A Tutorial On Bilattices

Ofer Arieli

School of Computer Science,
The Academic College of Tel-Aviv, Israel

oarieli@mta.ac.il
http://www.cs.mta.ac.il/∼oarieli

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Bilattices – Three Perspectives

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- Algebra
  - Bilattice structures and their properties
  - General constructions

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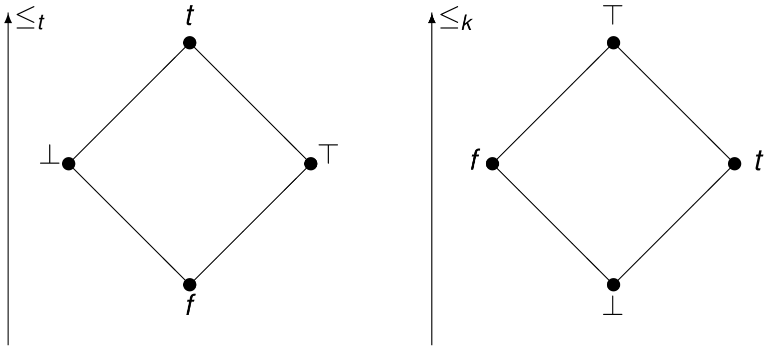
  - Semantics for proof systems
  - Inferences from incomplete and inconsistent data
- Computer Science
  - Fixpoint semantics for logic programs
  - Preferential modeling
  - Reasoning with uncertainty

N. Belnap, ”How A Computer Should Think?”

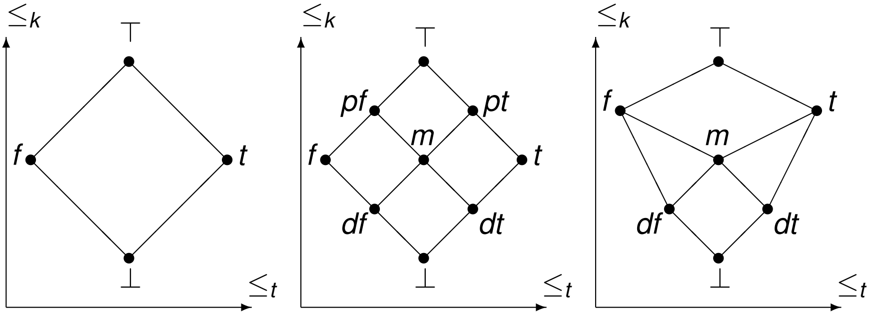


t – truth, f – falsity, $\perp$ – none, $\top$ – both

$\leq_t$ – the ‘truth’ order

$\leq_k$ – the ‘information’ (knowledge) order

Combining the Partial Orders



$\mathcal{FOUR}$, $\mathcal{NINE}$ and $\mathcal{SEVEN}$

Pre-Bilattices

Definition (Fitting)

A *pre-bilattice* is a triple $\mathcal{PB} = \langle B, \leq_t, \leq_k \rangle$, where

- B is a set containing at least four elements,
- $\langle B, \leq_t \rangle$, $\langle B, \leq_k \rangle$ are complete lattices.

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Notations:

t – the $\leq_t$-greatest element, f – the $\leq_t$-least element,
$\top$ – the $\leq_k$-greatest element, $\bot$ – the $\leq_k$-least element.

Basic operators:

- $\leq_t$-meet and join: $\wedge$ ('conjunction'), $\vee$ ('disjunction'),
- $\leq_k$-meet and join: $\otimes$ ('consensus'), $\oplus$ ('accept all').

Bilattices: Relating the Orders Through Negation

Definition (Ginsberg)

A *bilattice* is a quadruple $\mathcal{B} = \langle B, \leq_t, \leq_k, \neg \rangle$, where

- $\langle B, \leq_t, \leq_k \rangle$ is a pre-bilattice,
- $\neg$ is a $\leq_t$-*negation* on B.

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Properties of a $\leq_t$-negation:

- order reversing w.r.t. $\leq_t$: $a \leq_t b \Rightarrow \neg a \geq_t \neg b$,
- order preserving w.r.t $\leq_k$: $a \leq_k b \Rightarrow \neg a \leq_k \neg b$,
- involution: $\neg\neg a = a$.

Other Ways of Relating the Partial Orders

Definition

A (pre-) bilattice is *distributive* if all the (twelve) possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, $\oplus$ hold.

(e.g., $a \vee (b \otimes c) = (a \otimes b) \vee (a \otimes c)$)

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Definition (Fitting)

A (pre-) bilattice is *interlaced* if $\wedge$, $\vee$, $\otimes$, $\oplus$ are monotonic w.r.t. $\leq_t$ and $\leq_k$.

- $a \leq_t b$ implies that $a \otimes c \leq_t b \otimes c$ and $a \oplus c \leq_t b \oplus c$,
- $a \leq_k b$ implies that $a \wedge c \leq_k b \wedge c$ and $a \vee c \leq_k b \vee c$.

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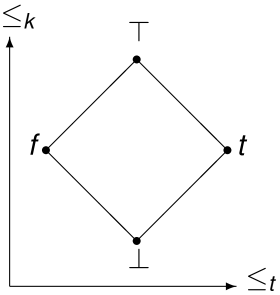
- $a \leq_k b$ implies that $a \wedge c \leq_k b \wedge c$ and $a \vee c \leq_k b \vee c$.

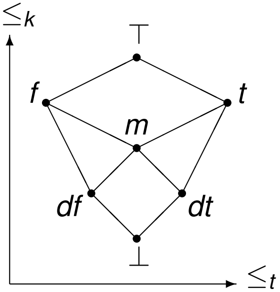
Note: Every distributive (pre-)bilattice is interlaced.

Example 1 – $\mathcal{FOUR}$



- The smallest bilattice
  $(\neg t = f, \neg f = t, \neg\top = \top, \neg\bot = \bot)$.
- Distributive (hence interlaced).

