Pricing and Modelling in Electricity Markets

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Electricity prices

Over the past 20 years a number of countries have deregulated their electricity markets allowing energy suppliers and users to buy and sell power in a financial market.

- power generators indicate what price they can supply different quantities of power. This is combined into a bid stack.

- users put in requests for how much power they need.

- Prices are formed through supply and demand as determined via the bid stack.
Some financial time series
Modelling

A distinctive feature of electricity markets is the formation of price spikes. These are typically due to sudden movements of demand relative to supply and may be due to

- generator failures,
- network problems,
- unexpected weather events.

The occurrence of spikes has far reaching consequences for risk management and motivates the introduction of swing options to help control this risk.
Swing Options

Swing contracts are a broad class of path dependent options allowing the holder to exercise a certain right multiple times over a specified period. They offer protection against price spikes. Examples are that on each day the holder may receive the payoff of

- a call option.

- a mixture of different payoff functions like calls and puts or calls with different strikes.

- a multiple of a call or put option at once, where the multiple is called volume. Restrictions on the volume are upper and lower bounds for each right and for the sum of all trades.
Aim

The aim of the talk is to introduce some models for the price of electricity and to examine the numerical techniques for the pricing of swing options. This is still work in progress and so will be a talk of two halves...

• We give models for prices arising from different approaches.
• We consider pricing multiple exercise contracts in simple cases.
Modelling the market

There are three natural approaches to modelling electricity markets that have been explored.

• Model the spot electricity price:
  Get a model which has the features of the observed phenomena.

• Model the forward prices:
  A mathematical finance approach - model the tradables.

• Model the supply and demand to obtain spot prices:
  Start from the economic fundamentals and derive the observed phenomena.
Modelling financial asset prices

An asset price $S_t$ is typically modelled by a stochastic differential equation driven by Brownian motion.

Bachelier (1900): Brownian motion with drift;

$$dS_t = \mu \, dt + \sigma \, dW_t.$$  

Samuelson (1960s): geometric Brownian motion;

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$$

as used by Black-Scholes.

Interest rates: Cox-Ingersoll-Ross model

$$dR_t = \alpha (\theta - R_t) \, dt + \sigma \sqrt{R_t} \, dW_t.$$
Modelling commodities

Commodities are also *usually* regarded as mean reverting as prices revert back to the cost of production. A basic mean reverting model is

\[ S_t = \exp(f(t) + X_t), \]
\[ dX_t = -\alpha X_t \, dt + \sigma \, dW_t, \]

where \( W \) is standard Brownian motion and \( f \) describes the seasonality. \( S \) is log-normally distributed which allows for analytic prices for simple options.

This can be extended by stochastic seasonality.

This model does not have spikes.
A natural extension to exhibit jumps is (Cartea and Figueroa)

\[ S_t = \exp(f(t) + X_t), \]
\[ dX_t = -\alpha X_t \, dt + \sigma \, dW_t + J_t \, dN_t, \]

where \((N_t)\) is a Poisson-process with intensity \(\lambda\) and \((J_t)\) is an independent identically distributed process representing the jump size.

It needs very high mean reversion to exhibit spikes.
It is OK for the UK market.
A spot price model with spikes

We model spikes with a mean reverting jump process (H, Howison and Kluge);

\[ S_t = \exp(f(t) + X_t + Y_t), \]
\[ dX_t = -\alpha X_t \, dt + \sigma \, dW_t, \]
\[ dY_t = -\beta Y_t^- \, dt + J_t \, dN_t. \]

We assume \((W_t), (N_t)\) and \((J_t)\) are mutually independent.
Solving for $X$ and $Y$, we have

$$X_t = X_0 e^{-\alpha t} + \sigma \int_0^t e^{-\alpha (t-s)} \, dW_s,$$

$$Y_t = Y_0 e^{-\beta t} + \sum_{i=1}^{N_t} e^{-\beta (t-\tau_i)} J_{\tau_i},$$

where $\tau_i$ is the $i$-th jump time. Thus, given $X_0 = x_0$, $X_t$ is normally distributed. Properties of the spike process $(Y_t)$ are not as obvious. However approximations allow us to calibrate the model to the forward curve.
Simulated sample paths of $X$, $Y$ and $S$ with $f(t) = \ln(100) + 0.5 \cos(2\pi t)$ arbitrary. Other parameters are as in Nordpool $\alpha = 7$, $\sigma = 1.4$, $\beta = 200$, $J_t \sim \exp(1/\mu_J)$, $\mu_J = 0.4$, $\lambda = 4$. 
Price formation

We develop a cartoon model of price formation.

- The power generators submit $N$ bids in which each bid is a pair $(a_i, p_i(t))$, where $a_i$ is a fixed quantity to be produced and $p_i(t)$ is the price required to produce $a_i$ at time $t$.

