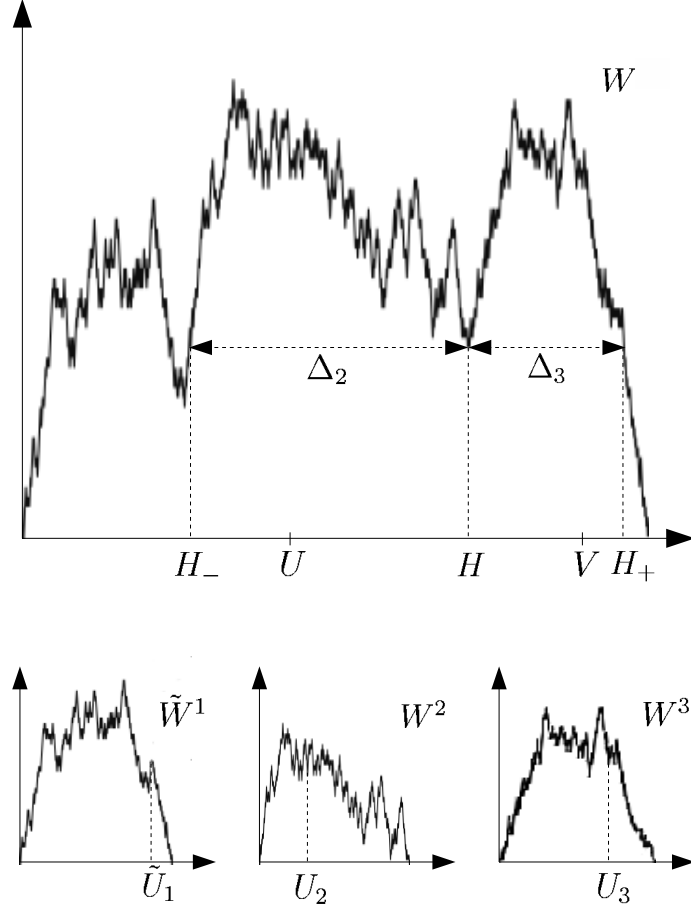


Figure 1: Brownian excursion decomposition.

Lemma 3.1 *The quantities $W^1, W^2, W^3, U_1, U_2, U_3$ and $(\Delta_1, \Delta_2, \Delta_3)$ are independent. Each W^i is a normalised Brownian excursion, each U_i is $U[0, 1]$, and $(\Delta_1, \Delta_2, \Delta_3)$ has the Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution.*

Describing the result in terms of the corresponding trees gives a much clearer picture of what the above decomposition does. Using the notation of Section 2, let $(\mathcal{T}, d_{\mathcal{T}}, \mu)$ be the continuum random tree associated with W , and $\rho = [0]$ its root. Again, we use $[t]$, for $t \in [0, 1]$, to represent the equivalence classes of $[0, 1]$ under the equivalence relation defined at (6). If we define $Z^1 := [U]$ and $Z^2 := [V]$, then Z^1 and Z^2 are two independent μ -random vertices of \mathcal{T} . We now split the tree \mathcal{T} at the branch-point $b(\rho, Z^1, Z^2)$, which may be checked to be equal to $[H]$, and denote by \mathcal{T}^1 , \mathcal{T}^2 and \mathcal{T}^3 the components of \mathcal{T} containing ρ , Z^1 and Z^2 respectively. Choose the root of each subtree to be equal to $b(\rho, Z^1, Z^2)$ and, for $i = 1, 2, 3$, let μ^i be the probability measure on \mathcal{T}^i defined by $\mu^i(A) = \mu(A)/\Delta_i$, for measurable $A \subseteq \mathcal{T}^i$, where $\Delta_i := \mu(\mathcal{T}^i)$. The previous result tells us precisely that $(\mathcal{T}^i, \Delta_i^{-1/2} d_{\mathcal{T}}, \mu^i)$, $i = 1, 2, 3$, are three independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu)$. Furthermore, if $Z_i := \rho, Z^1, Z^2$ for $i = 1, 2, 3$, respectively, then Z_i is a μ^i -random variable in \mathcal{T}^i . Finally, all these quantities are independent of the masses $(\mu(\mathcal{T}^1), \mu(\mathcal{T}^2), \mu(\mathcal{T}^3))$, which form a Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ triple. Although it is possible to deal with the subtrees directly using conditional definitions of the random variables to decompose the continuum

random tree in this way, the excursion description allows us to keep track of exactly what is independent more easily, and it is to this setting that we return. However, we shall not completely neglect the tree description of the algorithm we now introduce, and a summary in this vein appears after Proposition 3.4.

We continue by applying inductively the decomposition map from $U^{(1)} \times [0, 1]^2$ to $U^{(1)^3} \times [0, 1]^3 \times \Delta$ (where Δ is the standard 2-simplex) that takes the triple (W, U, V) to the collection $(W^1, W^2, W^3, U_1, U_2, U_3, (\Delta_1, \Delta_2, \Delta_3))$ of excursions and uniform and Dirichlet random variables. We shall denote this decomposition map by Υ . To label objects in our consideration it will be useful to use, as an address space, sequences of $\{1, 2, 3\}$. In particular, we will write the collections of finite sequences as, for $n \geq 0$,

$$\Sigma_n := \{1, 2, 3\}^n, \quad \Sigma_* := \bigcup_{m \geq 0} \Sigma_m,$$

where $\Sigma_0 := \{\emptyset\}$. Later, we will refer to the space of infinite sequences of $\{1, 2, 3\}$, which we denote by Σ , and also apply some further notation, which we introduce now. For $i \in \Sigma_m, j \in \Sigma_n, k \in \Sigma$, write $ij = i_1 \dots i_m j_1 \dots j_n$, and $ik = i_1 \dots i_m k_1 k_2 \dots$. For $i \in \Sigma_*$, denote by $|i|$ the integer n such that $i \in \Sigma_n$ and call this the *length* of i . For $i \in \Sigma_n \cup \Sigma$, $n \geq m$, the *truncation* of i to length m is written as $i|m := i_1 \dots i_m$.

Now, suppose we are given an independent collection $(W, U, (V_i)_{i \in \Sigma_*})$, where W is a normalised Brownian excursion, U is $U[0, 1]$, and $(V_i)_{i \in \Sigma_*}$ is a family of independent $U[0, 1]$ random variables. Set $(W^\emptyset, U_\emptyset) := (W, U)$. Given (W^i, U_i) , define

$$(W^{i1}, W^{i2}, W^{i3}, U_{i1}, U_{i2}, U_{i3}, (\Delta_{i1}, \Delta_{i2}, \Delta_{i3})) := \Upsilon(W^i, U_i, V_i),$$

and denote the filtration associated with $(\Delta_i)_{i \in \Sigma_* \setminus \{\emptyset\}}$ by $(\mathcal{F}_n)_{n \geq 0}$. In particular, $\mathcal{F}_n := \sigma(\Delta_i : |i| \leq n)$. The subsequent result is easily deduced by applying the previous lemma repeatedly.

Theorem 3.2 *For each n , $((W^i, U_i, V_i))_{i \in \Sigma_n}$ is an independent collection of independent triples consisting of a normalised Brownian excursion and two $U[0, 1]$ random variables, and moreover, the entire family of random variables is independent of \mathcal{F}_n .*

Resulting from this construction, the collection of random variables $(\Delta_i)_{i \in \Sigma_* \setminus \{\emptyset\}}$ have some particularly useful independence properties, which we will use in the next section to build a random self-similar fractal related to \mathcal{T} . Furthermore, Lemma 3.1 implies that each triple of the form $(\Delta_{i1}, \Delta_{i2}, \Delta_{i3})$ has the Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ distribution. Subsequently we will also be interested in the collection $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$, where for each i , we define

$$w(i) := \Delta_i^{1/2},$$

and will write $l(i)$ to represent the product $w(i|1)w(i|2) \dots w(i||i|)$, where $l(\emptyset) := 1$. The reason for considering such families is that, in our decomposition of the continuum random tree, $(\Delta_i)_{i \in \Sigma_* \setminus \{\emptyset\}}$ and $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$ represent the mass and length scaling factors respectively.

By viewing the inductive procedure for decomposing excursions as the repeated splitting of trees in the way described after Lemma 3.1, it is possible to use the above algorithm to break the continuum random tree into smaller components, with the subtrees in the

n th level of construction being described by the excursions $(W^i)_{i \in \Sigma_n}$. The maps we now introduce will make this idea precise. For the remainder of this section, the arguments that we give hold \mathbf{P} -a.s. First, denote by H^i , H_-^i and H_+^i the random variables in $[0, 1]$ associated with (W^i, U_i, V_i) by the formulae at (8) and (9). Let $i \in \Sigma_*$. Define, for $t \in [0, 1]$,

$$\phi_{i1}(t) := (H_+^i + t\Delta_{i1})\mathbf{1}_{\{t < U_{i1}\}} + (t - U_{i1})\Delta_{i1}\mathbf{1}_{\{t \geq U_{i1}\}},$$

and if $U_i < V_i$, define ϕ_{i2} and ϕ_{i3} to be the linear contractions from $[0, 1]$ to $[H_-^i, H^i]$ and $[H^i, H_+^i]$ respectively. If $U_i > V_i$, the images of ϕ_{i2} and ϕ_{i3} are reversed. Note that, for each i , the map ϕ_i satisfies, for any measurable $A \subseteq [0, 1]$,

$$\ell(\phi_i(A)) = \Delta_i \ell(A), \quad (10)$$

where ℓ is the usual Lebesgue measure on $[0, 1]$. Importantly, these maps also satisfy a certain distance scaling property. In particular, it is elementary to check from the definitions of the excursions that, for any $i \in \Sigma_*$, $j \in \{1, 2, 3\}$,

$$d_{W^i}(\phi_{ij}(s), \phi_{ij}(t)) = w(ij)d_{W^{ij}}(s, t), \quad \forall s, t \in [0, 1], \quad (11)$$

where d_{W^i} is the distance on $[0, 1]$ associated with W^i by the definition at (5). This equality allows us to define a map on the trees related to the excursions. Let $(\tilde{\mathcal{T}}_i, d_{\tilde{\mathcal{T}}_i})$ be the metric space dendrite determined from W^i by the equivalence relation given at (6). Denote the corresponding equivalence classes $[t]_i$ for $t \in [0, 1]$. Now define, for $i \in \Sigma_*$, $j \in \{1, 2, 3\}$,

$$\begin{aligned} \tilde{\phi}_{ij} : \tilde{\mathcal{T}}_{ij} &\rightarrow \tilde{\mathcal{T}}_i \\ [t]_{ij} &\mapsto [\phi_{ij}(t)]_i. \end{aligned}$$

The following result is readily deduced from the distance scaling property at (11), and so we state it without proof.