Example 2 – $\mathcal{SEVEN}$



- A bilattice introduced by Ginsberg for default reasoning ($\neg dt = df, \neg df = dt, \neg m = m$).
- Not even interlaced (e.g., $f <_t df$ but $f \otimes m >_t df \otimes m$).

Some Basic Properties

$\mathcal{B} = \langle B, \leq_t, \leq_k, \neg \rangle$ – a bilattice.

Lemma

a) $\neg(a \wedge b) = \neg a \vee \neg b,\ \neg(a \vee b) = \neg a \wedge \neg b,$
$\neg(a \otimes b) = \neg a \otimes \neg b,\ \neg(a \oplus b) = \neg a \oplus \neg b.$

b) $\neg f = t,\ \neg t = f,\ \neg \bot = \bot,\ \neg \top = \top.$

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b) $\neg f = t,\ \neg t = f,\ \neg\bot = \bot,\ \neg\top = \top.$

Note: If $\mathcal{B}$ has a $\leq_k$-negation "$-$" (conflation), then:

a) $-(a \wedge b) = -a \wedge -b,\ -(a \vee b) = -a \vee -b,$
$-(a \otimes b) = -a \oplus -b,\ -(a \oplus b) = -a \otimes -b.$

b) $-f = f,\ -t = t,\ -\bot = \top,\ -\top = \bot.$

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b) $-f = f$, $-t = t$, $-\bot = \top$, $-\top = \bot$.

Lemma

If $\mathcal{B}$ is interlaced, then $\bot \wedge \top = f$, $\bot \vee \top = t$, $f \otimes t = \bot$, $f \oplus t = \top$.

The Bilattice Product $\mathcal{L} \odot \mathcal{L}$

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Definition (Ginsberg)

Let $\mathcal{L} = \langle L, \leq_L \rangle$ be a complete lattice.

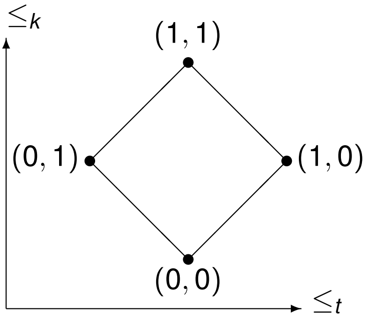
The bilattice $\mathcal{L} \odot \mathcal{L} = \langle L \times L, \leq_t, \leq_k, \neg \rangle$ is defined as follows:

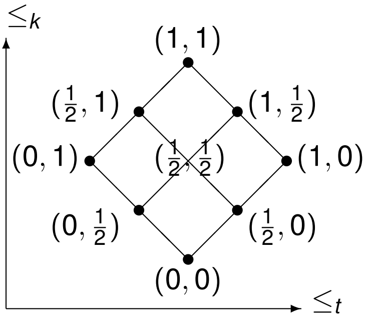
- $(b_1, b_2) \geq_t (a_1, a_2)$ iff $b_1 \geq_L a_1$ and $b_2 \leq_L a_2$,
- $(b_1, b_2) \geq_k (a_1, a_2)$ iff $b_1 \geq_L a_1$ and $b_2 \geq_L a_2$,
- $\neg(a_1, a_2) = (a_2, a_1)$.

Intuition: If $(x, y) \in L \times L$, then x represents the information *for* some assertion, and y is the information *against* it.

Note: Interlaced pre-bilattices may be constructed by $\mathcal{L}_1 \odot \mathcal{L}_2$.

Examples of $\mathcal{L} \odot \mathcal{L}$





$\mathcal{FOUR} = \mathcal{TWO} \odot \mathcal{TWO}$

$(\mathcal{TWO} = \langle \{0, 1\}, 0 < 1 \rangle)$

$\mathcal{NINE} = \mathcal{THREE} \odot \mathcal{THREE}$

$(\mathcal{THREE} = \langle \{0, \frac{1}{2}, 1\}, 0 < \frac{1}{2} < 1 \rangle)$

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Lemma

Let $\mathcal{L}$ be a complete lattice with a join $\sqcap_L$ and a meet $\sqcup_L$. Then:

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a) *$\mathcal{L} \odot \mathcal{L}$ is a bilattice with the following basic operations:*

$$(a,b) \vee (c,d) = (a \sqcup_L c, b \sqcap_L d),$$
$$(a,b) \wedge (c,d) = (a \sqcap_L c, b \sqcup_L d),$$
$$(a,b) \oplus (c,d) = (a \sqcup_L c, b \sqcup_L d),$$
$$(a,b) \otimes (c,d) = (a \sqcap_L c, b \sqcap_L d),$$
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$$(a,b) \otimes (c,d) = (a \sqcap_L c, b \sqcap_L d),$$
$$\neg(a,b) = (b,a).$$

b) *The four basic elements of $\mathcal{L} \odot \mathcal{L}$ are the following:*

$$\bot_{L \odot L} = (\inf(L), \inf(L)), \quad \top_{L \odot L} = (\sup(L), \sup(L)),$$
$$t_{L \odot L} = (\sup(L), \inf(L)), \quad f_{L \odot L} = (\inf(L), \sup(L)).$$

More Facts About $\mathcal{L} \odot \mathcal{L}$

Theorem

a) *$\mathcal{L} \odot \mathcal{L}$ is always interlaced* [Fitting]

b) *$\mathcal{L} \odot \mathcal{L}$ is distributive if so is $\mathcal{L}$* [Ginsberg]

c) *Every distributive bilattice is isomorphic to $\mathcal{L} \odot \mathcal{L}$ for some complete distributive lattice $\mathcal{L}$* [Ginsberg]

d) *Every interlaced bilattice is isomorphic to $\mathcal{L} \odot \mathcal{L}$ for some complete lattice $\mathcal{L}$* [Avron]

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d) *Every interlaced bilattice is isomorphic to $\mathcal{L} \odot \mathcal{L}$ for some complete lattice $\mathcal{L}$* [Avron]

Corollary: The number of elements of a finite interlaced bilattice is a perfect square.

