Canonical extensions of double quasioperator algebras: an algebraic perspective on duality for certain algebras with binary operations

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Abstract

The context for this paper is a class of distributive lattice expansions, called double quasioperator algebras (DQAs). The distinctive feature of these algebras is that their operations preserve or reverse both join and meet in each coordinate. Algebras of this type provide algebraic semantics for certain non-classical propositional logics. In particular, MV-algebras, which model the Łukasiewicz infinite-valued logic, are DQAs.

Varieties of DQAs are here studied through their canonical extensions. A variety of this type having additional operations of arity at least 2 may fail to be canonical; it is already known, for example, that the variety of MV-algebras is not. Non-canonicity occurs when basic operations have two distinct canonical extensions and both are necessary to capture the structure of the original algebra. This obstruction to canonicity is different in nature from that customarily found in other settings. A generalized notion of canonicity is introduced which is shown to circumvent the problem. In addition, generalized canonicity allows one to capture on the canonical extensions of DQAs the algebraic operations in such a way that the laws that these obey may be translated into first-order conditions on suitable frames. This correspondence may be seen as the algebraic component of duality, in a way which is made precise. (In this paper the term 'frames' is used in the sense in which it is employed in possible world semantics, rather than denoting the category opposite to that of locales.)

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In many cases of interest, binary residuated operations are present. An operation h which, coordinatewise, preserves $\vee$ and 0 lifts to an operation which is residuated, even when h is not. If h also preserves binary meet then the upper adjoints behave in a functional way on the frames.

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1 Introduction and overview

The motivation for this paper comes principally from algebraic logic. Specifically we seek to develop algebraic techniques involving canonical extensions in order to obtain dualities. Such dualities can profitably be used to analyse and exploit the algebraic semantics of certain non-classical propositional logics. The algebraic context for our investigation is a particular class of distributive lattice expansions (DLEs). Lattice meet and join in a DLE (denoted $\wedge$ and $\vee$) interpret logical conjunction and disjunction; we assume also that the lattices possess bounds 0 and 1, modelling universal falsity and truth. Further logical connectives, such as a negation (unary) or an implication (binary) are modelled by additional operations of the corresponding arities. We focus here on a particular class of DLEs, which we shall call double quasioperator algebras (DQAs); the formal definition is given in the next section. One important example of a variety of DQAs, and the one which led us to investigate them, is provided by MV-algebras. MV-algebras were introduced by C. C. Chang to model the Łukasiewicz infinite-valued logic. They have since been very thoroughly studied both for their logical applications and as algebraic structures in their own right. We note also that there is an important symbiotic relationship between MV-algebras and lattice-ordered groups. An account of MV-algebras can be found in [2], and an update on recent developments in [17].

The operations of a DQA, to be known as double quasioperators, will be allowed to be of any arity. These operations are required to interact well with *both* $\wedge$ and $\vee$ but only to do so coordinatewise. Specifically, they are required, a coordinate at a time, to preserve join and meet or to interchange these. Join and meet in a lattice L correspond respectively to meet and join in its order dual L^∂. Thus the operations with which we deal are those which, after order-reversal ('flipping') in certain coordinates, are simultaneously operators and dual operators, in the sense of the definitions in [10]. Of course, $\wedge$ and $\vee$ in a distributive lattice are themselves double quasioperators since each is an operator and a dual operator (no flipping needed). We emphasize the difference between the condition for a double quasioperator, involving only pairs of elements of the domain which differ in at most one coordinate, and the corresponding condition involving *arbitrary* pairs of elements. For operations of arity $\geqslant 2$, the latter requirement is stronger. This highlights the distinction between DQAs having (besides $\wedge$ and $\vee$) one or more non-unary basic operations and the distributive modal algebras studied in [11]. Each additional basic operation in a DMA is required, after order-reversal in the domain if necessary, to be *either* join- *or* meet-preserving. But, because each operation is assumed to be unary, this preservation occurs globally rather than merely coordinatewise.

One objective which has driven our investigations has been our desire to contribute to the understanding of binary operations. There are several reasons for our wishing to do this. Unary operations have been intensively studied, notably in the context of modal logic, while the theory of operations of higher arity is much more fragmentary. Yet arguably the most fundamental connective in logic—implication—is binary. Here, most attention has been paid to intuitionistic implication. This may be modelled in a Heyting algebra A by the algebraic operation $\to$ given by $a \to b = \bigvee\{\, c \in A \;:\; a \wedge c \leqslant b \,\}$. Important though the intuitionistic case undoubtedly is, it is also extremely special. The operation $\to$ in a Heyting algebra is determined by the underlying order of the lattice reduct. This simplifies matters considerably. Nonetheless the theory of Heyting algebras is both rich and highly complex. The intuitionistic implication sends joins (meets) in the first (second) coordinate to meets but is not in general a double quasioperator. In a linear Heyting algebra implication is a double quasioperator [21]. However the structure of such algebras is too simple for them to be of major significance in the present context.