Lemma 3.3 *\mathbf{P} -a.s., for every $i \in \Sigma_*$, $j \in \{1, 2, 3\}$, $\tilde{\phi}_{ij}$ is well-defined and moreover,*

$$d_{\tilde{\mathcal{T}}_i}(\tilde{\phi}_{ij}(x), \tilde{\phi}_{ij}(y)) = w(ij)d_{\tilde{\mathcal{T}}_{ij}}(x, y), \quad \forall x, y \in \tilde{\mathcal{T}}_{ij}.$$

By iterating the functions $(\tilde{\phi}_i)_{i \in \Sigma_* \setminus \{\emptyset\}}$, we can map any $\tilde{\mathcal{T}}_i$ to the original continuum random tree, $\mathcal{T} \equiv \tilde{\mathcal{T}}_\emptyset$, which is the object of interest. We will denote the map from $\tilde{\mathcal{T}}_i$ to \mathcal{T} by $\tilde{\phi}_{*i} := \tilde{\phi}_{i|1} \circ \tilde{\phi}_{i|2} \circ \dots \circ \tilde{\phi}_i$, and its image by $\mathcal{T}_i := \tilde{\phi}_{*i}(\tilde{\mathcal{T}}_i)$. It is these sets that form the basis of our decomposition of \mathcal{T} . We will also have cause to refer to the following points in \mathcal{T}_i :

$$\rho_i := \tilde{\phi}_{*i}([0]_i), \quad Z_i^1 := \tilde{\phi}_{*i}([U_i]_i), \quad Z_i^2 := \tilde{\phi}_{*i}([V_i]_i).$$

Although it has been quite hard work arriving at the definition of $(\mathcal{T}_i)_{i \in \Sigma_*}$, the properties of this family of sets that we will need are derived without too many difficulties from the construction. The proposition we now prove includes the following results: the sets $(\mathcal{T}_i)_{i \in \Sigma_n}$ cover \mathcal{T} ; \mathcal{T}_i is simply a rescaled copy of $\tilde{\mathcal{T}}_i$ with μ -measure $l(i)^2$; the overlaps of sets in the collection $(\mathcal{T}_i)_{i \in \Sigma_n}$ are small; and also describes various relationships between points of the form ρ_i , Z_i^1 and Z_i^2 .

Proposition 3.4 *P-a.s., for every $i \in \Sigma_*$,*

- (a) $\mathcal{T}_i = \cup_{j \in \Sigma_n} \mathcal{T}_{ij}$, for all $n \geq 0$.
- (b) $(\mathcal{T}_i, d_{\mathcal{T}})$ and $(\tilde{\mathcal{T}}_i, l(i)d_{\tilde{\mathcal{T}}_i})$ are isometric.
- (c) $\rho_{i1} = \rho_{i2} = \rho_{i3} = b(\rho_i, Z_i^1, Z_i^2)$.
- (d) $Z_{ij}^1 = \rho_i, Z_i^1, Z_i^2$, for $j = 1, 2, 3$ respectively.
- (e) $\rho_i \notin \mathcal{T}_{i2} \cup \mathcal{T}_{i3}$, $Z_i^1 \notin \mathcal{T}_{i1} \cup \mathcal{T}_{i3}$ and $Z_i^2 \notin \mathcal{T}_{i1} \cup \mathcal{T}_{i2}$.
- (f) if $|j| = |i|$, but $j \neq i$, then $\mathcal{T}_i \cap \mathcal{T}_j \subseteq \{\rho_i, Z_i^1\}$.
- (g) $\mu(\mathcal{T}_i) = l(i)^2$.

Proof: By induction, it suffices to show that (a) holds for $n = 1$. By definition, we have $\cup_{j \in \{1,2,3\}} \phi_{ij}([0, 1]) = [0, 1]$, and so

$$\tilde{\mathcal{T}}_i = \{[t]_i : t \in [0, 1]\} = \cup_{j \in \{1,2,3\}} \{[\phi_{ij}(t)]_i : t \in [0, 1]\} = \cup_{j \in \{1,2,3\}} \tilde{\phi}_{ij}(\tilde{\mathcal{T}}_{ij}),$$

where we apply the definition of $\tilde{\phi}_{ij}$ for the final equality. Applying $\tilde{\phi}_{*i}$ to both sides of this equation completes the proof of (a). Part (b) is an immediate consequence of the definition of \mathcal{T}_i and the distance scaling property of $\tilde{\phi}_{*i}$ proved in Lemma 3.3.

Analogous to the remark made after Lemma 3.1, the point $[H^i]_i$ represents the branch-point of $[0]_i$, $[U_i]_i$ and $[V_i]_i$ in $\tilde{\mathcal{T}}_i$. Thus, since $\tilde{\phi}_{*i}$ is simply a rescaling map, we have that

$$b(\rho_i, Z_i^1, Z_i^2) = b(\tilde{\phi}_{*i}([0]_i), \tilde{\phi}_{*i}([U_i]_i), \tilde{\phi}_{*i}([V_i]_i)) = \tilde{\phi}_{*i}([H^i]_i).$$

Now, note that for any $j \in \{1, 2, 3\}$, we have by definition that $\phi_{ij}(0) \in \{H^i, H_-^i, H_+^i\}$, and so $[\phi_{ij}(0)]_i = [H^i]_i$. Consequently,

$$\tilde{\phi}_{*i}([H^i]_i) = \tilde{\phi}_{*i}([\phi_{ij}(0)]_i) = \tilde{\phi}_{*ij}([0]_{ij}) = \rho_{ij}, \quad (12)$$

which proves (c). Part (d) and (e) are easy to check from the construction using similar ideas and so their proof is omitted.

Now note that, for $k \in \Sigma_*$, the decomposition of the excursions, and the fact that the local minima of a Brownian excursion are distinct, implies that for $j_1, j_2 \in \{1, 2, 3\}$, $j_1 \neq j_2$, we have $\tilde{\phi}_{kj_1}(\tilde{\mathcal{T}}_{kj_1}) \cap \tilde{\phi}_{kj_2}(\tilde{\mathcal{T}}_{kj_2}) = \{[H^k]_k\}$. Applying the injection $\tilde{\phi}_{*k}$ to this equation yields

$$\mathcal{T}_{kj_1} \cap \mathcal{T}_{kj_2} = \{\tilde{\phi}_{*k}([H^k]_k)\} = \{\rho_{k1}\}, \quad (13)$$

with the second equality following from (12). This fact will allow us to prove (f) by induction on the length of i . Obviously, there is nothing to prove for $|i| = 0$. Suppose now that $|i| \geq 1$ and the desired result holds for any index of length strictly less than $|i|$. Suppose $|j| = |i|$, but $j \neq i$, and define $k := i(|i| - 1)$. If $j(|j| - 1) \neq k$, then the inductive hypothesis implies that $\mathcal{T}_i \cap \mathcal{T}_j \subseteq \mathcal{T}_k \cap \mathcal{T}_{j(|j|-1)} \subseteq \{\rho_k, Z_k^1\}$, where we apply part (a) to obtain the first inclusion. Using parts (d) and (e) of the proposition it is straightforward to deduce from this that $\mathcal{T}_i \cap \mathcal{T}_j \subseteq \{Z_i^1\}$ in this case. If $j(|j| - 1) = k$, then we can apply the equality at (13) to obtain that $\mathcal{T}_i \cap \mathcal{T}_j = \{\rho_{k1}\} = \{\rho_i\}$, which completes the proof of part (f).

Finally, μ is non-atomic and so $\mu(\mathcal{T}_i) = \mu(\mathcal{T}_i \setminus \{\rho_i, Z_i^1\})$. Hence, by the disjointness of the sets and the fact that μ is a probability measure, we have $1 \geq \sum_{i \in \Sigma_n} \mu(\mathcal{T}_i \setminus \{\rho_i, Z_i^1\}) = \sum_{i \in \Sigma_n} \mu(\mathcal{T}_i)$. Now, by definition, for each i ,

$$\mathcal{T}_i = \{\tilde{\phi}_{*i}([t]_i) : t \in [0, 1]\} = \{[t] : t \in \phi_{i|1} \circ \phi_{i|2} \circ \dots \circ \phi_i([0, 1])\}.$$

Thus, since μ is the projection of Lebesgue measure, this implies that $\mu(\mathcal{T}_i)$ is no smaller than $\ell(\phi_{i|1} \circ \phi_{i|2} \circ \dots \circ \phi_i([0, 1]))$. By repeated application of (10), this lower bound is equal to $\Delta_{i|1} \Delta_{i|2} \dots \Delta_i = l(i)^2$. Now observe that, because $(\Delta_{i1}, \Delta_{i2}, \Delta_{i3})$ are Dirichlet $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ random variables, we have $\Delta_{i1} + \Delta_{i2} + \Delta_{i3} = 1$ for every $i \in \Sigma_*$, and from this it is simple to show that $\sum_{i \in \Sigma_n} l(i)^2 = 1$. Hence $\sum_{i \in \Sigma_n} \mu(\mathcal{T}_i) \geq \sum_{i \in \Sigma_n} l(i)^2 = 1$. Thus $\sum_{i \in \Sigma_n} \mu(\mathcal{T}_i)$ is actually equal to 1, and moreover, (g) must hold. \square

This result is summarised in Figure 2. Note that the fact that sets from $(\mathcal{T}_{ij})_{j \in \{1,2,3\}}$ only intersect at ρ_{i1} was shown at (13), and so the diagram is representative of the set structure of the decomposition. Furthermore, it is clear that the sets \mathcal{T}_i are all compact dendrites, because they are simply rescaled versions of the compact dendrites $\tilde{\mathcal{T}}_i$.