The Interval-based Construction $\mathcal{I}(\mathcal{L})$

Definition (Fitting)

Let $\mathcal{L} = \langle L, \leq_L \rangle$ be a complete lattice.

The structure $\mathcal{I}(\mathcal{L}) = \langle \mathrm{I}(\mathcal{L}), \leq_t, \leq_k \rangle$ is defined as follows:

- $\mathrm{I}(\mathcal{L}) = \{[a, b] \mid a \leq_L b\}$, where $[a, b] = \{c \mid a \leq_L c \leq_L b\}$,
- $[b_1, b_2] \geq_t [a_1, a_2]$ iff $b_1 \geq_L a_1$ and $b_2 \geq_L a_2$,
- $[b_1, b_2] \geq_k [a_1, a_2]$ iff $b_1 \geq_L a_1$ and $b_2 \leq_L a_2$.

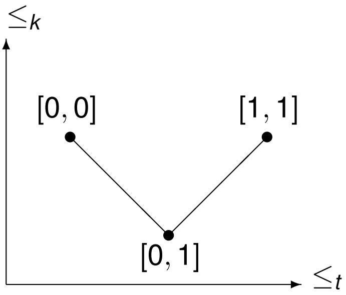
Intuition:

$\mathrm{I}(\mathcal{L})$: the 'intervals' of $\mathcal{L}$ (uncertain measurements).

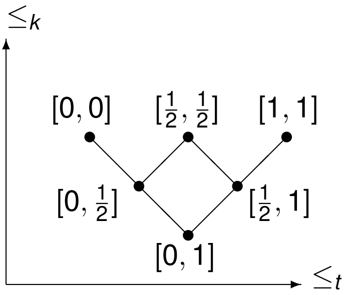
$\leq_t$: higher degree of truth; shift rightwards.

$\leq_k$: better approximations; interval narrowing ($[c, d] \geq_k [a, b] \Leftrightarrow [c, d] \subseteq [a, b]$).

$\mathcal{I}(\mathcal{L})$ – Examples and Applications

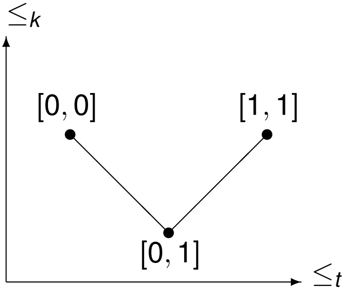


$\mathcal{I}(\mathcal{TWO})$



$\mathcal{I}(\mathcal{THREE})$

$\mathcal{I}(\mathcal{L})$ – Examples and Applications



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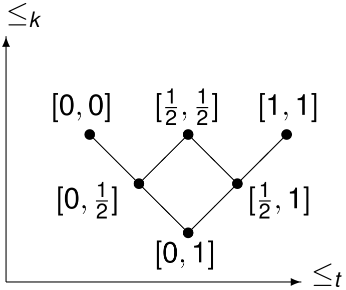

$\mathcal{I}(\mathcal{THREE})$

Applications:

- A generalization of Kleene's 3-valued structure.
- Interval-valued structures for fuzzy reasoning (using [0, 1]) .

Some Properties of $\mathcal{I}(\mathcal{L})$

Lemma

Let $\mathcal{L}$ be a complete lattice with a join $\sqcap_L$ and a meet $\sqcup_L$. Then:

a) $\mathcal{I}(\mathcal{L})$ *is a $\leq_k$-lower pre-bilattice, where:*

$$[a,b] \vee [c,d] = [a \sqcup_L c, b \sqcup_L d],$$
$$[a,b] \wedge [c,d] = [a \sqcap_L c, b \sqcap_L d],$$
$$[a,b] \otimes [c,d] = [a \sqcap_L c, b \sqcup_L d].$$

b) *The three basic elements of $\mathcal{I}(\mathcal{L})$ are the following:*

$$t_{\mathcal{I}(\mathcal{L})} = [\sup(L), \sup(L)], \quad f_{\mathcal{I}(\mathcal{L})} = [\inf(L), \inf(L)],$$
$$\perp_{\mathcal{I}(\mathcal{L})} = [\inf(L), \sup(L)].$$

Note: $\mathcal{I}(\mathcal{L})$ is not closed w.r.t. $\oplus$.

Relating $\mathcal{L} \odot \mathcal{L}$ and $\mathcal{I}(\mathcal{L})$

$\mathcal{L}$: a complete lattice with an order-reversing involution,

a^-: the $\leq_L$-involute of a.

- A *conflation* $-$ is defined on $\mathcal{L} \odot \mathcal{L}$ by $-(a,b) = (b^-, a^-)$.
  (This is a $\leq_k$-negation on $\mathcal{L} \odot \mathcal{L}$: involutive, $\leq_k$-reversing, $\leq_t$-preserving)
- An element $(a,b) \in L \times L$ is *coherent*, if $(a,b) \leq_k -(a,b)$.

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Proof.

The function $f_{\mathcal{L}} : \mathcal{I}(\mathcal{L}) \to \mathcal{L} \odot \mathcal{L}$, defined by $f_{\mathcal{L}}([a, b]) = (a, b^-)$, is an isomorphism between the structures. □

Bilattice-based Logics

What Is a Logic?

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Definition

A (Tarskian) *consequence relation* for a language L is a relation $\vdash$ between set of formulas in L and formulas in L, satisfying:

Reflexivity: $\psi \vdash \psi$.

Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma \vdash \psi$.

Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$, then $\Gamma \cup \Gamma' \vdash \phi$.

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Definition

A (propositional) *logic* is a pair $\mathbf{L} = \langle \mathsf{L}, \vdash \rangle$, where L is a propositional language and $\vdash$ is a consequence relation for L.

