Dualities for locally hypercompact, stably hypercompact and hyperspectral spaces

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Several central notions of lattice and domain theory are or may be defined in terms of the Scott topology. For example, an element of a poset P is compact iff it generates a Scott-open principal filter; or, a poset is continuous iff its Scott topology is supercontinuous (= completely distributive). Replacing in such definitions the Scott topology by the weaker upper topology vP (generated by the complements of principal ideals), one arrives at analogous "hyper-notions". Thus, an element is hypercompact iff it generates an v-open principal filter; specifically, a subset of a topological space is hypercompact iff its saturation is finitely generated; and a poset is hypercontinuous iff its upper topology is supercontinuous. Similarly, replacing "compact" with "hypercompact" in the respective definitions, one passes from locally compact spaces to locally hypercompact spaces, from stably compact spaces to stably hypercompact spaces, from spectral spaces to hyperspectral spaces, and so on. At first glance, the "hyper-versions" look quite similar to their classical counterparts. But a closer inspection reveals dramatic differences. For example, the category of locally compact sober spaces is dual to the category of continuous frames, and analogously, the category of locally hypercompact sober spaces is dual to the category of hypercontinuous frames. But the crucial difference between these two dualities is that there is an abundance of non-homeomorphic locally compact sober (or even T2) spaces with the same specialization order, whereas a locally hypercompact sober space is entirely determined by its specialization order: there is a concrete isomorphism between the category of locally hypercompact sober spaces and quasicontinuous domains—and a locally hyper-compact T1-space is already discrete. So one might guess that the topological theory of locally hypercompact spaces etc. is rather specific. But for Scott spaces (and even for the wider class of monotone determined spaces) a surprising coincidence arises: here every compact open set is already hypercompact. Moreover, the category of quasialgebraic domains is not only dual to the category of hyperalgebraic frames, but also concretely isomorphic to the category of compactly based sober Scott spaces—which answers in the negative Priestley's question of whether there might exist spectral spaces that carry the Scott topology but are not quasicontinuous. Indeed, the category of quasicoherent domains is concretely isomorphic to that of hyperspectral spaces and to the category of Priestley spaces carrying the Lawson topology; and these categories are dual to the category of hypercoherent frames and to the category of distributive lattices whose principal ideals are finite intersections of prime ideals.