- We assume that the cost of the production of power $p_i(t)$, is a mean reverting process driven by the cost of the underlying fuel used to generate the power

$$dp_i = \kappa (\theta - p_i) dt + \sigma \rho dW^f + \sigma \sqrt{1 - \rho^2} dW_i,$$

where $\theta$ is the fixed cost associated with production, $\kappa$ is the speed of mean reversion in prices, $\sigma$ is the volatility in prices, $W^f$ is the Brownian motion for the fuel price and $W_i$ is a Brownian motion for other aspects of the production process. The fuel price SDE is

$$dF = \kappa_f (\theta_f - F) dt + \sigma_F dW^f.$$
The bid-stack

In order to find the price we match supply and demand. The first part is to construct the supply curve from the bids - this is the bid-stack.

- We look at the empirical measure of the bids. Let the total capacity of the generators be \( A = \sum_{i=1}^{N} a_i \).

- The empirical probability measure is

\[
\nu_{N,t} = \frac{1}{A} \sum_{i=1}^{N} a_i \delta_{p_i(t)} = \frac{1}{N} \sum_{i=1}^{N} \tilde{a}_i \delta_{p_i(t)}.
\]

Our formulation ensures the individual bids are exchangeable,

- Let the number of bids \( N \to \infty \) (and assume that the total capacity remains bounded in that each bid is really regarded as an infinitesimal bid) we have by de Finetti the existence of a limit empirical measure

\[
\nu_t = \lim_{N \to \infty} \nu_{N,t} \in C([0, \infty), \mathcal{P}_{\mathbb{R}}).
\]
Empirical measure evolution

Take a test function \( \phi \in C_\infty \). We write

\[
< \phi, \nu_t > = \int_{-\infty}^{\infty} \phi(x) \nu_t(dx),
\]

and

\[
< \phi, \nu_{N,t} > = \frac{1}{N} \sum_{i=1}^{N} \tilde{a}_i \phi(p_i(t)).
\]

Then, using Ito’s formula, we have

\[
d < \phi, \nu_{N,t} > = \frac{1}{N} \sum_{i=1}^{N} \tilde{a}_i (\phi'(p_i)dp_i + \frac{1}{2} \phi''(p_i)dt < p_i >) \\
= < \kappa(\theta - \cdot)\phi', \nu_{N,t} > dt + < \sigma \rho \phi', \nu_{N,t} > dM \\
+ \frac{1}{N} \sum_{i=1}^{N} \sigma \sqrt{1 - \rho^2} \phi'(p_i) dW_t^i.
\]
The idiosyncratic term

\[ R_N(t) = \frac{1}{N} \sum_{i=1}^{N} \sigma \sqrt{1 - \rho^2} \phi'(p_i) dW_t^i, \]

has quadratic variation 0 as \( N \to \infty \) and thus we have an equation for the evolution of the empirical measure of prices in the limit as \( N \to \infty \),

\[ d < \phi, \nu_t > = < \kappa(\theta - \cdot) \phi', \nu_t > dt + < \sigma \rho \phi', \nu_N > dM. \]

If the measure has a density with respect to Lebesgue measure in that \( \nu_t(dx) = V_t(x)dx \), then integrating by parts, Fubini,...

\[ dV_t(x) = \left( -\frac{\partial}{\partial x}(\kappa(\theta - x)V_t) + \frac{1}{2} \sigma^2 \frac{\partial^2 V_t}{\partial x^2} \right) dt - \sigma \rho \frac{\partial V_s}{\partial x} dM_t. \]
This is a Zakai equation for a filtering problem and the existence and uniqueness of solutions for such an equation are standard. This equation is sufficiently simple that it can be solved by setting

\[ V_t(x) = u(t, x - \sigma \rho \bar{M}_t), \]

where

\[ d\bar{M} = -\kappa \bar{M} dt + dW, \]

and

\[ \frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(\kappa(\theta - x)u) + \frac{1}{2} \sigma^2(1 - \rho^2) \frac{\partial^2 u}{\partial x^2}. \]

with \( u(0, x) = V_0(x). \)
This PDE is the forward equation for the Ornstein-Uhlenbeck process. The solution is given by

\[ u(t, x) = \int_{-\infty}^{\infty} V_0(y) k(t, y, x) dy, \]

where, setting \( \tilde{\sigma} = \sigma \sqrt{1 - \rho^2} \),

\[ k(t, y, x) = \frac{1}{\sqrt{2\pi \frac{\tilde{\sigma}^2}{2\kappa} (1 - e^{-2\kappa t})}} \exp \left( -\frac{(x - \theta - (y - \theta) e^{-\kappa t})^2}{\frac{\tilde{\sigma}^2}{\kappa} (1 - e^{-2\kappa t})} \right). \]