Matrices

A semantic (model-theoretic) way of defining logics:

Definition

A (multi-valued) *matrix* for L is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where:

- $\mathcal{V}$ – the *truth values*,
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Standard definitions for the induced semantic notions:

$\mathcal{M}$-valuations: $\Lambda_{\mathcal{M}} = \{\nu \mid \nu : \text{Atoms}(\mathcal{L}) \to \mathcal{V}\}$.

$\mathcal{M}$-models of ψ: $\text{mod}_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. $(\nu \models_{\mathcal{M}} \psi)$

$\mathcal{M}$-models of Γ: $\text{mod}_{\mathcal{M}}(\Gamma) = \bigcap_{\psi \in \Gamma} \text{mod}(\psi)$. $(\nu \models_{\mathcal{M}} \Gamma)$

Matrix-Based Logics

$\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ – a matrix for a language L.

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Next, we consider logics that are induced by bilattice-based matrices (i.e., whose truth values are elements of a bilattice and the connectives are defined by bilattice operators).

Bilattices and Logicality

Why bilattices?

- Incorporation of information considerations.
- Simple ways of representing different levels of inconsistency and incompleteness.

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Further considerations in defining logics:

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- The choice of the designated elements.

What should be the designated elements?

- We need dual notions for lattice filters and prime-filters.

Bifilters and Satisfiability

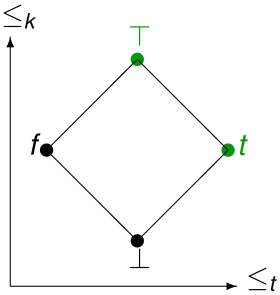
Definition (Arieli, Avron)

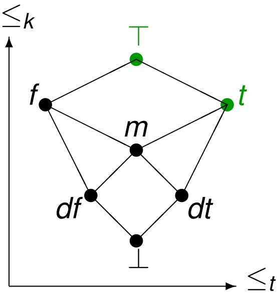
Let $\mathcal{B} = (B, \leq_t, \leq_k)$ be a bilattice.

a) A *bifilter* of $\mathcal{B}$ is a nonempty subset $\mathcal{F} \subset B$, such that:
   1. $a \wedge b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
   2. $a \otimes b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$.

b) A bifilter $\mathcal{F}$ is *prime*, if it satisfies the following conditions:
   1. $a \vee b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
   2. $a \oplus b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

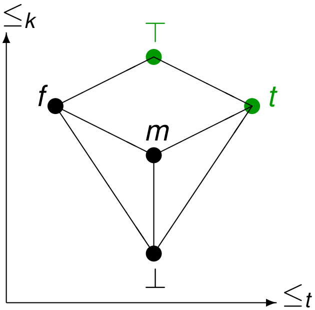
Examples of Bifilters





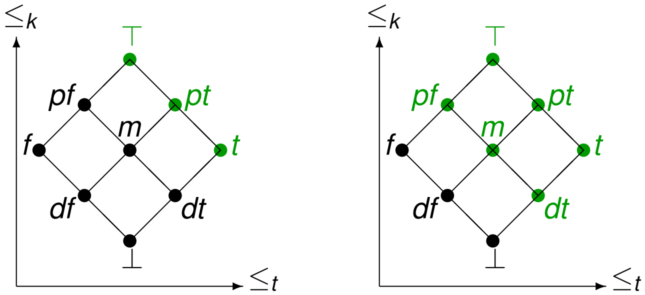
- Exactly one bifilter in $\mathcal{FOUR}$ and $\mathcal{SEVEN}$: $\mathcal{F} = \{t, \top\}$.
- This bifilter is prime in both cases.

Examples of Bifilters (Cont'd.)



- $\mathcal{F} = \{t, \top\}$ is also the unique bifilter of $\mathcal{FIVE}$.
- This time, it is *not* prime: $m \vee \bot \in \mathcal{F}$ but $m \notin \mathcal{F}$ and $\bot \notin \mathcal{F}$.

Examples of Bifilters (Cont'd.)



$\mathcal{NINE}$ has two bifilters, both are prime:

$\mathcal{F}_1 = \{t, pt, \top\}$,

$\mathcal{F}_2 = \{t, pt, dt, \top, m, pf\}$.

Bifilters – Some Facts (Arieli, Avron)

Lemma

Let $\mathcal{F}$ be a bifilter in $\mathcal{B}$. Then:

a) *$\mathcal{F}$ is upward-closed w.r.t. both $\leq_t$ and $\leq_k$.*

b) *$t, \top \in \mathcal{F}$ while $f, \bot \notin \mathcal{F}$.*

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Lemma

Let $\mathcal{B} = (B, \leq_t, \leq_k)$ be an interlaced (pre-)bilattice.

a) *A subset $\mathcal{F}$ of B is a (prime) bifilter iff it is a (prime) filter relative to $\leq_t$, and $\top \in \mathcal{F}$.*
b) *A subset $\mathcal{F}$ of B is a (prime) bifilter iff it is a (prime) filter relative to $\leq_k$, and $t \in \mathcal{F}$.*

Bifilters – More Facts

Notation: $\mathcal{F}_k(a) = \{b \mid b \geq_k a\}$, $\mathcal{F}_t(a) = \{b \mid b \geq_t a\}$.

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Lemma

In every interlaced bilattice it holds that $\mathcal{F}_k(t) = \mathcal{F}_t(\top)$, and this is the smallest bifilter.

Bifilters in $\mathcal{L} \odot \mathcal{L}$

Lemma

Let $\mathcal{L} = \langle L, \leq_L \rangle$ be a complete lattice. Then $\mathcal{F}$ is a [prime-] bifilter in $\mathcal{L} \odot \mathcal{L}$ iff $\mathcal{F} = \mathcal{F}_{\mathcal{L}} \times L$, where $\mathcal{F}_{\mathcal{L}}$ is a [prime-] filter in $\mathcal{L}$.

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Notation: Let $a \in L$, $a \neq \inf(L)$. We denote:
$\mathcal{F}(a) = \{(b_1, b_2) \mid b_1 \geq_L a, b_2 \in L\}$, $\mathcal{F}_{\mathcal{L}}(a) = \{y \in L \mid y \geq_L a\}$.