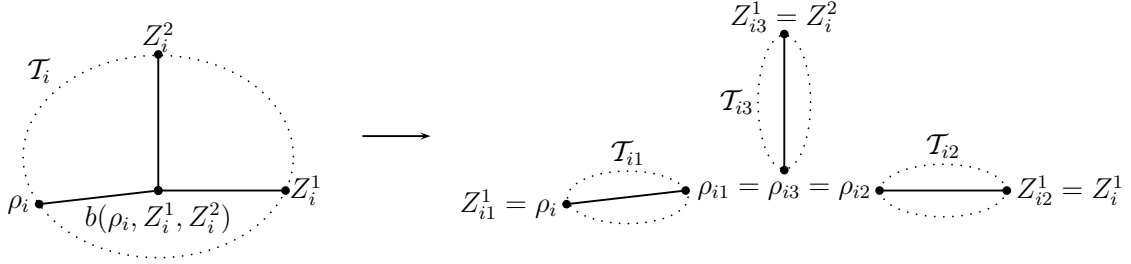


Figure 2: Continuum random tree decomposition.

The tree description of the inductive algorithm runs as follows. Suppose that the triples $((\mathcal{T}_i, l(i)^{-1}d_{\mathcal{T}}, \mu^i))_{i \in \Sigma_n}$ are independent copies of $(\mathcal{T}, d_{\mathcal{T}}, \mu)$, independent of \mathcal{F}_n , where $\mu^i(A) := \mu(A)/\mu(\mathcal{T}_i)$ for measurable $A \subseteq \mathcal{T}_i$. Furthermore, suppose \mathcal{T}_i has root ρ_i , and Z_i^1 and Z_i^2 are two μ^i -random variables in \mathcal{T}_i . For $j = 1, 2, 3$, define \mathcal{T}_{ij} to be the component of \mathcal{T}_i (when split at $b(\rho_i, Z_i^1, Z_i^2)$) containing ρ_i, Z_i^1, Z_i^2 respectively. Define $\Delta_{ij} := \mu^i(\mathcal{T}_{ij})$, and equip the sets with the metrics $\Delta_{ij}^{-1/2}l(i)^{-1}d_{\mathcal{T}} = l(ij)^{-1}d_{\mathcal{T}}$ and measures μ^{ij} , defined by

$$\mu^{ij}(A) := \frac{\mu^i(A)}{\Delta_{ij}} = \frac{\mu(A)}{\mu(\mathcal{T}_{ij})}.$$

Then the triples $((\mathcal{T}_i, l(i)^{-1}d_{\mathcal{T}}, \mu^i))_{i \in \Sigma_{n+1}}$ are independent copies of the continuum random tree, independent of \mathcal{F}_{n+1} . Moreover, for $i \in \Sigma_{n+1}$, the algorithm gives us the root ρ_i of \mathcal{T}_i and also a μ^i -random vertex, Z_i^1 . To continue the algorithm, we pick independently for each $i \in \Sigma_{n+1}$ a second μ^i -random vertex, Z_i^2 . Note that picking this extra μ^i -random vertex is the equivalent of picking the $U[0, 1]$ random variable V_i in the excursion picture.

To complete this section, we introduce one further family of variables associated with the decomposition of the continuum random tree. From Proposition 3.4(f), observe that the sets in $(\mathcal{T}_i)_{i \in \Sigma_n}$ only intersect at points of the form ρ_i or Z_i^1 . Consequently, it is possible to consider the two point set $\{\rho_i, Z_i^1\}$ to be the boundary of \mathcal{T}_i . Denote the renormalised distance between boundary points by, for $i \in \Sigma_*$,

$$D_i := l(i)^{-1}d_{\mathcal{T}}(\rho_i, Z_i^1).$$

By construction, we have that $d_{\mathcal{T}}(\rho_i, Z_i^1) = l(i)d_{W^i}(0, U_i)$. Hence we can also write $D_i = d_{W^i}(0, U_i)$, and so, for each n , $(D_i)_{i \in \Sigma_n}$ is a collection of independent random variables, independent of \mathcal{F}_n . Moreover, the random variables $(D_i)_{i \in \Sigma_*}$ are identically

distributed as D_\emptyset , which represents the height of a μ -random vertex in \mathcal{T} . It is known that such a random variable has mean $\sqrt{\pi/8}$, and finite variance (see [1], Section 3.3). Finally, we have the following recursive relationship

$$D_i = w(i1)D_{i1} + w(i2)D_{i2}, \quad (14)$$

which may be deduced by decomposing the path from ρ_i to Z_i^1 at $b(\rho_i, Z_i^1, Z_i^2)$, and applying parts (c) and (d) of Proposition 3.4.

4 Self-similar dendrite in \mathbb{R}^2

The subset of \mathbb{R}^2 to which we will map the continuum random tree is a simple self-similar fractal, and is described as the fixed point of a collection of contraction maps. In particular, for $(x, y) \in \mathbb{R}^2$, set

$$F_1(x, y) := \frac{1}{2}(1 - x, y), \quad F_2(x, y) := \frac{1}{2}(1 + x, -y), \quad F_3(x, y) := \left(\frac{1}{2} + cy, cx \right),$$

where $c \in (0, 1/2)$ is a constant, and define T to be the unique non-empty compact set satisfying $A = \bigcup_{i=1}^3 F_i(A)$. The existence and uniqueness of T , which is shown in Figure 3, is guaranteed by an extension of the usual contraction principle for metric spaces, see [11], Theorem 1.1.4. For a wide class of self-similar fractals, which includes T , there is now a well-established approximation procedure for defining an intrinsic Dirichlet form and associated resistance metric on the relevant space, see [4] and [11] for details. However, to capture the randomness of the continuum random tree, we will need to randomise this construction, and it is to describing how this is done that this section is devoted.

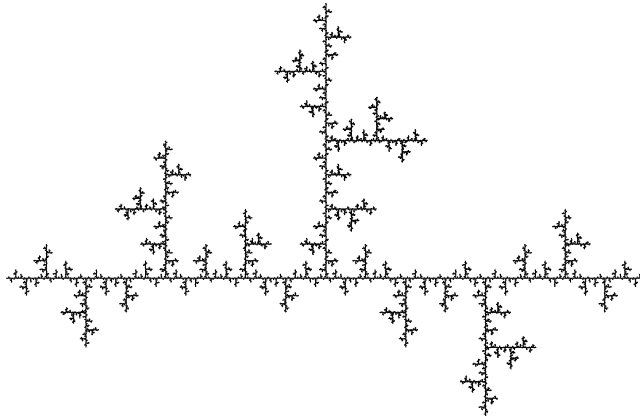


Figure 3: Self-similar dendrite.

The scaling factors that will be useful in defining a sequence of compatible Dirichlet forms on subsets T will be the family $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$, as defined in the previous section. Although we would like to simply replace the deterministic scaling factors that are used in the method of [11] with this collection of random variables, following this course of action would result in a sequence of non-compatible quadratic forms, and taking limits

would not be straightforward. To deal with the offending tail fluctuations caused by using random scaling factors, we introduce another collection of random variables

$$R_i := \lim_{n \rightarrow \infty} \sum_{j \in \{1,2\}^n} \frac{l(ij)}{l(i)}, \quad i \in \Sigma_*, \quad (15)$$

which we shall term resistance perturbations. Clearly these are identically distributed, and, by appealing to the independence properties of $(w(i))_{i \in \Sigma_* \setminus \{\emptyset\}}$, various questions regarding the convergence and distribution of the $(R_i)_{i \in \Sigma_*}$ may be answered by standard multiplicative cascade techniques. Consequently we provide only a brief explanation and suitable references for the proof of the following result. Crucially, part (d) reveals an important identity between the resistance perturbations and the family $(D_i)_{i \in \Sigma_*}$, which was defined from the continuum random tree.

Lemma 4.1 (a) *\mathbf{P} -a.s., the limit at (15) exists in $(0, \infty)$ for every $i \in \Sigma_*$.*

(b) *$\mathbf{E}R_\emptyset = 1$, and $\mathbf{E}R_\emptyset^d < \infty$ for every $d \geq 0$.*

(c) *\mathbf{P} -a.s., for every $i \in \Sigma_*$, the identity $R_i = w(i1)R_{i1} + w(i2)R_{i2}$ holds.*

(d) *\mathbf{P} -a.s., $(R_i)_{i \in \Sigma_*} \equiv (HD_i)_{i \in \Sigma_*}$, where $H := \sqrt{8/\pi}$.*

Proof: The finite limit result of (a) and part (b) are immediate applications of Theorem 2.0 of [15]. Part (c) is immediate from the definition of $(R_i)_{i \in \Sigma_*}$. Using the identical distribution of the family of resistance perturbations, part (c) implies that $\mathbf{P}(R_i = 0) = \mathbf{P}(R_i = 0)^2$. Since $\mathbf{E}R_i = 1$, it follows that $\mathbf{P}(R_i = 0) = 0$, which completes the proof of (a). Checking the equivalence of (d) is straightforward. First, from an elementary application of a conditional version of Chebyshev's inequality it may be deduced that, for each i ,

$$\mathbf{P} \left(\left| HD_i - \sum_{j \in \{1,2\}^n} \frac{l(ij)}{l(i)} \right| > \lambda \right) \leq H^2 \lambda^{-2} (\mathbf{E}(\Delta_1 + \Delta_2))^n \text{Var} D_\emptyset,$$

where we use the fact that $\mathbf{E}(HD_i) = 1$, and the identity at (14) to enable us to condition on \mathcal{F}_n . As remarked in the previous section, D_\emptyset has finite variance. Furthermore, a simple symmetry argument yields that the expectation in the right hand side is precisely $2/3$. Hence the sum of probabilities over n is finite, and applying a Borel-Cantelli argument yields the result. \square