As the OU process is positive recurrent, there is a stationary distribution which is \( N(\theta, \tilde{\sigma}^2 / 2\kappa) \). If we assume that the bids are in the stationary state at time 0 we have

\[ V_t(x) = \frac{1}{\sqrt{2\pi \frac{\tilde{\sigma}^2}{2\kappa}}} \exp \left( -\frac{(x - \theta - \sigma \rho \bar{M}_t)^2}{\frac{\tilde{\sigma}^2}{\kappa}} \right). \]
The bid-stack

The electricity price itself is formed from the bid-stack. This is the cost of producing a given amount of power and is obtained by integrating our model for the underlying bids. We set

\[ S_t(x) = \int_{-\infty}^{x} \nu_t(dy) = \int_{-\infty}^{x} V_t(y) dy. \]

Hence for our current model, starting from stationarity we get

\[ S_t(x) = A\Phi\left(\frac{x - \theta - \sigma \rho \bar{M}_t}{\bar{\sigma} / \sqrt{2\kappa}}\right), \]

where \( \Phi \) is the cumulative distribution function for the standard normal distribution and we recall that \( A \) is the total power generating capacity in the market.
The price

Let $\tilde{D}$ be the demand process for electricity. We model the demand as a proportion of the total power generating capacity, $D = \tilde{D}/A$. This is a diffusion on $[0, 1]$ and thus it is natural to choose

$$dD_t = \eta(g(t) - D_t)dt + \xi \sqrt{D(1-D)}dW^D,$$

with $g$ a time varying seasonality function. In order to determine the price we must match demand

$$\tilde{D}_t = S_t(\pi_t).$$

In the set up we have so far this becomes

$$D_t = \Phi\left(\frac{\pi_t - \theta - \sigma \rho M_t}{\sigma \sqrt{1 - \rho^2}/\sqrt{2\kappa}}\right).$$
The price

Theorem
The electricity price $\pi_t$ at stationarity satisfies

$$\pi_t = \theta + \sigma \rho \tilde{M}_t + \frac{\sigma \sqrt{1 - \rho^2}}{\sqrt{2\kappa}} \Phi^{-1}(D_t).$$

- There are two mean reverting components. Firstly $\tilde{M}$, which is driven by the underlying fuel price, and the second which is the inverse normal CDF of the demand. We note that spikes may occur if the proportionate demand process can approach very close to 1. If it can hit 1, we will have a black out as the price will go to $+\infty$!

- We can easily take a log transform $\pi \rightarrow \log \pi$ to remove negative prices (though negative prices can occur).

- If $\rho = 1$ we will have propagation of the initial density along characteristics.
Extensions

We can extend this to multiple fuels types, where each type has its own costs of production and fuel price, but there is difficulty in inverting the bid-stack as

\[ D_t = \sum_{j=1}^{\tau} \alpha_j \Phi\left( \frac{\pi - \theta_j - \sigma_j \rho_j \bar{M}_j}{\sigma_j \sqrt{1 - \rho_j^2 / \sqrt{2\kappa_j}}} \right). \]

Other extensions are also possible, for example to varying levels of production cost.

The issues of concern here are

1. calibration - we need a model which produces forward price curves which can be calibrated.

2. We need sufficient complexity to capture multiple fuels type and the effect of carbon emissions targets.
Pricing swing options

Assumptions

• We have an economy in discrete time with a finite time horizon $T$.

• We assume a financial market described by the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0,1,...,T}, \mathbb{P})$ with $(X_t)_{t=0,1,...,T} \in \mathbb{R}^d$ a discrete $\mathcal{F}_t$-adapted Markov chain describing the state of the economy - the price of the underlying assets and any other variables that affect the dynamics of the underlyings.

• $\mathbb{P}$ is a pricing measure and write $\mathbb{E}_t(X) = \mathbb{E}(X | \mathcal{F}_t)$.

• We will always assume the interest rate is 0.
Multiple exercise options

We will consider multiple exercise options in which the holder has:

- a single exercise opportunity at each time $t$.
- up to a fixed discrete number of exercise opportunities at each $t$.
- a continuous volume that can be exercised at each $t$.

We begin with the single exercise case and write $Z_t = h_t(X_t) \geq 0$ for the payoff from the exercise of the option at time $t$ when the asset price is $X_t$. 
Valuation

A policy $\pi$ is a set of $n$ stopping times $0 \leq \tau_n < \tau_{n-1} < \cdots < \tau_1 \leq T$.

The value function $V_{t}^{*,n}$, with $n$ remaining exercise opportunities is

$$V_{t}^{*,n}(x) = \sup_{\pi} \mathbb{E}_t \left[ \sum_{m=1}^{n} Z_{\tau_m} | X_t = x \right].$$

We denote the corresponding optimal policy by $\pi^* = \{\tau_n^*, \ldots, \tau_1^*\}$.