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Lemma

a) *$\mathcal{F}(a)$ is a prime bifilter of $\mathcal{L} \odot \mathcal{L}$ iff $\mathcal{F}_{\mathcal{L}}(a)$ is a prime filter in $\mathcal{L}$.*
b) *$\mathcal{F}(\sup(L))$ is a minimal prime bifilters of $\mathcal{L} \odot \mathcal{L}$ if $\sup(L)$ is join irreducible ($a \vee_L b = \sup(L) \Rightarrow a = \sup(L)$ or $b = \sup(L)$).*

Logical Billatices

Definition (Arieli, Avron)

A *logical bilattice* is a pair $\langle \mathcal{B}, \mathcal{F} \rangle$, where $\mathcal{B}$ is a bilattice and $\mathcal{F}$ is a prime bifilter of $\mathcal{B}$.

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Back to Logic

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In turn, $\mathcal{M}_{\mathfrak{B}}$ induces a corresponding *logic* $\mathbf{L}_{\mathfrak{B}} = \langle \mathsf{L}, \vdash_{\mathfrak{B}} \rangle$. We call it *the basic logic induced by the logical bilattice* $\mathfrak{B}$.

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We recall that in this logic, $\Gamma \vdash_{\mathfrak{B}} \phi$ means that for every $\nu \in \Lambda_{\mathfrak{B}}$, if $\nu(\psi) \in \mathcal{F}$ for every $\psi \in \Gamma$ then $\nu(\phi) \in \mathcal{F}$ as well.

Adding Implication Connectives

Note: In the language of $\{\vee, \wedge, \oplus, \otimes, \neg\}$, $\vdash_{\mathfrak{B}}$ has no tautologies.
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We add an implication connective $\supset$ for introducing tautologies and for reducing deducibility to theoremhood:

$$a \supset b = \begin{cases} b & \text{if } a \in \mathcal{F}, \\ t & \text{if } a \notin \mathcal{F}. \end{cases}$$

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- This connective is a generalization of the classical implication $a \to b = \neg a \vee b$ (they are identical on $\{t, f\}$).
- Modus ponens and the deduction theorem are valid for $\vdash_{\mathfrak{B}}$: $\Gamma, \psi \vdash_{\mathfrak{B}} \phi$ iff $\Gamma \vdash_{\mathfrak{B}} \psi \supset \phi$.

Tweety Dilemma

$$\Gamma = \left\{ \begin{array}{l} bird(tweety) \rightarrow fly(tweety) \\ penguin(tweety) \supset bird(tweety) \\ penguin(tweety) \supset \neg fly(tweety) \\ bird(tweety) \\ penguin(tweety) \end{array} \right\}$$

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Model No.	bird(tweety)	fly(tweety)	penguin(tweety)
$\nu_1 - \nu_2$	$\top$	$\top$	$\top, t$
$\nu_3 - \nu_4$	$\top$	f	$\top, t$
$\nu_5 - \nu_6$	t	$\top$	$\top, t$

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$\Gamma \vdash_4 bird(tweety)$, $\Gamma \not\vdash_4 \neg bird(tweety)$,

$\Gamma \vdash_4 penguin(tweety)$, $\Gamma \not\vdash_4 \neg penguin(tweety)$,

$\Gamma \vdash_4 \neg fly(tweety)$, $\Gamma \not\vdash_4 fly(tweety)$.

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Theorem (Arieli, Avron)

$\Gamma \vdash_{\mathfrak{B}} \psi$ iff $\Gamma \vdash_4 \psi$.

Proof Theory

The system *GBS*

(Gentzen-type Bilattice-based System)

Axioms:

$$\Gamma, \psi \Rightarrow \Delta, \psi$$

Structural Rules:

Permutation:

$$\frac{\Gamma_1, \psi, \phi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \phi, \psi, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, \psi, \phi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \phi, \psi, \Delta_2}$$

Contraction:

$$\frac{\Gamma_1, \psi, \psi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \psi, \Gamma_2 \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta_1, \psi, \psi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \psi, \Delta_2}$$

The Proof System GBS

Logical Rules:

$[\neg\neg\Rightarrow]$ $\dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta}$ $\dfrac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}$ $[\Rightarrow\neg\neg]$

$[\wedge\Rightarrow]$ $\dfrac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta}$ $\dfrac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}$ $[\Rightarrow\wedge]$

$[\neg\wedge\Rightarrow]$ $\dfrac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \phi) \Rightarrow \Delta}$ $\dfrac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \phi)}$ $[\Rightarrow\neg\wedge]$

$[\vee\Rightarrow]$ $\dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta}$ $\dfrac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}$ $[\Rightarrow\vee]$

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The Proof System GBS

Logical Rules (Cont'd.):

$[\otimes\Rightarrow]$ $$\frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}$$ $[\Rightarrow\otimes]$

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$[\oplus\Rightarrow]$ $$\frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}$$ $[\Rightarrow\oplus]$

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The Proof System GBS

Logical Rules (Cont'd.):

$[\supset\Rightarrow]$	$\dfrac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta}$	$\dfrac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}$	$[\Rightarrow\supset]$
$[\neg\supset\Rightarrow]$	$\dfrac{\Gamma, \psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \phi) \Rightarrow \Delta}$	$\dfrac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \phi), \Delta}$	$[\Rightarrow\neg\supset]$
$[\neg\mathsf{t}\Rightarrow]$	$\Gamma, \neg\mathsf{t} \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, \mathsf{t}$	$[\Rightarrow\mathsf{t}]$
$[\mathsf{f}\Rightarrow]$	$\Gamma, \mathsf{f} \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, \neg\mathsf{f}$	$[\Rightarrow\neg\mathsf{f}]$
$[\bot\Rightarrow]$	$\Gamma, \bot \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, \top$	$[\Rightarrow\top]$
$[\neg\bot\Rightarrow]$	$\Gamma, \neg\bot \Rightarrow \Delta$	$\Gamma \Rightarrow \Delta, \neg\top$	$[\Rightarrow\neg\top]$

Main Results

Definition

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If $\Gamma_1 \vdash_{GBS} \psi$ *and* $\Gamma_2, \psi \vdash_{GBS} \phi$*, then* $\Gamma_1, \Gamma_2 \vdash_{GBS} \phi$*.*

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Corollary: $\Gamma \vdash_4 \psi$ iff $\Gamma \vdash_{GBS} \psi$. Thus, the $\{\wedge, \vee, \supset, \mathsf{t}, \mathsf{f}\}$-fragment of $\vdash_4$ is identical to the $\{\wedge, \vee, \supset, \mathsf{t}, \mathsf{f}\}$-fragment of classical logic.