The sequence of vertices upon which we will define our Dirichlet forms will be that which is commonly used for a p.c.f.s.s. fractal, see [11] for more examples. Thus we shall not detail the reason for the choice, but start by simply stating that the boundary of T may be taken to be the two point set $V^0 := \{(0,0), (1,0)\}$. Our initial Dirichlet form is defined by

$$D(f, f) := \sum_{x, y \in V^0, x \neq y} H(f(x) - f(y))^2, \quad \forall f \in C(V^0),$$

where, for a countable set, A , we denote $C(A) := \{f : A \rightarrow \mathbb{R}\}$. The constant H is defined as in the previous lemma and is necessary to achieve the correct scaling in the

metric we shall later define. We now introduce an increasing family of subsets of T by setting $V^n := \bigcup_{i \in \Sigma_n} F_i(V^0)$, where for $i \in \Sigma_*$, $F_i := F_{i_1} \circ \dots \circ F_{i_{|i|}}$. By defining

$$\mathcal{E}^n(f, f) := \sum_{i \in \Sigma_n} \frac{1}{l(i)R_i} D(f \circ F_i, f \circ F_i), \quad \forall f \in C(V^n),$$

we obtain Dirichlet forms on each of the appropriate finite subsets of T , \mathbf{P} -a.s. By applying the identity of Lemma 4.1(c), it is straightforward to check that the family (V^n, \mathcal{E}^n) is compatible in the sense of [11], Definition 2.2.1, and from this fact we may take a limit in a sensible way. Specifically, let

$$\mathcal{E}'(f, f) := \lim_{n \rightarrow \infty} \mathcal{E}^n(f, f), \quad \forall f \in \mathcal{F}',$$

where \mathcal{F}' is the set of functions on the countable set $V^* := \bigcup_{n \geq 0} V^n$ for which this limit exists finitely. Note that we have abused notation slightly by using the convention that if a form \mathcal{E} is defined for functions on a set A and f is a function defined on $B \supseteq A$, then we write $\mathcal{E}(f, f)$ to mean $\mathcal{E}(f|_A, f|_A)$.

The quadratic form $(\mathcal{E}', \mathcal{F}')$ is actually a resistance form (see [11], Definition 2.3.1), and we can use it to define a (resistance) metric R' on V^* using a formula analogous to (7)

$$R'(x, y)^{-1} = \inf\{\mathcal{E}'(f, f) : f \in \mathcal{F}', f(x) = 0, f(y) = 1\},$$

for $x, y \in V^*$, $x \neq y$, and setting $R'(x, x) = 0$. We note that for sets of the form $F_i(V_0)$ with $i \in \Sigma_*$ we have

$$R'(F_i(0, 0), F_i(1, 0)) = \frac{l(i)R_i}{H}. \quad (16)$$

To prove that this metric may be extended to T in a natural way (at least \mathbf{P} -a.s.) requires a similar argument to the deterministic case, and so we omit the full details here. The most crucial fact that is needed is the following:

$$\lim_{n \rightarrow \infty} \sup_{i \in \Sigma_n} \text{diam}_{R'} F_i(V^*) = 0, \quad \mathbf{P}\text{-a.s.}, \quad (17)$$

where, in general, $\text{diam}_d(A)$ represents the diameter of a set A with respect to a metric d . The proof follows the chaining argument of [4], Proposition 7.10, and full details of the proof of the following Proposition can be found in [5].

Proposition 4.2 *There exists a unique metric R on T such that (T, R) is the completion of (V^*, R') , \mathbf{P} -a.s. Moreover, the topology induced upon T by R is the same as that induced by the Euclidean metric, \mathbf{P} -a.s.*

To complete this section, we introduce the natural stochastic self-similar measure on T , and note that $(\mathcal{E}', \mathcal{F}')$ may be extended to a Dirichlet form on the corresponding L^2 space. In particular, by proceeding exactly as in the deterministic case, see [11], Section 1.4, it is possible to prove that, \mathbf{P} -a.s., there exists a unique non-atomic Borel probability measure, μ^T say, on (T, R) that satisfies

$$\mu^T(F_i(T)) = l(i)^2, \quad \forall i \in \Sigma_*. \quad (18)$$

Again, full details of this result are given in [5]. If we extend $(\mathcal{E}', \mathcal{F}')$ in the natural way by setting $\mathcal{E}(f, f) := \mathcal{E}'(f, f)$, for $f \in \mathcal{F} := \{f \in C(T) : f|_{V^*} \in \mathcal{F}'\}$, where we use $C(T)$ to represent the continuous functions on T (with respect to the Euclidean metric or R), then the following result holds (for a proof, see [5]).

Proposition 4.3 *\mathbf{P} -a.s., $(\mathcal{E}, \mathcal{F})$ is a local, regular Dirichlet form on $L^2(T, \mu^T)$ and, moreover, it may be associated with the metric R through*

$$R(x, y)^{-1} = \inf\{\mathcal{E}(f, f) : f \in \mathcal{F}, f(x) = 0, f(y) = 1\}.$$

5 Equivalence of measure-metric spaces

In this section, we demonstrate how the decomposition of the continuum random tree presented in Section 3 allows us to define an isometry from the continuum random tree to the random self-similar dendrite, (T, R) , described in the previous section. An important consequence of the decomposition is that it allows us to label points in \mathcal{T} using the shift space of infinite sequences, $\Sigma := \{1, 2, 3\}^{\mathbb{N}}$. The following lemma defines the projection $\pi_{\mathcal{T}} : \Sigma \rightarrow \mathcal{T}$ that we will use, which is analogous to the well-known projection map for self-similar fractals, see [4], Lemma 5.10. We include the result for the corresponding projection $\pi_T : \Sigma \rightarrow T$ to allow us to introduce the necessary notation, and provide a direct comparison of the two maps. Henceforth, we shall use the notation $T_i := F_i(T)$, for $i \in \Sigma_*$.

Lemma 5.1 (a) *There exists a map $\pi_T : \Sigma \rightarrow T$ such that $\pi_T \circ \sigma_i(\Sigma) = T_i$, for every $i \in \Sigma_*$, where $\sigma_i : \Sigma \rightarrow \Sigma$ is defined by $\sigma_i(j) = ij$ for $j \in \Sigma$. Furthermore, this map is continuous, surjective and unique.*

(b) *\mathbf{P} -a.s., there exists a map $\pi_{\mathcal{T}} : \Sigma \rightarrow \mathcal{T}$ such that $\pi_{\mathcal{T}} \circ \sigma_i(\Sigma) = \mathcal{T}_i$, for every $i \in \Sigma_*$, where σ_i is defined as in (a). Furthermore, this map is continuous, surjective and unique.*

Proof: Part (a) is proved in [4] and [11], so we prove only (b). \mathbf{P} -a.s., for each $i \in \Sigma$, the sets in the collection $(\mathcal{T}_{i|n})_{n \geq 0}$ are compact, non-empty subsets of $(\mathcal{T}, d_{\mathcal{T}})$, and by Proposition 3.4(a), the sequence is decreasing. Hence, to show that $\bigcap_{n \geq 0} \mathcal{T}_{i|n}$ contains exactly one point for each $i \in \Sigma$, \mathbf{P} -a.s., it will suffice to demonstrate that, \mathbf{P} -a.s.,

$$\lim_{n \rightarrow \infty} \sup_{i \in \Sigma_n} \text{diam}_{d_{\mathcal{T}}} \mathcal{T}_i = 0. \quad (19)$$

From Proposition 3.4(b), we have that $\text{diam}_{d_{\mathcal{T}}} \mathcal{T}_i = l(i) \text{diam}_{d_{\tilde{\mathcal{T}}_i}} \tilde{\mathcal{T}}_i$. Using the similarity that this implies, the above result may be proved in the same way as (17). To enable us to apply this argument, we note that $\text{diam}_{d_{\tilde{\mathcal{T}}_i}} \tilde{\mathcal{T}}_i \leq 2 \sup_{t \in [0,1]} W_t^i$. The upper bound here is simply twice the maximum of a normalised Brownian excursion, and has finite positive moments of all orders as required (see [1], for example).

Using the result of the previous paragraph, it is \mathbf{P} -a.s. possible to define a map $\pi_{\mathcal{T}} : \Sigma \rightarrow \mathcal{T}$ such that, for $i \in \Sigma$, $\{\pi_{\mathcal{T}}(i)\} = \bigcap_{n \geq 0} \mathcal{T}_{i|n}$. That $\pi_{\mathcal{T}}$ satisfies the claims of the lemma, and is the unique map to do so, may be proved in exactly the same way as in the self-similar fractal case. \square

Heuristically, the isometry that we will define between the two dendrites under consideration can be thought of as simply “ $\varphi = \pi_T \circ \pi_{\mathcal{T}}^{-1}$ ”. However, to introduce the map rigorously, so that it is well-defined, we first need to prove some simple, but fundamental, results about the geometry of the sets and the maps π_T and $\pi_{\mathcal{T}}$.

Lemma 5.2 *P*-a.s.,

(a) $\pi_T^{-1}(\rho_{k1}) = \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$, for all $k \in \Sigma_*$.

(b) For every $i, j \in \Sigma$, $\pi_T(i) = \pi_T(j)$ if and only if $\pi_T(i) = \pi_T(j)$.

Proof: The proof we give holds on the *P*-a.s. set for which the decomposition of \mathcal{T} and the definition of π_T is possible. Recall that $\rho_{k1} = b(\rho_k, Z_k^1, Z_k^2)$. For this branch-point to equal ρ_k or Z_k^1 , we would require at least two of its arguments to be equal, which happens with zero probability. Thus $\rho_{k1} \in \mathcal{T}_k \setminus \{\rho_k, Z_k^1\}$, and so Proposition 3.4(f) implies that if $\pi_T(i) = \rho_{k1}$ for some $i \in \Sigma$, then $i||k| = k$. Given this fact, it is elementary to apply the defining property of π_T and the results about ρ_i and Z_i^1 that were deduced in Proposition 3.4 to deduce that part (a) of this lemma also holds. It now remains to prove part (b).

