Duality Theory in Algebra, Logik and Computer Science

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Dualities for locally hypercompact, stably hypercompact and hyperspectral spaces

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Part I

Variants of compactness and sobriety
Variants of compactness and sobriety

1. Order-theoretical and topological preliminaries

2. Topological concepts of compactness

3. Sobriety and well-filtration
Let $P = (X, \leq)$ be a preordered set ($\leq$ reflexive and transitive).

- **lower set** and **upper set** generated by $Y \subseteq X$, respectively:
  $$\uparrow Y = \{x \in X : \exists y \in Y (x \geq y)\},$$
  $$\downarrow Y = \{x \in X : \exists y \in Y (x \leq y)\}.$$  
- **principal ideal** and **principal filter** of $y \in X$, respectively:
  $$\downarrow y = \{x \in X : x \leq y\},$$
  $$\uparrow y = \{x \in X : x \geq y\}.$$  
- A **foot** is a finitely generated upper set (FUS) $\uparrow F$.
- The upper sets form the (upper) Alexandroff topology $\alpha P$.
- **The lower sets are exactly the closed sets w. r. t. $\alpha P$.**
- **The feet are exactly the compact open sets w. r. t. $\alpha P$.**
Let \( X \) be a topological space.

- The **specialization order** of \( X \) is given by
  \[ x \leq y \iff x \in \overline{\{y\}} \iff \text{for all open } U (x \in U \Rightarrow y \in U). \]
  *All order-theoretical statements about spaces refer to the specialization order, unless otherwise stated.*

- The **saturation** of a subset \( Y \subseteq X \) is the intersection of all neighborhoods of \( Y \); hence it is the upset \( \uparrow Y \) generated by \( Y \).

- Specifically, the (neighborhood) **core** of a point \( x \in X \) is the principal filter \( \uparrow x \), the intersection of all neighborhoods of \( x \).

- Dually, the **point closure** of \( x \) is the principal ideal \( \downarrow x \).
Definition
Let $X$ be a $T_0$-space.

- A **monotone net** in $X$ is a map $\nu$ from a directed set $N$ into $X$ such that $m \leq n$ implies $\nu(m) \leq \nu(n)$ (with respect to the specialization order).
- $X$ is a **monotone convergence space** if each monotone net in $X$ has a join to which it converges.
- $X$ is **monotone determined** if a subset $U$ is open whenever any monotone net converging to a point of $U$ is residually in $U$. Equivalently, $U$ is open whenever for all directed $D \subseteq X$,
  \[ \overline{D} \cap U \neq \emptyset \text{ implies } D \cap U \neq \emptyset. \]
The Scott topology

Definition

- A subset $U$ of a poset $P = (X, \leq)$ is **Scott open** if for all directed subsets $D$ possessing a join, $D \cap U \neq \emptyset \iff \bigvee D \in U$.
- The **Scott topology** $\sigma P$ consists of all Scott-open sets.
- The **Scott space** associated with $P$ is $\Sigma P = (X, \sigma P)$.
- The **weak Scott topology** $\sigma_2 P$ consists of all upsets $U$ that meet each directed set $D$ with $D^{\uparrow\downarrow} \cap U \neq \emptyset$, where $D^{\uparrow\downarrow} = \bigcap \{ \downarrow y : D \subseteq \downarrow y \}$ is the cut closure of $D$.
- The poset $P$ is **up-complete**, a **dcpo** or a **domain** if all directed subsets of $P$ have joins. In that case, $\sigma P$ coincides with $\sigma_2 P$. 
Theorem
(ME 2009)

- The topology of a monotone convergence space is always coarser than the Scott topology.
- The topology of a monotone determined space is always finer than the weak Scott topology.
- The Scott spaces of domains are exactly the monotone determined monotone convergence spaces.
The Separation Lemma for locales

Definition

- A **locale** or **frame** is a complete lattice \( L \) satisfying the infinite distributive law \( a \land \bigvee B = \bigvee \{ a \land b : b \in B \} \).
- A locale \( L \) enjoys the Separation Lemma or **Strong Prime Element Theorem** if each element outside a Scott-open filter \( U \) in \( L \) lies below a prime element outside \( U \).