Hilbert-type Proof Systems

The system *HBS*

(Hilbert-type Bilattice-based System)

Defined Connective:

$$\psi \equiv \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\phi \supset \psi)$$

Inference Rule:

$$\frac{\psi \qquad \psi \supset \phi}{\phi}$$

The Proof System HBS

Axioms:

$[\supset 1]$	$\psi \supset \phi \supset \psi$	$[\supset 2]$	$(\psi \supset \phi \supset \tau) \supset (\psi \supset \phi) \supset (\psi \supset \tau)$
$[\supset 3]$	$((\psi \supset \phi) \supset \psi) \supset \psi$		
$[\wedge \supset]$	$\psi \wedge \phi \supset \psi \quad \psi \wedge \phi \supset \phi$	$[\supset \wedge]$	$\psi \supset \phi \supset \psi \wedge \phi$
$[\otimes \supset]$	$\psi \otimes \phi \supset \psi \quad \psi \otimes \phi \supset \phi$	$[\supset \otimes]$	$\psi \supset \phi \supset \psi \otimes \phi$
$[\supset \vee]$	$\psi \supset \psi \vee \phi \quad \phi \supset \psi \vee \phi$	$[\vee \supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \vee \phi \supset \tau)$
$[\supset \oplus]$	$\psi \supset \psi \oplus \phi \quad \phi \supset \psi \oplus \phi$	$[\oplus \supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \oplus \phi \supset \tau)$
$[\neg \wedge]$	$\neg(\psi \wedge \phi) \equiv \neg\psi \vee \neg\phi$	$[\neg \vee]$	$\neg(\psi \vee \phi) \equiv \neg\psi \wedge \neg\phi$
$[\neg \otimes]$	$\neg(\psi \otimes \phi) \equiv \neg\psi \otimes \neg\phi$	$[\neg \oplus]$	$\neg(\psi \oplus \phi) \equiv \neg\psi \oplus \neg\phi$
$[\neg \supset]$	$\neg(\psi \supset \phi) \equiv \psi \wedge \neg\phi$	$[\neg\neg]$	$\neg\neg\psi \equiv \psi$

Main Results

Theorem (*HBS* is well-axiomatized)

A complete and sound axiomatization of every fragment of $\vdash_{\mathfrak{B}}$ that includes $\supset$, is given by the axioms of HBS that mention only the connectives of this fragment.

Theorem (*GBS* and *HBS* are equivalent)

$\psi_1, \ldots, \psi_n \vdash_{GBS} \phi$ *iff* $\vdash_{HBS} \psi_1 \wedge \ldots \wedge \psi_n \supset \phi$.

Theorem (Soundness and Completeness)

$\Gamma \vdash_4 \psi$ *iff* $\Gamma \vdash_{HBS} \psi$.

Adding Quantifiers

- Standard extensions to first order languages, taking e.g., $\forall$ as a generalization of $\wedge$.
- For a structure D, $\nu(\forall x \psi(x)) = \inf_{\leq_t}\{\nu(\psi(d)) \mid d \in D\}$.

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Corresponding Gentzen-type rules:

$$[\forall\Rightarrow] \quad \frac{\Gamma, \psi(d) \Rightarrow \Delta}{\Gamma, \forall x \psi(x) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi(y), \Delta}{\Gamma \Rightarrow \forall x \psi(x), \Delta} \quad [\Rightarrow\forall]$$

$$[\neg\forall\Rightarrow] \quad \frac{\Gamma, \neg\psi(y) \Rightarrow \Delta}{\Gamma, \neg\forall x \psi(x) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \neg\psi(d), \Delta}{\Gamma \Rightarrow \neg\forall x \psi(x), \Delta} \quad [\Rightarrow\neg\forall]$$

Assuming, as usual, that the variable y is not free in $\Gamma \cup \Delta$.

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Corresponding Gentzen-type rules:

$$[\forall\Rightarrow]\quad \frac{\Gamma, \psi(d) \Rightarrow \Delta}{\Gamma, \forall x \psi(x) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi(y), \Delta}{\Gamma \Rightarrow \forall x \psi(x), \Delta}\quad [\Rightarrow\forall]$$

$$[\neg\forall\Rightarrow]\quad \frac{\Gamma, \neg\psi(y) \Rightarrow \Delta}{\Gamma, \neg\forall x \psi(x) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \neg\psi(d), \Delta}{\Gamma \Rightarrow \neg\forall x \psi(x), \Delta}\quad [\Rightarrow\neg\forall]$$

Assuming, as usual, that the variable y is not free in $\Gamma \cup \Delta$.

Quantifiers for $\oplus$ and $\otimes$ can be introduced in a similar way.

Drawbacks of $\vdash_{\mathfrak{B}}$

- Strictly weaker than classical logic even for consistent theories.
- Rejects some very useful (and intuitively justified) inference rules, such as the Disjunctive Syllogism: $\neg p, p \vee q \nvdash_{\mathfrak{B}} q$.

Preferential Reasoning by the Information Order

$\mathfrak{B} = \langle \mathcal{B}, \mathcal{F} \rangle$ – a logical bilattice, where $\leq_k$ is well-founded in $\mathcal{B}$.