The marginal value $\Delta V_{t}^{\pi,n}$ is for policy $\pi$ and $n \geq 1$:

$$\Delta V_{t}^{\pi,n}(x) = V_{t}^{\pi,n}(x) - V_{t}^{\pi,n-1}(x).$$

Under an optimal policy $\pi^*$ the marginal value is denoted $\Delta V_{t}^{*,n}$.

The marginal value is the additional payoff that can be expected from having one more exercise right.
Dynamic programming

It is simple to write down the dynamic programming equations for the value function

$$V_{t,n}^*(x) = \max\{h_t(X_t) + E(V_{t+1,n}^*(X_{t+1})|\mathcal{F}_t), E(V_{t+1,n}^*(X_{t+1})|\mathcal{F}_t)\}.$$ 

The *continuation value* $Q_{t,n}^*$ is the expectation of the value function one timestep later, $Q_T^* = 0$ and for $t < T$

$$Q_{t,n}^*(x) = \mathbb{E}[V_{t+1,n}^*|X_t = x]$$

Thus all we need to do is solve

$$V_{t,n}^*(x) = \max\{h_t(x) + Q_{t,n}^*(x+1), Q_{t,n}^*(x)\}.$$
Numerical results

We use the phenomenolgical model with spikes and assume that the mean-reversion process \( (X_t) \) and the spike process \( (Y_t) \) are individually observable. The value function \( V^n_t(x, y) \) of our swing option can be expressed using dynamic programming as

\[
\max \left\{ e^{-r\Delta t} \mathbb{E}^Q \left[ V^n_{t+\Delta t}(X_{t+\Delta t}, Y_{t+\Delta t}) \right| X_t = x, Y_t = y \right], \right.
\]

\[
\left. e^{-r\Delta t} \mathbb{E}^Q \left[ V^{n-1}_{t+\Delta t}(X_{t+\Delta t}, Y_{t+\Delta t}) \right| X_t = x, Y_t = y \right] + (e^f(t) + x + y - K)^+. \]

We need transition probabilities to obtain the conditional expectations. We need to compute the conditional expectations for the grid. This can be done via approximations for the spike process.
We price a swing contract over 365 days with 100 exercise opportunities,

Swing option value with and without spikes.

This is time consuming to calculate. What happens if we add more stochastic factors?
Monte Carlo for swing options

Techniques for pricing high dimensional American options via Monte Carlo have been developed over the past 10 years. We extend these ideas to swing contracts. We view these as multiple optimal stopping problems (Meinshausen-Hambly (2004)).

A policy $\pi$ is a set of stopping times $\{\tau_n, \ldots, \tau_1\}$ with $\tau_n < \ldots < \tau_1$. $\tau_m$ determines the time where the $m$-th remaining exercise opportunity is used under policy $\pi$. The expected payoff under policy $\pi$ is,

$$V_{t, n}^{\pi}(x) = \mathbb{E}_t \left[ \sum_{m=1}^{n} Z_{\tau_m} | X_t = x \right].$$
The standard approach

In order to tackle this problem by Monte Carlo it is natural to use the least squares basis function regression approach to determine the exercise boundary and hence a policy $\pi$ for exercise.

Thus, using this policy $\pi$, we will have a lower bound for the price of the multiple exercise option as

$$V_{t}^{\pi, n}(x) \leq V_{t}^{*, n}(x) = \sup_{\pi} V_{t}^{\pi, n}(x).$$

We give examples with the phenomenological spike model but How do we find an upper bound to ensure our result is accurate?
Total swing option value as jump sizes vary with parameters
\[ f(t) = \ln(100) + 0.1 \cos(3\pi t), \quad \alpha = 7, \quad \sigma = 0.4, \quad \beta = 500, \quad J_t \sim \exp(1/\mu_J), \]
\[ \lambda = 250. \]
Total swing option value as jump frequencies vary with parameters

\[ f(t) = \ln(100) + 0.1 \cos(3\pi t), \alpha = 7, \sigma = 0.4, \beta = 500, J_t \sim \exp\left(\frac{1}{\mu_J}\right), \mu_J = 5. \]
Duality

From the form of the price we see that choosing any policy gives a lower bound on the price. An upper bound can be found by duality. For the case of American options (1 exercise) we have the following argument due to Rogers

\[
V_0^* = \sup_{0 \leq \tau \leq T} \mathbb{E} Z_\tau \\
= \sup_{0 \leq \tau \leq T} \mathbb{E} (Z_\tau - M_\tau) \\
\leq \mathbb{E} \left( \sup_{0 \leq t \leq T} (Z_t - M_t) \right)
\]