Fix $i, j \in \Sigma$, $i \neq j$, and let m be the unique integer satisfying $i|m = j|m$ and $i_{m+1} \neq j_{m+1}$. Furthermore, define $k = i_1 \dots i_m \in \Sigma_*$. Now by standard arguments for p.c.f.s.s. fractals (see [11], Proposition 1.2.5 and the subsequent remark) we have that $\pi_T(i) = \pi_T(j)$ implies that $\sigma^m(i), \sigma^m(j) \in \mathcal{C}$, where \mathcal{C} is the critical set for the self-similar structure, T , as defined in [11], Definition 1.3.4. Here, we use the notation σ to represent the shift map, which is defined by $\sigma(i) = i_2 i_3 \dots$. Note that it is elementary to calculate that $\mathcal{C} = \{11\dot{2}, 21\dot{2}, 31\dot{2}\}$ for this structure. Thus $i, j \in \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$, and so, by part (a), $\pi_T(i) = \rho_{k1} = \pi_T(j)$, which completes one implication of the desired result.

Now suppose $\pi_T(i) = \pi_T(j)$. From the definition of π_T , we have that $\pi_T(i) \in \mathcal{T}_{ki_{m+1}}$ and also $\pi_T(j) \in \mathcal{T}_{kj_{m+1}}$. Hence $\pi_T(i), \pi_T(j) \in \mathcal{T}_{ki_{m+1}} \cap \mathcal{T}_{kj_{m+1}} = \{\rho_{k1}\}$, where we use (13) to deduce the above equality. In particular, this allows us to apply part (a) to deduce that $i, j \in \{k11\dot{2}, k21\dot{2}, k31\dot{2}\}$. Applying the shift map to this m times yields $\sigma^m(i), \sigma^m(j) \in \mathcal{C}$. It is easy to check that $\pi_T(\mathcal{C})$ contains only the single point $(\frac{1}{2}, 0)$. Thus $\pi_T(i) = F_k \circ \pi_T(\sigma^m(i)) = F_k \circ \pi_T(\sigma^m(j)) = \pi_T(j)$, which completes the proof. \square

We are now able to define the map φ precisely on a *P*-a.s. set by

$$\begin{aligned} \varphi : \mathcal{T} &\rightarrow T \\ x &\mapsto \pi_T(i), \quad \text{for any } i \in \Sigma \text{ with } \pi_T(i) = x. \end{aligned}$$

By part (b) of the previous lemma, this is a well-defined injection. Furthermore, since π_T is surjective, so is φ . Hence we have constructed a bijection from \mathcal{T} to T and it remains to show that it is also an isometry. We start by checking that φ is continuous, which will enable us to deduce that it maps geodesic paths in \mathcal{T} to geodesic paths in T . However, before we proceed with the lemma, we introduce the following notation for $x \in \mathcal{T}$, $n \geq 0$,

$$\mathcal{T}_n(x) := \bigcup \{\mathcal{T}_i : i \in \Sigma_n, x \in \mathcal{T}_i\}.$$

Define $(T_n(x))_{x \in T, n \geq 0}$ similarly, replacing \mathcal{T}_i with T_i in the above definition where appropriate. From the properties $\pi_T(i\Sigma) = T_i$, $\pi_T(i\Sigma) = \mathcal{T}_i$, and the definition of φ , it is straightforward to deduce that

$$\varphi(\mathcal{T}_i) = T_i, \quad \forall i \in \Sigma_*, \quad (20)$$

on the *P*-a.s. set that we can define all the relevant objects.

Lemma 5.3 *P*-a.s., φ is a continuous map from $(\mathcal{T}, d_{\mathcal{T}})$ to (T, R) .

Proof: By [11], Proposition 1.3.6, for each $x \in T$, the collection $(T_n(x))_{n \geq 0}$ is a base of

neighbourhoods of x with respect to the Euclidean metric on \mathbb{R}^2 . Since, by Proposition 4.2, R is topologically equivalent to this metric, \mathbf{P} -a.s., then the same is true when we consider the collections of neighbourhoods with respect to R , \mathbf{P} -a.s. Similarly, we may use (19), \mathbf{P} -a.s., to imitate the proofs of these results to deduce that \mathbf{P} -a.s., for each $x \in \mathcal{T}$, the collection $(\mathcal{T}_n(x))_{n \geq 0}$ is a base of neighbourhoods of x with respect to $d_{\mathcal{T}}$.

The remaining argument applies \mathbf{P} -a.s. Let U be an open subset of (T, R) and $x \in \varphi^{-1}(U)$. Define $y = \varphi(x) \in U$. Now, since U is open, there exists an n such that $T_n(y) \subseteq U$. Also, by (20), for each $i \in \Sigma_n$, we have that $x \in \mathcal{T}_i$ implies that $y \in T_i$. Hence

$$\varphi(\mathcal{T}_n(x)) = \varphi(\cup_{i \in \Sigma_n, x \in \mathcal{T}_i} \mathcal{T}_i) \subseteq \cup_{i \in \Sigma_n, y \in T_i} T_i = T_n(y) \subseteq U.$$

Consequently, $\mathcal{T}_n(x) \subseteq \varphi^{-1}(U)$. Since $\mathcal{T}_n(x)$ is a $d_{\mathcal{T}}$ -neighbourhood of x it follows that $\varphi^{-1}(U)$ is open in $(\mathcal{T}, d_{\mathcal{T}})$. The lemma follows. \square

We are now ready to proceed with the main result of this section. In the proof, we will use the notation $\gamma_{xy}^{\mathcal{T}} : [0, 1] \rightarrow \mathcal{T}$ to denote a geodesic path (continuous injection) from x to y , where x and y are points in the dendrite \mathcal{T} . Clearly, because φ is a continuous injection, $\varphi \circ \gamma_{xy}^{\mathcal{T}}$ describes a geodesic path from $\varphi(x)$ to $\varphi(y)$ in T .

Theorem 5.4 *\mathbf{P} -a.s., the map φ is an isometry, and the metric spaces $(\mathcal{T}, d_{\mathcal{T}})$ and (T, R) are isometric.*

Proof: Obviously, the second statement of the theorem is an immediate consequence of the first. The following argument, in which we demonstrate that φ is indeed an isometry, holds \mathbf{P} -a.s. Given $\varepsilon > 0$, by (17) and (19), we can choose an $n \geq 1$ such that

$$\sup_{i \in \Sigma_n} \text{diam}_{d_{\mathcal{T}}} \mathcal{T}_i, \sup_{i \in \Sigma_n} \text{diam}_R T_i < \frac{\varepsilon}{4}.$$

Now, fix $x, y \in \mathcal{T}$, define $t_0 := 0$ and set

$$t_{m+1} := \inf\{t > t_m : \gamma_{xy}^{\mathcal{T}}(t) \notin \mathcal{T}_n(\gamma_{xy}^{\mathcal{T}}(t_m))\},$$

where $\inf \emptyset := 1$. We will also denote $x_m := \gamma_{xy}^{\mathcal{T}}(t_m)$. Since, for each $x' \in \mathcal{T}$, the collection $(\mathcal{T}_n(x'))_{n \geq 0}$ forms a base of neighbourhoods of x' , we must have that $t_{m-1} < t_m$ whenever $t_{m-1} < 1$. We now claim that for any m with $t_{m-1} < 1$ there exists a unique $i(m) \in \Sigma_n$ such that

$$\gamma_{xy}^{\mathcal{T}}(t) \in \mathcal{T}_{i(m)}, \quad t_{m-1} \leq t \leq t_m. \quad (21)$$

Let m be such that $t_{m-1} < 1$. By the continuity of $\gamma_{xy}^{\mathcal{T}}$, we have that $x_m \in \mathcal{T}_n(x_{m-1})$, and hence there exists an $i(m) \in \Sigma_n$ such that $x_{m-1}, x_m \in \mathcal{T}_{i(m)}$. Clearly, the image of $\gamma_{xy}^{\mathcal{T}}$ restricted to $t \in [t_{m-1}, t_m]$ is the same as the image of $\gamma_{x_{m-1}x_m}^{\mathcal{T}}$, which describes the unique path in \mathcal{T} from x_{m-1} to x_m . Note also that $\mathcal{T}_{i(m)}$ is a path-connected subset of \mathcal{T} , and so the path from x_{m-1} to x_m lies in $\mathcal{T}_{i(m)}$. Consequently, the set $\gamma_{xy}^{\mathcal{T}}([t_{m-1}, t_m])$ is contained in $\mathcal{T}_{i(m)}$. Thus to prove the claim at (21), it remains to show that $i(m)$ is unique. Suppose that there exists $j \in \Sigma_n$, $j \neq i(m)$ for which the inclusion at (21) holds. Then the uncountable set $\gamma_{xy}^{\mathcal{T}}([t_{m-1}, t_m])$ is contained in $\mathcal{T}_{i(m)} \cap \mathcal{T}_j$, which, by Proposition 3.4(f), contains at most two points. Hence no such j can exist.

Now assume that $m_1 < m_2$ and that $t_{m_2-1} < 1$. Suppose that $i(m_1) = i(m_2)$, then $x_{m_1-1}, x_{m_2} \in \mathcal{T}_{i(m_1)}$. By a similar argument to the previous paragraph, it follows that

$\gamma_{xy}^{\mathcal{T}}([t_{m_1-1}, t_{m_2}]) \subseteq \mathcal{T}_{i(m_1)}$. By definition, this implies that $t_{m_1} \geq t_{m_2}$, which cannot be true. Consequently, we must have that $i(m_1) \neq i(m_2)$. Since Σ_n is a finite set, it follows from this observation that $N := \inf\{m : t_m = 1\}$ is finite, and moreover, the elements of $(i(m))_{m=1}^N$ are distinct.