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a) ν_1 is $\leq_k$*-smaller* than ν_2, if for each atom p, $\nu_1(p) \leq_k \nu_2(p)$.

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Definition

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Intuition: Do not assume anything that is not really known; As long as one keeps the redundant information as minimal as possible, the tendency of getting into conflicts decreases.

Tweety Dilemma, Revisited

$$\Gamma = \left\{ \begin{array}{l} bird(tweety) \rightarrow fly(tweety) \\ penguin(tweety) \supset bird(tweety) \\ penguin(tweety) \supset \neg fly(tweety) \\ bird(tweety) \\ penguin(tweety) \end{array} \right\}$$

Two $\leq_k$-minimal models (out of six models)

Model No.	bird(tweety)	fly(tweety)	penguin(tweety)
ν_1	$\top$	f	t
ν_2	t	$\top$	t

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Model No.	bird(tweety)	fly(tweety)	penguin(tweety)
ν_1	$\top$	f	t
ν_2	t	$\top$	t

$\Gamma \vdash_4^{\leq_k} bird(tweety)$, $\quad \Gamma \not\vdash_4^{\leq_k} \neg bird(tweety)$,

$\Gamma \vdash_4^{\leq_k} penguin(tweety)$, $\quad \Gamma \not\vdash_4^{\leq_k} \neg penguin(tweety)$,

$\Gamma \vdash_4^{\leq_k} \neg fly(tweety)$, $\quad \Gamma \not\vdash_4^{\leq_k} fly(tweety)$.

Basic Properties of $\vdash_{\mathfrak{B}}^{\leq_k}$

Theorem

$\mathfrak{B} = \langle \mathcal{B}, \mathcal{F} \rangle$ – a logical bilattice where $\mathcal{B}$ is interlaced.
If ψ is in the language without $\supset$, then $\Gamma \vdash_{\mathfrak{B}} \psi$ iff $\Gamma \vdash_{\mathfrak{B}}^{\leq_k} \psi$.

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Note: $\vdash_{\mathfrak{B}}^{\leq_k}$ is in general nonmonotonic, but it is cautiously monotonic: If $\Gamma \vdash_{\mathfrak{B}}^{\leq_k} \phi$ then $\Gamma, \psi \vdash_{\mathfrak{B}}^{\leq_k} \phi$ when $\Gamma \vdash_{\mathfrak{B}}^{\leq_k} \psi$.

Basic Properties of $\vdash_{\mathfrak{B}}^{\leq_k}$

Theorem

*$\mathfrak{B} = \langle \mathcal{B}, \mathcal{F} \rangle$ – a logical bilattice where $\mathcal{B}$ is interlaced.
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Theorem (Characterization in $\mathcal{FOUR}$)

*$\mathfrak{B} = \langle \mathcal{B}, \mathcal{F} \rangle$ – a logical bilattice such that $\inf_{\leq_k} \mathcal{F} \in \mathcal{F}$.
Then: $\Gamma \vdash_{\mathfrak{B}}^{\leq_k} \psi$ iff $\Gamma \vdash_{4}^{\leq_k} \psi$.*

Bilattices and Logic Programming

In a series of papers, Melving Fitting has shown that bilattices are very useful for defining and analyzing fixpoint semantics for logic programs.

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Some advantages of using bilattices in this context:

- Extended languages for logic programs.
- Generalizations of standard two- and three-valued semantics to finite-valued semantics.
- Accommodation of incompleteness and inconsistency.
- Characterizing structures of standard approaches for negation handling (stable-model semantics, well-founded semantics).

(Bilattice-based) Logic Programs

A *clause* is an expression of the form $A \leftarrow \Psi$, where

- The clause's *head* A is a (first-order) atomic formula.
- The clause's *body* Ψ is a formula built-up from literals (atoms or negated atoms) using $\{\wedge, \vee, \otimes, \oplus, \exists, \forall\}$ and constants from $\mathcal{B}$, and whose free variables occur in A.

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A *logic program* $\mathcal{P}$ is a finite set of clauses. If there are no negations in the clauses' bodies, $\mathcal{P}$ is called *positive*.

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Note: We may assume that there are no different clauses with the same head, and that the bodies are non-empty, since $A \leftarrow \Psi_1$ and $A \leftarrow \Psi_2$ are replaced by $A \leftarrow \Psi_1 \vee \Psi_2$, and A is replaced by $A \leftarrow \mathsf{t}$.

Example

$$A \leftarrow \mathrm{t},$$
$$C \leftarrow B \oplus A,$$
$$D \leftarrow B \otimes C,$$
$$E \leftarrow C \vee D \vee E$$

The Immediate Consequence Operator $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

Notations:

$\mathcal{P}^*$: the grounding of $\mathcal{P}$ over the Herbard base,
$\Lambda_{\mathcal{B}}^{\mathcal{P}} = \{\nu \mid \nu : \text{Atoms}(\mathcal{P}^*) \to B\}$: the B-valuations for $\mathcal{P}$.

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Note: $\langle \Lambda_{\mathcal{B}}^{\mathcal{P}}, \leq_t, \leq_k \rangle$ is a pre-bilattice:

- $\nu_1 \leq_t \nu_2$ iff $\nu_1(A) \leq_t \nu_2(A)$ for every ground atom A,
- $\nu_1 \leq_k \nu_2$ iff $\nu_1(A) \leq_k \nu_2(A)$ for every ground atom A.

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Definition (Fitting, Apt, van-Emden, Kowalski)

$\mathcal{T}_{\mathcal{B}}^{\mathcal{P}} : \Lambda_{\mathcal{B}}^{\mathcal{P}} \to \Lambda_{\mathcal{B}}^{\mathcal{P}}$ is defined by:

$$\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu)(A) = \begin{cases} \nu(\Psi) & \text{if } A \leftarrow \Psi \in \mathcal{P}^*, \\ f & \text{otherwise.} \end{cases}$$

Fixpoints of $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

Note: If $\mathcal{B}$ is an interlaced pre-bilattice, then

- $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$ is $\leq_k$-monotone on $\Lambda_{\mathcal{B}}^{\mathcal{P}}$: $\nu_1 \leq_k \nu_2 \Rightarrow \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_1) \leq_k \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_2)$.
- If $\mathcal{P}$ is positive, then $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$ is $\leq_t$-monotone on $\Lambda_{\mathcal{B}}^{\mathcal{P}}$.