Thus

\[
V_0^* \leq \inf_M \mathbb{E} \left( \sup_{0 \leq t \leq T} (Z_t - M_t) \right).
\]
By the Doob decomposition of the supermartingale $V_t^* = V_0^* + M_t^* - A_t^*$ where $M$ is a martingale and $A$ is an increasing process. Then

$$\inf_M \mathbb{E}( \sup_{0 \leq t \leq T} (Z_t - M_t)) \leq \mathbb{E}( \sup_{0 \leq t \leq T} (Z_t - M_t^*))$$

$$\leq \mathbb{E}( \sup_{0 \leq t \leq T} (V_t^* - M_t^*))$$

$$= \mathbb{E}( \sup_{0 \leq t \leq T} (V_0^* - A_t^*))$$

$$= V_0^*$$

Hence the dual problem for $n = 1$ is

$$V_0^* = \inf_M \mathbb{E}( \sup_{0 \leq t \leq T} (Z_t - M_t)).$$
Theorem
The marginal value $\Delta V_{0}^{*,n}$ is equal to:

$$\Delta V_{0}^{*,n} = \inf_{\pi} \inf_{M \in H_0} \mathbb{E}_0 \left[ \max_{u \in (T \setminus \{\tau_{n-1}, ..., \tau_1\})} (Z_u - M_u) \right],$$

where $T = \{0, \ldots, T\}$ is the set of possible exercise dates, $0 \leq \tau_{n-1} < \ldots < \tau_1$ are stopping times, $\{M_t\}$ is a martingale and $H_0$ is the set of all martingales that are zero at time $t = 0$. The infimum is attained for the policy $\pi^*$ and martingale $M^*$ whose increment at time $t$ is the martingale part of the marginal value function:

$$M^*_t - M^*_{t-1} = \Delta V_{t}^{*,m} - \mathbb{E}_{t-1}[\Delta V_{t}^{*,m}],$$

where $m := \max\{n : t \leq \tau_n\}$. 
Optimal Policy Approximation

Assuming that the true value function is known, the optimal policy is given by exercising the option if and only if the payoff from immediate exercise is larger than the expected marginal value under continuation.

An approximation to the optimal policy can be obtained by replacing the marginal value $\Delta V^{*,n}_t$ by an approximation $\Delta V^n_t$.

The policy $\pi = \{\tau_n, \ldots, \tau_1\}$ is defined by the stopping time

$$\tau_n = \min\{t : Z_t > \mathbb{E}_t[\Delta V^n_{t+1}]\},$$

and, for $m = 1, \ldots, n - 1$, by

$$\tau_m = \min\{t > \tau_{m+1} : Z_t > \mathbb{E}_t[\Delta V^m_{t+1}]\},$$

In the case that the approximation to the value function is the true value function, we clearly have that $\pi = \pi^*$ and the optimal policy is obtained.
Martingale Approximation.

We approximate the optimal martingale \( M^* \), with \( M_0^* = 0 \) and increments

\[
M_t^* - M_{t-1}^* = \Delta V_{t,m} - \mathbb{E}_{t-1}[\Delta V_{t,m}^*],
\]

(where \( m \) is the largest number such that \( t \leq \tau_m \)), by \( M \) with increments

\[
M_t - M_{t-1} = \Delta V_{t,m}^* - \mathbb{E}_{t-1}^{m},
\]

where \( m \) is the largest natural number such that \( t \leq \tau_m \) and \( \mathbb{E}_{t-1}^{m} \) is a Monte Carlo approximation to \( \mathbb{E}_{t-1}[\Delta V_{t,m}] \). If \( X_i^i, i = 1, \ldots, k \) are independent samples from the distribution of \( X_t \) conditional on \( X_{t-1} \), then

\[
\mathbb{E}_{t-1}^{m} = \frac{1}{k} \sum_{i=1}^{k} \Delta V_{t,m}^*(X_t^i).
\]
A simple swing example

The lifetime $T = 1000$ days with up to $n = 100$ exercise opportunities.

$$\log S_{t+1} = (1 - k) \log S_t + \sigma(\Delta W_t),$$

The constants in the model are set to $\sigma = 0.5$, $k = 0.9$, $K = 0$ and $S_0 = 1$. 
Swing options with volume

The holder now has $k_t$ exercises, at time $t$, where $k_t$ will be an $\mathbb{N}$-valued random variable. Let $Z^i_t = h^i_t(X_t)$ be the payoff from the $i$–th exercise of the option $i = 1, 2, \ldots, k_t$ at time $t$ when the asset price is $X_t$.

We assume that the payoffs are non-negative $h^i_t(x) \geq 0$ and non-increasing $h^i_t(x) \geq h^{i+1}_t(x)$. The total payoff at time $t$ is $H^k_t = \sum_{i=1}^k Z^i_t$.

For a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_m \leq \tau_{m-1} \leq \cdots \leq \tau_1$ taking values in $\{0, 1, \ldots, T\}$ we write $N^m_t(\tau_m, \ldots, \tau_1) = \#\{j : \tau_j = t, j = 1, \ldots, m\}$ for the number of stopping times in the set $\{\tau_1, \ldots, \tau_m\}$ taking the value $t$.