**Theorem**

(ME 1986, BB+ME 1993)

*The Ultrafilter Principle (UP) resp. Prime Ideal Theorem (PIT) is equivalent to the Separation Lemma for locales (or quantales).*
Hyper- and supercompactness in spaces

Definition

- A subset $C$ of a space $X$ is **hypercompact** if $\uparrow C$ is a foot $\uparrow F$.
- A subset $C$ of a space $X$ is **supercompact** if $\uparrow C$ is a core $\uparrow x$.
- A space is **compactly based** resp. **hypercompactly based** resp. **supercompactly based** if it has a base of compact resp. hypercompact resp. supercompact open sets.
- A space is **locally compact** resp. **locally hypercompact** resp. **locally supercompact** if each point has a neighborhood base of compact resp. hypercompact resp. supercompact sets.
Definition
A $T_0$-space is sober if each irreducible closed subset is a point closure.

Theorem
(ME 2009)
(1) Compact open subsets of monotone determined spaces are hypercompact.
(2) The hypercompactly based spaces are exactly the compactly based monotone determined spaces.
(3) The hypercompactly based sober spaces are exactly the compactly based Scott spaces of domains.
Nets and filters
Definition
The **Lawson dual** $\delta P$ of a poset $P$ consists of all Scott-open filters.

A $T_0$-space $X$ is

- **$\delta$-sober** if each Scott-open filter of open sets (i.e. each $\mathcal{V} \in \delta \mathcal{O}X$) contains all open neighborhoods of its intersection.

- **well-filtered** (resp. **$\mathcal{H}$-well-filtered**) if for any filter base $\mathcal{B}$ of compact (resp. hypercompact) saturated sets, each open neighborhood of the intersection $\bigcap \mathcal{B}$ contains a member of $\mathcal{B}$.

Lemma
(1) $X$ is $\delta$-sober iff its open locale enjoys the Separation Lemma.
(2) Every $\delta$-sober space is sober and well-filtered.
(3) Every locally compact well-filtered space is $\delta$-sober.
Tychonoff’s Product Theorem

Compact unit cube with a partial open covering
The power of $\delta$-sobriety and Tychonoff’s Theorem

**Theorem**

(ME 2012) Each of the following statements is equivalent to **UP**:

- The Hofmann-Mislove Theorem: sober spaces are $\delta$-sober.
- Sober spaces are well-filtered.
- Every filter base of compact saturated sets in a sober space has nonempty intersection.
- Every filter base of compact saturated sets in a sober space has compact intersection.
- Tychonoff’s Product Theorem for sober spaces.
- Tychonoff’s Product Theorem for Hausdorff spaces.
- Tychonoff’s Product Theorem for two-element discrete spaces.
Power of sobriety

A sober space and a non-sober neighborhood

Sobrification
The power of Rudin’s Lemma

Definition
For any system $\mathcal{Y}$ of sets, each member of the crosscut system

$$\mathcal{Y}^\# = \{Z \subseteq \bigcup \mathcal{Y} : \forall Y \in \mathcal{Y} (Y \cap Z \neq \emptyset)\}$$

is a cutset or transversal of $\mathcal{Y}$.

Theorem

(ME 2012) Each of the following statements is equivalent to UP:

- Every system of compact sets whose saturations form a filter base has an irreducible transversal.
- Rudin’s Lemma: Every system of finite sets generating a filter base of feet has a directed transversal.
- Every monotone convergence space is $\mathcal{H}$-well-filtered.
Part II

Stone-Priestley dualities
Stone-Priestley dualities

4. Order-theoretical concepts of compactness

5. Coherence in spaces and ordered structures

6. Duals of hyper- and superspectral spaces

7. Stone-Priestley dualities
Compactness properties in posets

Definition

Let $P$ be a poset and $c$ an element of $P$.

- $c$ is **compact** if $P \uparrow c$ is Scott closed.
- $c$ is **hypercompact** if $P \uparrow c$ is a finitely generated lower set.
- $c$ is **supercompact** if $P \uparrow c$ is a principal ideal.
- $P$ is **algebraic** resp. **hyperalgebraic** resp. **superalgebraic** if each element is a directed join of compact resp. hypercompact resp. supercompact elements.

*These definitions generalize the corresponding topological ones.*
(Quasi-)algebraic and (quasi-)continuous domains

Definition
A domain $P$ is

- **algebraic** iff each principal filter $\uparrow x$ is the intersection of a filterbase of Scott-open cores.
- **quasialgebraic** iff each principal filter $\uparrow x$ is the intersection of a filterbase of Scott-open feet.
- **continuous** iff each principal filter $\uparrow x$ is the intersection of a filterbase of cores having $x$ in their Scott-interior.
- **quasialgebraic** iff each principal filter $\uparrow x$ is the intersection of a filterbase of feet having $x$ in their Scott-interior.
Topology meets algebra
Theorem

1. The category of algebraic domains is isomorphic to the category of supercompactly based sober spaces and dual to the category of superalgebraic (= completely distributive algebraic) frames (ZF).