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- If $\mathcal{P}$ is positive, then $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$ is $\leq_t$-monotone on $\Lambda_{\mathcal{B}}^{\mathcal{P}}$.

Thus: If $\mathcal{P}$ is positive and $\mathcal{B}$ is interlaced,

- $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$ has a $\leq_t$-least fixpoint ν_t and a $\leq_t$-greatest fixpoint V_t,
- $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$ has a $\leq_k$-least fixpoint ν_k and a $\leq_k$-greatest fixpoints V_k.

(by Knaster-Tarski Theorem)

Computing the Fixpoints of $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

$$A \leftarrow \mathrm{t},$$
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Computing the $\leq_k$-least fixpoint, ν_k:

Start with the $\leq_k$-smallest valuation, and iterate over $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$:

Computing the Fixpoints of $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

$$
\begin{aligned}
A &\leftarrow \mathrm{t}, \\
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\end{aligned}
$$

Computing the $\leq_k$-least fixpoint, ν_k:

Start with the $\leq_k$-smallest valuation, and iterate over $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$:

1. $A : \bot,\ B : \bot,\ C : \bot,\ D : \bot,\ E : \bot \quad \nu_0$

Computing the Fixpoints of $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

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2. $A : t,\ \ B : f,\ \ C : \bot,\ D : \bot,\ E : \bot \quad \nu_1 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_0)$

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2. $A : t,\ \ B : f,\ \ C : \bot,\ D : \bot,\ E : \bot \quad \nu_1 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_0)$
3. $A : t,\ \ B : f,\ \ C : \top,\ D : \bot,\ E : \bot \quad \nu_2 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_1)$

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3. $A : t,\ B : f,\ C : \top,\ D : \bot,\ E : \bot \quad \nu_2 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_1)$
4. $A : t,\ B : f,\ C : \top,\ D : f,\ E : t \quad \nu_3 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(\nu_2) \quad \nu_k = \nu_3$

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Computing the $\leq_k$-greatest fixpoint, V_k:

Start with the $\leq_k$-largest valuation, and iterate over $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$:

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2. $A : t,\ B : f,\ C : \top,\ D : \top,\ E : \top \quad V_1 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(V_0)$
3. $A : t,\ B : f,\ C : \top,\ D : f,\ E : \top \quad V_2 = \mathcal{T}_{\mathcal{B}}^{\mathcal{P}}(V_1) \quad V_k = V_2$

Relating the Fixpoints of $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

ν_t and V_t are computed similarly, starting with the $\leq_t$-smallest and the $\leq_t$-largest valuation (respectively) and iterating over $\mathcal{T}_{\mathcal{B}}^{\mathcal{P}}$

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In our example $\nu_t = V_k$ and $V_t = \nu_k$, thus:

- $\nu_k = V_t = \{A : t,\ B : f,\ C : \top,\ D : f,\ E : t\}$,
- $\nu_t = V_k = \{A : t,\ B : f,\ C : \top,\ D : f,\ E : \top\}$.

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- $\nu_k = V_t = \{A : t,\ B : f,\ C : \top,\ D : f,\ E : t\}$,
- $\nu_t = V_k = \{A : t,\ B : f,\ C : \top,\ D : f,\ E : \top\}$.

In general, we have:

Theorem (Fitting)

If $\mathcal{P}$ is positive and $\mathcal{B}$ is interlaced, then:
$\nu_k = \nu_t \otimes V_t, \quad V_k = \nu_t \oplus V_t, \quad \nu_t = \nu_k \wedge V_k, \quad V_t = \nu_k \vee V_k.$

Another Example

$$even(0) \leftarrow \mathrm{t},$$
$$even(s(x)) \leftarrow odd(x),$$
$$odd(s(x)) \leftarrow even(x)$$

Another Example

$$even(0) \leftarrow t,$$
$$even(s(x)) \leftarrow odd(x),$$
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Grounding:

$$even(0) \leftarrow t,$$
$$even(s(0)) \leftarrow odd(0),$$
$$odd(s(0)) \leftarrow even(0),$$
$$even(s(s(0))) \leftarrow odd(s(0)),$$
$$odd(s(s(0))) \leftarrow even(s(0)), \ \ldots$$

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$$even(s(s(0))) \leftarrow odd(s(0)),$$
$$odd(s(s(0))) \leftarrow even(s(0)), \ \ldots$$

A unique fixpoint: $\nu(even(s^n(0)) = t$ iff n is even.

Other Generalizations to the Bilattice Setting of Fixpoint Operators

The Gelfond-Lifschitz transformation for handling negation as failure in logic programs can also be generalized to the bilattice setting, yielding a *bilattice-based stable operator*.

Other Generalizations to the Bilattice Setting of Fixpoint Operators

The Gelfond-Lifschitz transformation for handling negation as failure in logic programs can also be generalized to the bilattice setting, yielding a *bilattice-based stable operator*.

Again, this operator is $\leq_k$-monotonic, so it has a $\leq_k$-least fixpoint s_k and a $\leq_k$-greatest fixpoint S_k. Fitting has shown that these $\leq_k$-external fixpoints are related to the $\leq_t$-external (oscillation) points s_t and S_t as follows:

- $s_k = s_t \otimes S_t$, $S_k = s_t \oplus S_t$.
- $s_t = s_k \wedge S_k$, $S_t = s_k \vee S_k$.

(Details are given in Fitting's papers on the family of stable models and the survey on logic programming, the references to which appear in the next slides).

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Thank you!