We define an exercise policy $\pi_k$ to be a set of stopping times $\{\tau_i\}_{i=1}^m$ with $\tau_m \leq \tau_{m-1} \leq \cdots \leq \tau_1$ such that $N^m_t \leq k_t$. The value of the policy $\pi_k$ at time $t$ is given by

$$V_{t}^{\pi_k,m} = \mathbb{E}_t(\sum_{s=t}^{T} H_{s}^{N^m_s}(X_s)).$$
The value function is defined to be

\[ V_{t,m}^* = \sup_{\pi_k} V_{t}^{\pi_k, m} = \sup_{\pi_k} \mathbb{E}_t \left( \sum_{s=t}^{T} H_{s}^{N_{s}^{m}} (X_s) \right). \]

We denote the corresponding optimal policy \( \pi^* = \{\tau_{m}^*, \tau_{m-1}^*, \ldots, \tau_1^*\} \).

The continuation value \( Q_{t,m,k}^* \) at time \( t \) is given by

\[ Q_{t,m}^* = \mathbb{E}_t [V_{t+1,m}^*] \]

The marginal value of one additional exercise opportunity is denoted by \( \Delta V_{t,m}^* \) for \( m \geq 1 \):

\[ \Delta V_{t,m}^* = V_{t,m}^* - V_{t,m-1}^* \]
Dynamic Programming

The price $V^*_t, m$ at time $t$ of an option with payoff function
$
\{Z^i_s, t \leq s \leq T, 1 \leq i \leq k_s \}$ which could be exercised up to $k_s$ times per
single exercise time $s \in \{t, \ldots, T\}$ with $m$ exercise opportunities in total is
given by

\[
V^*_T, m = H_T^{\min\{k_T, m\}},
\]

\[
V^*_t, m = \max\{H_t^{\min\{k_t, m\}} + \mathbb{E}_t[V^*_t, m - \min\{k_t, m\}],
\]

\[
H_t^{\min\{k_t, m\} - 1} + \mathbb{E}_t[V^*_t, m - (\min\{k_t, m\} - 1)],
\]

\[
\ldots, H_t^1 + \mathbb{E}_t[V^*_t, m - 1], \mathbb{E}_t[V^*_t, m]\}.
\]
A simple case

If the option is allowed to be exercised equal number of times i.e. 
\( k_1 = k_2 = \ldots = k_T = k \), then the option is equivalent to \( k \) options which can be exercised once at a time.

\[
V_{t,*}^{*,km,k} = kV_{t,*}^{*,m,1}.
\]
Lower bound on price

To compute lower bounds we use the obvious generalization of the basis function regression to approximate the marginal continuation values.

With the ordering of the marginal continuation values

\[ E_t[\Delta V_{t+1}^*,m] \leq E_t[\Delta V_{t+1}^{*,m-1}], \quad \forall m, \forall t \]

we can determine the optimal exercise strategy at time level \( t \) as

if \( i = \max(j : Z_t^j \geq E_t[\Delta V_{t+1}^{*,m-j+1}]1 \leq j \leq \min\{m, k_t\}) \) - we exercise \( i \) times.

if \( E_t[\Delta V_{t+1}^{*,m}] > Z_t^1 \) - do not exercise.

In this way approximation to the optimal exercise strategy is found and consequently a lower bound for the option price.
The dual problem

The previous dual problem turns out to extend in a natural way to this setting as first shown by Bender (2009). We give the formulation from Aleksandrov-Hambly (2010).

Theorem

The marginal value $\Delta V^{*,m}_0$ is equal to

$$\Delta V^{*,m}_0 = \inf_{\pi} \inf_{\mathcal{M} \in \mathcal{M}_0} \mathbb{E}_0 \left[ \max_{u=0,1,\ldots,T} \left( Z_u^{N_u^{m-1}+1} - \mathcal{M}_u \right) \right],$$

The first $\inf$ is taken over all exercise policies with $m-1$ rights, which are denoted by $\pi = (\tau_{m-1}, \ldots, \tau_1)$. The second $\inf$ is taken over $\mathcal{M}_0$, the set of integrable martingales which are null at 0. The $\max$ is taken over the exercise times that have not been already used in $\pi$. 

The simple example revisited

Figure 1: The red, yellow, blue and green curves in this order are the values of options which can be exercised once, twice, twice on weekends and four times on weekdays, and four times a day as a function of the total number of exercise possibilities.
Continuous exercises

The holder of a volume option has the opportunity to exercise up to an amount \( k_t \), at time \( t \), where \( k_t \) will be an \( \mathbb{R}_+ \)-valued random variable measurable with respect to \( \mathcal{F}_t \). The total payoff at time \( t \) we denote by \( H_t^h \).