2. The category of quasialgebraic domains is isomorphic to the category of hypercompactly based sober spaces and dual to the category of hyperalgebraic frames (UP).

3. The category of continuous domains is isomorphic to the category of locally supercompact sober spaces and dual to the category of supercontinuous (completely distributive) frames (ZF).

4. The category of quasicontinuous domains is isomorphic to the category of locally hypercompact sober spaces and dual to the category of hypercontinuous (dually filter distributive) frames (UP).
Coherence for spaces

Definition
A topological space is

- **coherent** if finite intersections of compact upsets are compact.
- **quasicohrent** if finite intersections of cores are compact.
- **open coherent** if finite intersections of compact open sets are compact.

A spectral (hyperspectral, superspectral) space is a compactly (hypercompactly, supercompactly) based coherent sober space.

Lemma
*For Stone spaces the above three notions of coherence are equivalent.*
Coherence for domains and lattices

**Definition**

- A poset $P$ is **quasicoherent** if it is quasialgebraic and the Scott space $\Sigma P$ is coherent.
- A complete lattice is **coherent** (**hypercoherent**, **supercoherent**) if it is algebraic (**hyperalgebraic**, **superalgebraic**) and finite meets of compact elements are compact.

**Lemma**

*Not only every coherent, but even every algebraic complete lattice is quasicoherent.*
Finite prime decompositions

Definition

Let $S$ be a meet-semilattice.

- A **prime ideal** of $S$ is a directed proper downset whose complement is a subsemilattice.
- $S$ is an **np semilattice** if the poset of prime ideals is noetherian (all directed subsets have greatest elements).
- $S$ is an **fp semilattice** if each element is a finite meet of primes.
- $S$ is an **fpi semilattice** if each principal ideal is a finite meet of prime ideals.
- A ring or semilattice has **property $M_f$** if any element $a$ has only finitely many prime ideals maximal w.r.t. not containing $a$. 
Theorem

(ME 2009) *The following statements are equivalent for a space $X$:*

- $X$ is a hyperspectral space.
- $X$ is a spectral space and carries the Scott topology.
- $X$ is the Scott space of a quasicoherent domain.
- $X$ is the spectrum of a hypercoherent frame.
- $X$ is the upper space of a Priestley-Lawson space.
- $X$ is the prime ideal spectrum of a ring with property $M_f$.
- $X$ is the prime ideal spectrum of a distributive lattice with $M_f$.
- $X$ is the prime filter spectrum of an fpi lattice.
Duality for quasicoherent domains and hyperspectral spaces

Theorem (ME 2009)
The category of quasicoherent domains is isomorphic to the category of hyperspectral spaces and to the category of Priestley spaces with the Lawson topology.

These categories are dual to the category of hypercoherent frames (via the open set functor) and to the category of fpi lattices (via Priestley duality).

A similar isomorphism and duality holds for the categories of quasicoherent algebraic domains and superspectral spaces, with supercoherent frames and fp lattices as duals.
The General Stone Duality
The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.
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Stone-Priestley dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.

- Superspatial frames
- Compact elements
- Ideals
- Distributive np semi-lattices

- Prime spectrum with dual order
- Scott open sets
- Compact open sets
- Prime ideal spectrum

- Noetherian domains
- Alexandroff space
- Specialization poset
- Alexandroff sober spaces
Stone-Priestley dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.

- Superalgebraic frames
  - Compact elements
  - Ideals

- Distributive fp semi-lattices

- Prime spectrum
  - With dual order

- Scott space
  - Open sets

- Algebraic domains
  - Specialization poset

- Super-Stone spaces
  - Compact open sets

- Prime ideal spectrum

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The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.
Stone-Priestley dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.

- coherent frames
- compact elements
- ideals
- distributive lattices

- prime spectrum with patch top.
- upper open sets
- compact open sets
- prime ideal spectrum

- Priestley spaces
- upper space
- patch space
- spectral spaces
Stone-Priestley dualities

The two categories in the upper row are equivalent and dual to the isomorphic categories in the lower row.

- algebraic frames
- compact elements
- ideals
- distributive semi-lattices
- prime spectrum
- upper open sets
- compact open sets
- prime ideal spectrum
- upper space
- patch space
- ? spaces
- Stone spaces

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Open end: Mile-Stones of duality