Dualities induced by canonical extensions

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1 Introduction

Whenever a concept of canonical extension makes sense, it can usually be obtained as a double dual process. Using a duality in one direction and coming back with a discretized version of it. In fact the existence of the canonical extension is often shown to be equivalent to the existence of duality.

So the maxim "under each canonical extension is hidden a duality" may come as triviality. However it may be considered as interesting in certain circumstances. We sketch here a general setting for such dualities, and details in the Boolean case.

2 A general framework

We dispose of structures $A, B, ...$ with the following properties:

1) for each A we have a dual A^* that is a subset of P^A for some finite structure P,

2) for each A we have a canonical extension $e_A : A \to A^\sigma$, and A^σ receives a topology (the σ-topology) for which $e_A(A)$ is the set of isolated points of A^σ and is dense in A^σ.

We want to dualise maps $f : A \to B$. The idea is to do this via a relation $\mathcal{U}_f^\sigma \subseteq A^\sigma \times B^* \times P$ defined by

$$(x, \varphi, p) \in \mathcal{U}_f^\sigma \text{ iff } \exists V \quad x \in V \in \delta, \quad \varphi f e_A^{-1}(V) = \{p\}.$$

Note that to parallel the lattice-based case, an alternative option might be the relation $\mathcal{U}_f^\pi$ defined by

$$(x, \varphi, p) \in \mathcal{U}_f^\pi \text{ iff } \forall V \quad x \in V \in \delta \to \varphi f e_A^{-1}(V) \ni p.$$

3 The Boolean case

To lay emphasis on ideas an to avoid technicalities, we give an illustration in a particularly easy case, the Boolean case. The general ingredients are well-known. We first note that since the generating structure P is the 2-element algebra, we can get rid of it by means of zero (or co-zero) sets. Now for a Boolean algebra A, A^* is its usual spectrum (space of ultrafilters on A) and A^σ is the powerset of A^* (with $e_A(a) = \{x \in A^* | a \in x\}$). We also recall that if X is a topological space then $\sigma(\mathcal{P}(X))$, the σ-topology on $\mathcal{P}(X)$, is the topology generated by the $[F, O]$, where F is closed and O is open in X.

Definitions 3.1. 1) The category we propose to dualise is the category of *Boolean algebras with operations*, that is algebras $(A, f, ...)$ where A is a Boolean algebra and each f is a finitary operation on A (i.e., $f : A^n \to A$ for some n).

2) If $(A, f, ...)$ is Boolean algebra with operations, its *dual* is $(A^*, \mathcal{U}_f^\sigma, ...)$ where according to section 2, $\mathcal{U}_f^\sigma \subseteq \mathcal{P}(A^*)^n \times A^*$ is defined for $E \in \mathcal{P}(A^*)^n$ and $x \in A^*$ by

   $(E, x) \in \mathcal{U}_f^\sigma$ iff $\exists V \quad E \in V \in (\sigma(\mathcal{P}(A^*))^n$ with $f(a) \in x$ for each $a \in A^n$ such that $e_a(a) \in V$.

   And the *alternative dual* $\mathcal{U}_f^\pi$ is defined by

   $(E, x) \in \mathcal{U}_f^\pi$ iff $\forall V \quad E \in V \in \sigma(\mathcal{P}(A^*)^n) \rightarrow f(a) \in x$ for some $a \in A^n$ such that $e_A(a) \in V$.

As suggested in the title of the talk, both $\mathcal{U}_f^\sigma$ and $\mathcal{U}_f^\pi$ can be captured by the lower and upper extensions f^σ and f^π of f to A^σ as defined, for instance, by Gehrke and Jónsson [1] and we have

Proposition 3.1. With the previous notations,

$$(E, x) \in \mathcal{U}_f^\sigma \text{ iff } f^\sigma(E) \ni x$$

and

$$(E, x) \in \mathcal{U}_f^\pi \text{ iff } f^\pi(E) \ni x.$$

Also, rather suprisingly, $\mathcal{U}_f^\pi$ and $\mathcal{U}_f^\sigma$ are closely linked: one is the closure and the other the interior of the other.

Notation 3.2. For $E \in (A^\sigma)^n$ and $x \in A^*$ we note

$$\mathcal{U}_f^\sigma(E\_) = \{x | (E, x) \in \mathcal{U}_f^\sigma\}, \mathcal{U}_f^\sigma(\_x) = \{E | (E, x) \in \mathcal{U}_f^\sigma\},$$

similarly for $\mathcal{U}_f^\pi$.

Proposition 3.3. For $x \in A^*$, $\mathcal{U}^\sigma(\_x)$ is a regular open set of $(\mathcal{P}(A^*))^n$,

$$\mathcal{U}_f^\sigma(\_x)^- = \mathcal{U}_f^\pi(\_x) \text{ and consequently } \mathcal{U}_f^\pi(\_x)^\circ = \mathcal{U}_f^\sigma(\_x).$$

This proposition enables us to characterize the dual category.

Definition 3.4. By a *neighborhood space*, we mean a structure $(X, \mathcal{T}, \mathcal{U}, ...)$ where $(X, \mathcal{T})$ is a Boolean space and each $\mathcal{U}$ is (following the adapted terminology of Farley [2]) a *Boolean multirelation*, that is, a subset of $\mathcal{P}(X)^n \times X$, such that

1) if $O \in (Clop(X))^n$, then $\mathcal{U}(O\_)$ is clopen in X, and

2) if $x \in X$, then $\mathcal{U}(\_x)$ is a regular open subset of $\mathcal{P}(X)^n$ for the σ-topology on $\mathcal{P}(X)$.

Our terminology is taken from the realm of modal logic: when classical non-normal logics are considered, completeness results are obtained via the so-called neighborhood semantic, which deal exactly with the non-topological version of our neighborhood spaces ([3]).

Neighborhood spaces are made into a category by considering that a map $\varphi : X \rightarrow Y$ between neighborhood spaces $(X, \mathcal{U}, ...)$ and $(Y, \mathcal{U}', ...)$ is a morphism if it is continuous and for each $E \in (Clop(Y))^n$ and $x \in X$,

$$(E, \varphi(x)) \in \mathcal{U}' \text{ iff } (\varphi^{-1}(E), x) \in \mathcal{U}.$$

We have the expected theorem (which can be extended to lattice expansions and natural extensions).

Theorem 3.5. The categories of Boolean algebras with operations and that of neighborhood spaces are dually equivalent.

It would be interesting to develop a correspondence theory in this setting. We just give one example, that shows that Theorem 3.5 contains as a particular case the extended Stone duality for Boolean algebra with operators.

Proposition 3.6. The following assertions are equivalent:

1) the operation $f : A^n \rightarrow A$ is $\vee$-preserving,

2) for all $x \in A^*$, $\complement \mathcal{U}_f^\sigma(\_x)$ is an ideal of $\mathcal{P}(A^*)^n$.

In addition, f is an operator if and only if

$$\mathcal{U}_f^\sigma = \{(E, x) | \exists y \in E, \ f(y) \subseteq x\}.$$

4 Bibliography

[1] M. GEHRKE AND B. JÓNSSON, Bounded distributive lattices expansions, *Mathematica Scandinavia* **94**,2 (2004),13-45.

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[3] B.F. CHELLAS, Modal logic: an introduction, *Cambridge University Press*, 1980.