Let \( V_t^{\pi^*,y} \) be the value function at time \( t \) of an option for which up to an amount \( k = \{ k_0, k_1, k_2, \ldots, k_T \} \) can be exercised at the corresponding time points and has a volume \( y \) left to be exercised.

An exercise policy \( \pi_k \) is a set of exercise volumes \( \{ h_i \}_{i=0}^T \) with \( 0 \leq h_i \leq k_i \) and such that \( \sum_{i=0}^T h_i \leq y \). Then the value of the policy \( \pi_k \) at time \( t \) is given by

\[
V_t^{\pi_k, y} = \mathbb{E}_t \left( \sum_{s=t}^{T} H_s^{h_s}(X_s) \right).
\]

The value function is defined to be

\[
V_t^{\pi^*,y}(x) = \sup_{\pi_k} V_t^{\pi_k, y} = \sup_{\pi_k} \mathbb{E}_t \left( \sum_{s=t}^{T} H_s^{h_s}(X_s) | X_0 = x \right).
\]
Dynamic programming

The dynamic programming formulation of the problem can be written as follows.

The price $V_{t}^{*,y}$ at time $t$ of an option with payoff function

$$\{ Z_{s}^{h}, t \leq s \leq T, 0 \leq h \leq k_{s} \}$$

which could be exercised up to an amount $k_{s}$ per single exercise time $s \in \{ t, \ldots, T \}$ with a total volume $y$ to exercise is given by

$$V_{T}^{*,y} = H_{T}^{\min\{ k_{T}, y \}},$$

$$V_{t}^{*,y} = \sup\{ H_{t}^{h} + \mathbb{E}_{t}[V_{t+1}^{*,y-h}] | 0 \leq h \leq \min \{ k_{t}, y \} \}. $$
The dual

We now develop a dual for this problem as follows. Let \( \mathcal{M}_0 \) denote the space of martingales which are null at time 0. For a policy \( \pi \) we will write
\[
y_0 = y, \quad y_1 = y - h_0 \quad \text{and successively} \quad y_i = y_{i-1} - h_{i-1} = y - \sum_{j=0}^{i-1} h_j \quad \text{to be the amount left to be exercised at the } i\text{-th time point.}
\]

**Theorem**

We can write
\[
V^*_{0,y}(x) = \inf_{M^w : 0 \leq w \leq y} \mathbb{E}_0 \left( \sup_{\pi} \sum_{i=0}^{T-1} (H^i_{h_i} - M^{y_i+1}_{i+1} + M^{y_i+1}_i) + H^T_{T} | X_0 = x \right).
\]

The optimal martingales are obtained from the Doob decomposition of the value function
\[
M^{y}_{i+1} - M^{y}_i = V^{y}_{i+1} - \mathbb{E}_i V^{y}_{i+1}.
\]
Comparison of upper and lower bounds with parameters

\[ f(t) = \ln(100) + 0.1 \cos(3\pi t), \alpha = 7, \sigma = 0.4, \beta = 500, J_t \sim \exp(1/\mu_J), \]
\[ \mu_J = 5, \lambda = 250. \]
Final Remarks

The Monte Carlo approach to multiple exercise options is fairly easy to use in larger problems using the basis function regression approach. The dual method works in simple examples so far. There is a need to investigate its performance in more complex models such as those arising from the bid-stack model with multiple fuel types.

In the case of continuous exercises Greg Gyurko, Jan Witte and I are working on numerical techniques for solving this dual efficiently and obtaining good upper bounds.
Proof of Duality theorem

Following the same idea as before

\[
V_{0^*,y}(x) = \sup_{\pi_k} \mathbb{E}_0 \left( \sum_{s=0}^{T} H_{s}^{h_s}(X_s) | X_0 = x \right)
\]

\[
= \sup_{\pi_k} \mathbb{E}_0 \left( \sum_{s=0}^{T-1} (H_{s}^{h_s}(X_s) - M_{s+1}^{y_{s+1}} + M_{s}^{y_{s+1}}) + H_{T}^{h_T} | X_0 = x \right)
\]

\[
\leq \mathbb{E}_0 \left( \sup_{\pi_k} \sum_{s=0}^{T-1} (H_{s}^{h_s}(X_s) - M_{s+1}^{y_{s+1}} + M_{s}^{y_{s+1}}) + H_{T}^{h_T} | X_0 = x \right)
\]