The conclusion of the previous paragraph provides us with a useful decomposition of the path from x to y , which we will be able to use to complete the proof. The fact that $d_{\mathcal{T}}$ is a shortest path metric allows us to write $d_{\mathcal{T}}(x, y) = \sum_{m=1}^N d_{\mathcal{T}}(x_{m-1}, x_m)$. For $m \in \{2, \dots, N-1\}$, we have that $i(m) \neq i(m+1)$, and so by applying Proposition 3.4(f), we can deduce that $x_m \in \mathcal{T}_{i(m)} \cap \mathcal{T}_{i(m+1)} \subseteq \{\rho_{i(m)}, Z_{i(m)}^1\}$. Similarly, we have $x_{m-1} \in \mathcal{T}_{i(m-1)} \cap \mathcal{T}_{i(m)} \subseteq \{\rho_{i(m)}, Z_{i(m)}^1\}$. Thus, by the injectivity of $\gamma_{xy}^{\mathcal{T}}$, we must have that $\{x_{m-1}, x_m\} = \{\rho_{i(m)}, Z_{i(m)}^1\}$, which implies $d_{\mathcal{T}}(x_{m-1}, x_m) = d_{\mathcal{T}}(\rho_{i(m)}, Z_{i(m)}^1) = l(i(m))D_{i(m)}$. Hence we can conclude that

$$d_{\mathcal{T}}(x, y) - \sum_{m=2}^{N-1} l(i(m))D_{i(m)} = d_{\mathcal{T}}(x_0, x_1) + d_{\mathcal{T}}(x_{N-1}, x_N). \quad (22)$$

As remarked before this lemma, $\varphi \circ \gamma_{xy}^{\mathcal{T}}$ is a geodesic path from $\varphi(x)$ to $\varphi(y)$. Thus the shortest path property of R allows us to write

$$R(\varphi(x), \varphi(y)) = \sum_{m=1}^N R(\varphi(x_{m-1}), \varphi(x_m)). \quad (23)$$

Let $m \in \{2, \dots, N-1\}$. By applying φ to the expression for $\{x_{m-1}, x_m\}$ that was deduced above, we obtain that $\{\varphi(x_{m-1}), \varphi(x_m)\} = \{\varphi(\rho_{i(m)}), \varphi(Z_{i(m)}^1)\}$. Now, part (a) of Lemma 5.2 implies that

$$\varphi(\rho_{i(m)}) = \pi_T(k11\dot{2}) = F_k(\pi_T(11\dot{2})) = F_k((\frac{1}{2}, 0)) = F_{i(m)}((0, 0)),$$

where $k := i(m)|(|i(m)| - 1)$. In Proposition 3.4(d) it was shown that $Z_i^1 = Z_{i2}^1$, for every $i \in \Sigma_*$. It follows that $i(m)\dot{2} \in \pi_T^{-1}(Z_{i(m)}^1)$, and so

$$\varphi(Z_{i(m)}^1) = \pi_T(i(m)\dot{2}) = F_{i(m)}(\pi_T(\dot{2})) = F_{i(m)}((1, 0)).$$

Thus $R(\varphi(x_{m-1}), \varphi(x_m)) = R(F_{i(m)}((0, 0)), F_{i(m)}((1, 0)))$, and so from the expression at (16), we can deduce that $R(\varphi(x_{m-1}), \varphi(x_m)) = \sqrt{\pi/8}l(i(m))R_{i(m)} = l(i(m))D_{i(m)}$, where we have used Lemma 4.1(d) to obtain the second equality. Substituting this into (23), and combining the resulting equation with the equality at (22) yields

$$|d_{\mathcal{T}}(x, y) - R(\varphi(x), \varphi(y))| \leq \sum_{m \in \{1, N\}} (d_{\mathcal{T}}(x_{m-1}, x_m) + R(\varphi(x_{m-1}), \varphi(x_m))).$$

Now, x_0 and x_1 are both contained in $\mathcal{T}_{i(1)}$, and so the choice of n implies that $d_{\mathcal{T}}(x_0, x_1) < \varepsilon/4$. Furthermore, $\varphi(x_0)$ and $\varphi(x_1)$ are both contained in $\varphi(\mathcal{T}_{i(1)}) = T_{i(1)}$, and so we also have $R(\varphi(x_0), \varphi(x_1)) < \varepsilon/4$. Thus the summand with $m = 1$ is bounded by $\varepsilon/2$. Similarly for $m = N$. Hence $|d_{\mathcal{T}}(x, y) - R(\varphi(x), \varphi(y))| < \varepsilon$. Since the choice of x, y and ε was arbitrary, the proof is complete. \square

The final result that we present in this section completes the proof of the fact that $(\mathcal{T}, d_{\mathcal{T}}, \mu)$ and (T, R, μ^T) are equivalent measure-metric spaces, where we continue to use the notation μ^T to represent the stochastic self-similar measure on (T, R) , as defined in Section 4.

Theorem 5.5 *\mathbf{P} -a.s., the probability measures μ and $\mu^T \circ \varphi$ agree on the Borel σ -algebra of $(\mathcal{T}, d_{\mathcal{T}})$.*

Proof: That both $\mu^T \circ \varphi$ and μ are non-atomic Borel probability measures on $(\mathcal{T}, d_{\mathcal{T}})$, \mathbf{P} -a.s., is obvious. Recall from Proposition 3.4(g) that $\mu(\mathcal{T}_i) = l(i)^2$, for every $i \in \Sigma_*$, \mathbf{P} -a.s. Furthermore, from the identities of (18) and (20), we also have $\mu^T \circ \varphi(\mathcal{T}_i) = \mu^T(T_i) = l(i)^2$, for every $i \in \Sigma_*$, \mathbf{P} -a.s. The result is readily deduced from these facts. \square

6 Spectral asymptotics

Due to the construction of the natural Dirichlet form on the continuum random tree from the natural metric on the space, the results of the previous section imply that the spectrum of $(\mathcal{E}_{\mathcal{T}}, \mathcal{F}_{\mathcal{T}}, \mu)$ is \mathbf{P} -a.s. identical to that of $(\mathcal{E}, \mathcal{F}, \mu^T)$, the random Dirichlet form and self-similar measure on T , as defined in Section 4. Consequently, to deduce the results of the introduction, it will suffice to show that the analogous results hold for $(\mathcal{E}, \mathcal{F}, \mu^T)$, which is possible using techniques developed for related self-similar fractals. For this argument, it will be helpful to apply various decomposition and comparison inequalities for the Dirichlet and Neumann eigenvalues associated with this Dirichlet form, and we shall start by introducing these.

To define the Dirichlet eigenvalues for $(\mathcal{E}, \mathcal{F}, \mu^T)$, we first introduce the related Dirichlet form $(\mathcal{E}^D, \mathcal{F}^D)$ by setting

$$\mathcal{E}^D(f, f) := \mathcal{E}(f, f), \quad \forall f \in \mathcal{F}^D,$$

where

$$\mathcal{F}^D := \{f \in \mathcal{F} : f|_{V^0} = 0\}.$$

The Dirichlet eigenvalues of the original form, $(\mathcal{E}, \mathcal{F}, \mu^T)$, are then defined to be the eigenvalues of $(\mathcal{E}^D, \mathcal{F}^D, \mu^T)$. We shall use the title Neumann eigenvalues to refer to the usual eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu^T)$, defined analogously to (1).

Before continuing, note that the description of R in Proposition 4.3 easily leads to the well known inequality

$$|f(x) - f(y)|^2 \leq R(x, y)\mathcal{E}(f, f), \quad \forall x, y \in T, f \in \mathcal{F}. \quad (24)$$

By applying this fact (and using $\|\cdot\|_p$ to represent the corresponding $L^p(T, \mu^T)$ norm), we find that, for $x \in T$, $f \in \mathcal{F}$,

$$|f(x)|^2 \leq 2 \int_T (|f(x) - f(y)|^2 + |f(y)|^2) d\mu \leq 2 \text{diam}_R T \mathcal{E}(f, f) + 2\|f\|_2^2,$$

and so, \mathbf{P} -a.s., $\|f\|_{\infty}^2 \leq C(\mathcal{E}(f, f) + \|f\|_2^2)$, for some constant C . Combining this inequality with (24), we can imitate the argument of [12], Lemma 5.4, to deduce that the natural

inclusion map from $(\mathcal{F}, \mathcal{E} + \|\cdot\|_2^2)$ to $L^2(T, \mu^T)$ is a compact operator. It follows that the Dirichlet and Neumann spectra of $(\mathcal{E}, \mathcal{F}, \mu^T)$ are discrete, and so the associated eigenvalue counting functions, $N^D(\lambda)$ and $N^N(\lambda)$, are well-defined and finite for all $\lambda \in \mathbb{R}$.

From the definitions in the previous paragraph, we can easily see that $N(\lambda) = N^N(\lambda)$, \mathbf{P} -a.s., and so, using the terminology introduced above, the eigenvalues of $(\mathcal{E}_T, \mathcal{F}_T, \mu)$ may be thought of as Neumann eigenvalues. Of course, this definition does not provide any justification for using the name Neumann, so we will now give an explanation of why it is sensible to do so. By using a definition analogous to [11], Definition 3.7.1, it is possible to construct directly a Laplacian Δ on T , with domain \mathcal{D} , \mathbf{P} -a.s. This Laplacian is essentially the limit operator of the discrete Laplacians related to the sequence $(\mathcal{E}^n)_{n \geq 0}$. Furthermore, for $f \in \mathcal{D}$, we can also define a function, df say, with domain V^0 , which represents the Neumann derivative on the boundary of T . By using a Green's function argument as in the proof of [11], Theorem 3.7.9, it is possible to deduce that the Friedrichs extension of Δ on $\mathcal{D}_D := \{f \in \mathcal{D} : f|_{V^0} = 0\}$ is precisely Δ_D , the Laplacian associated with $(\mathcal{E}^D, \mathcal{F}^D, \mu^T)$. Similarly, the Friedrichs extension of Δ on $\mathcal{D}_N := \{f \in \mathcal{D} : (df)(x) = 0, \forall x \in V^0\}$ is Δ_N , the Laplacian associated with $(\mathcal{E}, \mathcal{F}, \mu^T)$. Note that the construction of the relevant Green's function may be accomplished more easily than in [11] by, instead of imitating the analytic definition used there, applying a probabilistic definition, with $g(x, y)$ being the Green's kernel for the Markov process associated with $(\mathcal{E}, \mathcal{F})$ killed on hitting V^0 (the existence of which follows from an argument similar to that used in [13], Proposition 4.2).