As this holds for all martingales \( M^w \) we have

\[
V_{0^*,y}(x) \leq \inf_{M^w:0 \leq w \leq y | M^w \in M_0} \mathbb{E}_0 \left( \sup_{\pi_k} \sum_{i=0}^{T-1} (H_{i}^{h_i} - M_{i+1}^{y_{i+1}} + M_{i}^{y_{i+1}}) + H_{T}^{h_T} | X_0 = x \right).
\]
We take \( \{M_i^{*,y_i}, i = 0, 1, \ldots, T - 1\} \) from the Doob decomposition of the value function. Increments are

\[
\Delta M_i^{*,y_i+1} = M_{i+1}^{*,y_i+1} - M_i^{*,y_i+1} = V_{i+1}^{*,y_i+1} - \mathbb{E}_i(V_{i+1}^{*,y_i+1}(X_{i+1})).
\]

Using this martingale we have

\[
\inf_{M^w:0 \leq w \leq y|M^w \in M_0} \mathbb{E}_0(\sup_\pi \sum_{i=0}^{T-1} (H_i^{h_i} - M_i^{y_i+1} + M_i^{y_i+1}) + H_T^{h_T} | X_0 = x) \\
\leq \mathbb{E}_0(\sup_\pi \sum_{i=0}^{T-1} (H_i^{h_i} - \Delta M_i^{*,y_i+1}) + H_T^{h_T} | X_0 = x) \\
= \mathbb{E}_0(\sup_\pi \sum_{i=0}^{T-1} (H_i^{h_i} - V_{i+1}^{*,y_i+1} + \mathbb{E}_i(V_{i+1}^{*,y_i+1}(X_{i+1})) + H_T^{h_T} | X_0 = x).
\]
From the dynamic programming equations we have

\[ V_i^{*}, y_i \geq H_{i}^{h} + E_i V_{i+1}^{*}, y_{i+1}. \]

Hence

\[
\inf_{M^w: 0 \leq w \leq y} \mathbb{E}_0 \left( \sup_{\pi} \sum_{i=0}^{T-1} (H_{i}^{h} - M_{i+1}^{y} + M_{i}^{y} + 1) + H_{T}^{h} | X_0 = x \right)
\leq \mathbb{E}_0 \left( \sup_{\pi} \sum_{i=0}^{T-1} (V_{i}^{*}, y_i - V_{i+1}^{*}, y_{i+1}) + H_{T}^{h} | X_0 = x \right)
= V_0^{*}, y + \mathbb{E}_0 \left( \sup_{\pi} H_{T}^{h} - V_T^{*, h_T}. \right).
\]

At \( T \) we must have \( V_T^{*}, y = H_T^{y} \), so we have the result.
The pure martingale dual of Schönmacher in the single exercise case, in which $h$ only takes values 0 or 1, is easily obtained. Thus we assume that $y = L$ is integer valued and the value of the payoff is constant in time. Hence we have a family of $L$ martingales and by only considering the times at which $h = 1$ we have the following dual.

In the single exercise case the dual value function is

$$V_{0}^{*,y}(x) = \inf_{M^{(w)}:0 \leq w \leq y | M^{w} \in \mathcal{M}_{0}} \mathbb{E}_{0}(\max_{0 \leq j_{1} < \ldots < j_{L} \leq T} \sum_{i=0}^{L-1} (Z_{j_{i}} - M_{j_{i+1}}^{(i+1)} + M_{j_{i}}^{(i+1)})),$$

$$\text{Pricing and Modelling in Electricity Markets – p. 56}$$
Continuous time

Assume that $X$ is a one dimensional diffusion process

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t,$$

where $\mu, \sigma^2$ are Lipschitz functions and $W$ is a standard Brownian motion. Pricing measure and write $\mathbb{E}_t(X) = \mathbb{E}(X|\mathcal{F}_t)$ for any random variable $X$ on our probability space.

receive payments at a certain rate $q_t$ at time $t$. As before we have a payoff function $Z^{q_t}_t(X_t)$ and now the total amount that we can receive over the lifetime of the option when exercising according to the function $q_t$ is

$$\int_0^T q_t Z^{q_t}_t(X_t)dt.$$
We define an exercise policy $\pi$ now to be a function $q_t, t \in [0, T]$ subject to the constraints that $0 \leq q_t \leq k_t$ and $\int_0^T q_u du \leq y$. Thus we can define the value function as

$$V_0^*(x, y) = \sup_{\pi} \mathbb{E}_0(\int_0^T q_t Z_t^q(X_t)dt | X_0 = x).$$

of all possible policies and we write $y_t = \int_t^T q_u du$.

$$\sup_{q \in \Pi} \left\{ \frac{\partial V}{\partial t} + \mu(x) \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2(x) \frac{\partial^2 V}{\partial x^2} + q(Z_t^q - \frac{\partial V}{\partial y}) \right\} = 0.$$
We can write a dual version as well.

\[ V_0^*(x, y) = \inf_{M^w: 0 \leq w \leq y} \mathbb{E}_0(\sup_{\pi} \int_0^T q_t Z_t^q(X_t) dt - \int_0^T dM_t^{y-f_0}) \]