Applying the relationships between the various operators introduced in the previous paragraph (and also the continuity of the Green's function), we are able to emulate the argument of [11], Proposition 4.1.2, to deduce that the eigenvalues of $(\mathcal{E}^D, \mathcal{F}^D, \mu^T)$ are precisely the solutions to

$$-\Delta u = \lambda u, \quad u|_{V^0} = 0,$$

for some eigenfunction $u \in \mathcal{D}$. Furthermore, the eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu^T)$ are precisely the solutions to

$$-\Delta u = \lambda u, \quad (du)|_{V^0} = 0, \tag{25}$$

for some eigenfunction $u \in \mathcal{D}$. From these characterisations, it is clear that the Dirichlet and Neumann eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu^T)$ that we have defined are exactly the eigenvalues of $-\Delta$ with the usual Dirichlet (zero function on boundary) and Neumann (zero derivative on boundary) boundary conditions respectively, where the analytic boundary of T is taken to be V^0 .

By mapping these results to the continuum random tree, we are able to deduce, \mathbf{P} -a.s., the existence of a Laplace operator $\Delta_{\mathcal{T}}$ on \mathcal{T} , and also a Neumann boundary derivative, so that the eigenvalues of $(\mathcal{E}, \mathcal{F}, \mu)$ satisfy a result analogous to (25). In the continuum random tree setting, observe that the natural analytic boundary is the two point set consisting of the root and one μ -random vertex, $\{\rho, Z_\emptyset^1\}$. Consequently, the results we prove also demonstrate the Dirichlet spectrum corresponding to this boundary satisfies the same asymptotics as the original (Neumann) spectrum. Another point of interest is that by replicating the argument of [11], Theorem 3.7.14, we are able to uniquely solve the Dirichlet problem for Poisson's equation (with respect to $\Delta_{\mathcal{T}}$) on the continuum random tree, again taking $\{\rho, Z_\emptyset^1\}$ as our boundary.

We now return to our main argument. From the construction of $(\mathcal{E}, \mathcal{F})$, it is possible

to deduce the following self-similar decomposition using the same proof as in Lemma 4.5 of [8].

Lemma 6.1 *\mathbf{P} -a.s., we have, for every $n \geq 1$,*

$$\mathcal{E}(f, g) = \sum_{i \in \Sigma_n} \frac{1}{l(i)} \mathcal{E}_i(f \circ F_i, g \circ F_i), \quad \forall f, g \in \mathcal{F},$$

where $(\mathcal{E}_i)_{i \in \Sigma_n}$ are independent copies of \mathcal{E} , independent of \mathcal{F}_n .

The operators of the above theorem each have a Dirichlet version, \mathcal{E}_i^D , defined in the same way as \mathcal{E}^D was from \mathcal{E} . We shall denote by $N_i^D(\lambda)$ and $N_i^N(\lambda)$ the corresponding Dirichlet and Neumann eigenvalue counting functions. Repeating the argument of [8], Lemma 5.1, it is possible to prove the following bounds involving these functions.

Lemma 6.2 *\mathbf{P} -a.s., we have, for every $\lambda > 0$,*

$$\sum_{i=1}^3 N_i^D(\lambda w(i)^3) \leq N^D(\lambda) \leq N^N(\lambda) \leq \sum_{i=1}^3 N_i^N(\lambda w(i)^3),$$

and also

$$N^D(\lambda) \leq N^N(\lambda) \leq N^D(\lambda) + 2.$$

For the remainder of this section, we will continue to follow [8], and proceed by defining a time-shifted general branching process, X . Although the results we shall prove will be in terms of N^D , the second set of inequalities in the above lemma imply that the asymptotics of N^N , and consequently N , are the same.

Define the functions $(\eta_i)_{i \in \Sigma_*}$ by, for $t \in \mathbb{R}$,

$$\eta_i(t) := N_i^D(e^t) - \sum_{j=1}^3 N_{ij}^D(e^t w(ij)^3),$$

and let $\eta := \eta_\emptyset$. Clearly, the paths of $\eta_i(t)$ are cadlag, and Lemma 6.2 implies that the functions take values in $[0, 6]$, \mathbf{P} -a.s. If we set $X_i(t) := N_i(e^t)$, and $X := X_\emptyset$, then it is possible to check that the following evolution equation holds:

$$X(t) = \eta(t) + \sum_{i=1}^3 X_i(t + 3 \ln w(i)); \quad (26)$$

and also that

$$X(t) = \sum_{i \in \Sigma_*} \eta_i(t + 3 \ln l(i)). \quad (27)$$

The equation at (26) is particularly important, as it will allow us to use branching process and renewal techniques to obtain the results of interest.

We start by investigating the mean behaviour of X , and will now introduce the notation necessary to do this. Set $\gamma = 2/3$, and define, for $t \in \mathbb{R}$,

$$m(t) := e^{-\gamma t} \mathbf{E}X(t), \quad u(t) := e^{-\gamma t} \mathbf{E}\eta(t). \quad (28)$$

Furthermore, let ν be the measure on $[0, \infty)$ defined by $\nu([0, t]) = \sum_{i=1}^3 \mathbf{P}(w(i) \geq e^{-t})$, and ν_γ the measure that satisfies $\nu_\gamma(dt) = e^{-\gamma t} \nu(dt)$. Some properties of these objects are collected in the following lemma.

Lemma 6.3 (a) *The function m is bounded and measurable, and $m(t) \rightarrow 0$ as $t \rightarrow -\infty$.*
(b) *The function u is in $L^1(\mathbb{R})$ and $u(t) \rightarrow 0$ as $|t| \rightarrow \infty$.*
(c) *The measure ν_γ is a Borel probability measure on $[0, \infty)$ and also $\int_0^\infty t \nu_\gamma(dt) < \infty$.*

Proof: A fact that may be deduced from (24), and will be important in proving parts (a) and (b), is that \mathbf{P} -a.s., $\|f\|_2^2 \leq \mathcal{E}(f, f) \text{diam}_R T$, for every $f \in \mathcal{F}^D$. In particular, this implies that the bottom of the Dirichlet spectrum is bounded below by $(\text{diam}_R T)^{-1}$, and consequently we must have $\eta(t) = 0$ for $t < -\ln \text{diam}_R T$, \mathbf{P} -a.s. Hence,

$$\mathbf{E}\eta(t) \leq 6\mathbf{P}(t \geq -\ln \text{diam}_R T). \quad (29)$$

Applying this result, the alternative representation of X at (27), and the independence of N_i^D and $\mathcal{F}_{|i|}$, we obtain

$$\begin{aligned} m(t) &= \sum_{i \in \Sigma_*} e^{-\gamma t} \mathbf{E}\eta_i(t + 3 \ln l(i)) \\ &\leq 6e^{-\gamma t} \mathbf{E}(\#\{i \in \Sigma_* : t + 3 \ln l(i) \geq -\ln \text{diam}_{R'} T\}), \end{aligned}$$

where R' is an independent copy of R . Applying standard branching process techniques to the process with particles $i \in \Sigma_*$, where $i \in \Sigma_*$ has offspring ij at time $-\ln w(ij)$ after its birth, $j = 1, 2, 3$, it is possible to show that $\mathbf{E}(\#\{i \in \Sigma_* : \ln l(i) \geq -t\}) \leq Ce^{2t}$, for every $t \in \mathbb{R}$; the exponent 2 that arises is the Malthusian parameter for the relevant branching process. Thus, for $t \in \mathbb{R}$,

$$m(t) \leq 6C\mathbf{E}((\text{diam}_R T)^\gamma).$$

Since $\text{diam}_R T \stackrel{d}{=} \text{diam}_{d_T} \mathcal{T}$, and, as remarked in the proof of Lemma 5.1, $\text{diam}_{d_T} \mathcal{T}$ has finite positive moments, we are able to deduce that the right hand side of the above inequality is finite. Thus, m is bounded. The measurability of m follows from the fact that X has cadlag paths, \mathbf{P} -a.s. To demonstrate the limit result, we recall the bound at (29), which we apply to (27) to obtain

$$m(t) \leq \sum_{i \in \Sigma_*} 6e^{-\gamma t} \mathbf{P}(l(i)^3 \text{diam}_{R'} T \geq e^{-t}),$$

where R' is again an independent copy of R . Applying Markov's inequality to this expression, we find that, for $\theta > 0$,

$$\begin{aligned} m(t) &\leq \sum_{i \in \Sigma_*} 6e^{(\theta-\gamma)t} (\mathbf{E}(w(i)^{3\theta}))^{|i|} \mathbf{E}((\text{diam}_R T)^\theta) \\ &= 6e^{(\theta-\gamma)t} \mathbf{E}((\text{diam}_R T)^\theta) \sum_{n \geq 0} 3^n (\mathbf{E}(w(i)^{3\theta}))^n \end{aligned}$$

Taking $\theta > \gamma$, we have $\mathbf{E}(w(i)^{3\theta}) < \mathbf{E}(w(i)^2) = \frac{1}{3}$, so the sum over n is finite, as is the expectation involving $\text{diam}_R T$. Consequently, the upper bound converges to zero as $t \rightarrow -\infty$, which completes the proof of (a).

That $u(t)$ is finite for $t \in \mathbb{R}$ follows from the fact that $\eta(t)$ is, and the measurability of u is a result of η having cadlag paths, \mathbf{P} -a.s. Again applying (29), we see that

$$\int_{-\infty}^{\infty} u(t) dt \leq 6 \int_{-\infty}^{\infty} e^{-\gamma t} \mathbf{P}(\text{diam}_R T > e^{-t}) dt. \quad (30)$$

Observe that, for $t \geq 0$,

$$\mathbf{P}(\text{diam}_R T > t) = \mathbf{P}(\text{diam}_{d_T} T > t) \leq \mathbf{P}\left(2 \sup_{s \in [0,1]} W_s > t\right) \leq C e^{-t^2/4}, \quad (31)$$

for some constant C , where the final inequality is obtained by applying the exact distribution of the supremum of a normalised Brownian excursion (see [1], Section 3.1). Hence, the integral at (30) is finite, so $u \in L^1(\mathbb{R})$. Clearly, by the boundedness of η , $u(t) \rightarrow 0$ as $t \rightarrow +\infty$. But also, applying (29) and (31), we have

$$u(t) \leq 6e^{-\gamma t} \mathbf{P}(\text{diam}_R T > e^{-t}) \leq 6C e^{-\gamma t - e^{-2t}/4}.$$

Since the upper bound converges to zero as $t \rightarrow -\infty$, the proof of (b) is complete.

Part (c) is easily deduced using simple properties of the Dirichlet distribution of the triple $(w(1)^2, w(2)^2, w(3)^2)$. \square

The importance of the previous lemma is that it allows us to apply the renewal theorem to deduce the mean behaviour of X , with the precise result being presented in the following proposition. Part (a) of Theorem 1.2 is an easy corollary of this.

Proposition 6.4 *The function m converges as $t \rightarrow \infty$ to the finite and non-zero constant*

$$m(\infty) := \frac{\int_{-\infty}^{\infty} u(t) dt}{\int_0^{\infty} t \nu_{\gamma}(dt)}.$$

Proof: After multiplying by $e^{-\gamma t}$ and taking expectations, the equation at (26) may be rewritten, for $t \in \mathbb{R}$,

$$m(t) = u(t) + \int_0^{\infty} m(t-s) \nu_{\gamma}(ds),$$

which is the double-sided renewal equation of [9]. The results that are proved about m , u and ν_{γ} in Lemma 6.3 mean that the conditions of the renewal theorem stated in [9] are satisfied, and the proposition follows from this. \square

To determine the \mathbf{P} -a.s. behaviour of X , and prove part (b) of Theorem 1.2, the argument of [8], Section 5, may be used. Note that this method is in turn an adaptation of Nerman's results on the almost-sure behaviour of general branching processes, see [16]. Since the steps of our proof are almost identical to those of [8], we shall omit many of the details here. One point that should be highlighted, however, is that in the proof of Lemma 5.7 of [8] there is an error, with several terms being omitted from consideration. We shall explain how to deal with these terms, and also correct the limiting procedure

that should be used at the end of the argument. For the purposes of the proof, we introduce the following notation to represent a cut-set of Σ_* : for $t > 0$,

$$\Lambda_t := \{i \in \Sigma_* : -3 \ln l(i) \geq t > -3 \ln l(i)(|i| - 1)\}.$$

We will also have cause to refer to the subset of Λ_t defined by, for $t, c > 0$,

$$\Lambda_{t,c} := \{i \in \Sigma_* : -3 \ln l(i) \geq t + c, t > -3 \ln l(i)(|i| - 1)\}.$$

Proposition 6.5 ***P**-a.s., we have*

$$e^{-\gamma t} X(t) \rightarrow m(\infty), \quad \text{as } t \rightarrow \infty,$$

where $m(\infty)$ is the constant defined in Proposition 6.4.

Proof: First, we truncate the characteristic functions η_i by defining, for fixed $c > 0$, $\eta_i^c(t) := \eta_i(t) \mathbf{1}_{\{t < n_0 c\}}$, where n_0 is an integer that will be chosen later in the proof. From these characteristics, construct the processes X_i^c , by

$$X_i^c(t) := \sum_{j \in \Sigma_*} \eta_{ij}^c(t + 3 \ln(l(ij)/l(i))),$$

and set $X^c := X_\emptyset^c$. The corresponding discounted mean process is $m^c(t) := e^{-\gamma t} \mathbf{E} X^c(t)$, and this may be checked to converge to $m^c(\infty) \in (0, \infty)$ as $t \rightarrow \infty$ using the argument of Proposition 6.4. From a branching process decomposition of X^c , we can deduce the following bound for $n_1 \geq n_0$, $n \in \mathbb{N}$,

$$|e^{-\gamma c(n+n_1)} X^c(c(n+n_1)) - m^c(\infty)| \leq S_1(n, n_1) + S_2(n, n_1) + S_3(n, n_1),$$

where

$$\begin{aligned} S_1(n, n_1) &:= \left| \sum_{i \in \Lambda_{cn} \setminus \Lambda_{cn, cn_1}} (e^{-\gamma c(n+n_1)} X_i^c(c(n+n_1) + 3 \ln l(i)) - l(i)^2 m^c(c(n+n_1) + 3 \ln l(i))) \right|, \\ S_2(n, n_1) &:= \left| \sum_{i \in \Lambda_{cn} \setminus \Lambda_{cn, cn_1}} l(i)^2 m^c(c(n+n_1) + 3 \ln l(i)) - m^c(\infty) \right|, \\ S_3(n, n_1) &:= e^{-c(n+n_1)} \sum_{i \in \Lambda_{cn, cn_1}} X_i^c(c(n+n_1) + 3 \ln l(i)). \end{aligned}$$

The first two of these terms are dealt with in [8], and using the arguments from that article, we have that, **P**-a.s.,

$$\lim_{n_1 \rightarrow \infty} \limsup_{n \rightarrow \infty} S_j(n, n_1) = 0, \quad \text{for } j = 1, 2.$$

We now show how $S_3(n, n_1)$ decays in a similar fashion. First, introduce a set of characteristics, ϕ_i^{c, n_1} , defined by

$$\phi_i^{c, n_1}(t) := \sum_{j=1}^3 X_{ij}(0) \mathbf{1}_{\{-3 \ln w(ij) > t + cn_1, t > 0\}},$$

and, for $t > 0$, set

$$Y^{c,n_1}(t) := \sum_{i \in \Sigma_*} \phi_i^{c,n_1}(t + 3 \ln l(i)).$$

Note that from the definition of the cut-sets Λ_{cn} and Λ_{cn,cn_1} we can deduce that

$$Y^{c,n_1}(cn) = \sum_{i \in \Lambda_{cn,cn_1}} X_i(0) \geq e^{\gamma c(n+n_1)} S_3(n, n_1),$$

where for the second inequality we apply the monotonicity of the X_i s. Now, Y^{c,n_1} is a branching process with random characteristics ϕ_i^{c,n_1} , and we are able to check the conditions of the extension of [16], Theorem 5.4, that is stated as [8], Theorem 3.2, are satisfied. By applying this result, we find that **P**-a.s.,

$$e^{-\gamma t} Y^{c,n_1}(t) \rightarrow \frac{\int_0^\infty e^{-\gamma t} \mathbf{E} \phi_\emptyset^{c,n_1}(t) dt}{\int_0^\infty t \nu_\gamma(dt)}, \quad \text{as } t \rightarrow \infty.$$

It is obvious that $\mathbf{E} \phi_\emptyset^{c,n_1}(t) \leq 3 \mathbf{E} X(0) \leq 3m(0) < \infty$, where m is the function defined at (28). Consequently, there exists a constant C that is an upper bound for the above limit uniformly in n_1 , and so **P**-a.s.,

$$\lim_{n_1 \rightarrow \infty} \limsup_{n \rightarrow \infty} S_3(n, n_1) \leq \lim_{n_1 \rightarrow \infty} \limsup_{n \rightarrow \infty} e^{-\gamma c(n+n_1)} Y^{c,n_1}(cn) \leq \lim_{n_1 \rightarrow \infty} C e^{-\gamma c n_1} = 0.$$

Combining the three limit results for S_1 , S_2 and S_3 , it is easy to deduce that **P**-a.s.,

$$\lim_{n \rightarrow \infty} |e^{-\gamma cn} X^c(cn) - m^c(\infty)| = 0.$$

We continue by showing how the process X , when suitably scaled, converges along the subsequence $(cn)_{n \geq 0}$. Applying the conclusion of the previous paragraph, we find that **P**-a.s.,

$$\limsup_{n \rightarrow \infty} |e^{-\gamma cn} X(cn) - m(\infty)| \leq |m(\infty) - m^c(\infty)| + \limsup_{n \rightarrow \infty} e^{-\gamma cn} |X(cn) - X^c(cn)|. \quad (32)$$

Recall that the process X^c and its discounted mean process m^c depend on the integer n_0 . By applying the dominated convergence theorem, it is straightforward to check that, as we let $n_0 \rightarrow \infty$, the first of the terms in the above estimate, which is deterministic, converges to zero. For the second term, observe that

$$|X(t) - X^c(t)| = \sum_{i \in \Sigma_*} \eta_i(t + 3 \ln l(i)) \mathbf{1}_{\{t + 3 \ln l(i) > cn_0\}} \leq 6 \# \{i \in \Sigma_* : t + 3 \ln l(i) > cn_0\}.$$

Applying standard branching process results to the process described in Lemma 6.3, we are able to deduce the existence of a finite constant C such that, as $t \rightarrow \infty$, we have $e^{-2t} \# \{i \in \Sigma_* : -\ln l(i) < t\} \rightarrow C$, **P**-a.s., from which it follows that **P**-a.s.,

$$\limsup_{n \rightarrow \infty} e^{-\gamma cn} |X(cn) - X^c(cn)| \leq 6C e^{-\gamma cn_0}.$$

Consequently, by choosing n_0 suitably large, the upper bound in (32) can be made arbitrarily small, which has as a result that $e^{-\gamma cn} X(cn) \rightarrow m(\infty)$ as $n \rightarrow \infty$, **P**-a.s., for each c . The proposition is readily deduced from this using the monotonicity of X . \square

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