DQAs provide a framework within which to study an implication which is not order-determined but nonetheless still tractable. The tractability here comes from the strong interaction of double quasioperators with *both* $\wedge$ and $\vee$. In [20] (see also [21]), N. G. Martínez introduces implicative lattices, his primary objective being to study MV-algebras and ℓ-groups, both of which, modulo term-equivalence, can be viewed as varieties of implicative lattices. These algebras are of the form $(A; \vee, \wedge, \to)$, where $(A; \vee, \wedge)$ is a distributive lattice (not necessarily bounded); the implication $\to$ switches join and meet in the first coordinate and preserves join and meet in the second. In Wajsberg's axiomatisation of MV-algebras (in the guise of Wajsberg algebras), the basic operations (together with a nullary operation, 1) are taken to be $\to$ and $\neg$. In an MV-algebra, each of the derived 'arithmetic' binary operations, $\oplus$, $\ominus$ and $\cdot$, is a double quasioperator. In summary, with MV-algebras we have a situation, 'orthogonal' to that in Heyting algebras, but which is nevertheless of importance in logic.

In this paper we consider DQAs in full generality. Applications to particular varieties of algebras of this type will be pursued elsewhere. However, by way of motivation, we give some further pointers to potential applications. We note that a negation, $\neg$, satisfying De Morgan's laws is a unary double quasioperator. More generally, DQAs arise from DLEs with a monoid of unary additional operations each of which is an endomorphism or dual endomorphism of the bounded lattice reduct. Algebras of this type were studied systematically by W. H. Cornish [3], where they are set in a categorical framework, and were further investigated by H. A. Priestley [22] (where they are called Cornish algebras); see also [23]. Here is a selection of well-known varieties coming under this umbrella: Ockham algebras and the subvarieties of Kleene, De Morgan and Stone algebras; double Stone algebras. Now consider an n-ary operation $f : A^n \to A$ on a distributive lattice A. If f is a double quasioperator then f is necessarily order-preserving or order-reversing in each coordinate. The converse is seldom true, but there is one important case in which it certainly is, namely when A is a chain. Many forms of fuzzy logic have a truth-value algebra obtained by equipping the closed unit interval $[0, 1]$ with basic operations of various types; often a De Morgan negation is present. In addition, there will typically be some binary operations meant to model some form of conjunction and disjunction. If these are related via the negation by De Morgan's laws, then the algebra is referred to as a De Morgan system, e.g. see [15] where these are studied from an algebraic point of view. A logic given by a truth-value algebra is modelled algebraically by the variety generated by that truth-value algebra. If the truth-value algebra is based on a chain, as is typically the case for fuzzy logics, then, so long as all the basic operations at least preserve or reverse the order in each coordinate, then the logic is modelled by a variety of DQAs. In fact, MV-algebras are associated with a form of fuzzy logic in just this way.

Although we do not consider them in this paper, we note that nowadays ordered algebras based on non-distributive lattices, or more generally on posets, provide models for logical systems in which one finds notions of implication and fusion that are not tightly tied to the order. Amongst these are various logics, for example BL-logics, which generalize the logic corresponding to MV-algebras. Thus our study here is not just an end in itself but also a first step towards the development of algebraic machinery for analysing substructural logics which can be modelled by lattice or poset expansions with non-unary operations.

Our work here focuses on canonical extensions for DQAs. The study of canonical extensions originated in the famous paper of B. Jónsson and A. Tarski [19] on Boolean algebras with operators (BAOs). Amongst BAOs are the modal algebras which supply semantic models in particular for modal logics; here the connectives include, as well as conjunction and disjunction, a classical negation and modalities, all of which are unary. Fifty years on from Jónsson and Tarski's pioneering work on BAOs, the definitions and theory of canonical extensions have been successfully extended to the much wider setting of DLEs by M. Gehrke, in collaboration with B. Jónsson (see [10] and the references therein) and latterly also with Y. Venema and others (see in particular [11]). R. Goldblatt's landmark paper [18] should also be noted.

Algebraic models of a propositional logic are often obtained very directly from the syntactic description of the logic. This may be through the associated Lindenbaum–Tarski algebras or through a transcription of Gentzen rules. However, semantic modelling by such algebras is often not far removed from the syntactic treatment of the logics, precisely because it is arrived at so directly. Relational semantics on the other hand, when available, are likely to give a significantly different and much more powerful tool. This phenomenon is akin to that whereby algebraists have exploited topological dualities to great advantage. It is well known that canonicity of a class of algebraic models associated with a non-classical logic provides a gateway to a complete relational semantics for the logic. To explain what is at stake here, and its relevance to our work, we outline how canonicity and correspondence can be exploited to this end.

We begin by recalling some basic facts about canonical extensions, which we may view as providing a covariant way of studying lattice expansions via concrete structures. Consider a variety $\mathcal{V}$ of DLEs of a common type. To form the canonical extension of an algebra $A \in \mathcal{V}$ we first construct the canonical extension A^σ of the bounded distributive lattice (DL) reduct of A; this construction is described, for example, in [8] or in Section 2 of [10]. Here we recall, for future reference, that $C = A^\sigma$ is in the class DL^+: it is a doubly algebraic lattice generated by $J^\infty(C)$ (the completely join-irreducible elements) or, equivalently, by $M^\infty(C)$ (the completely meet-irreducible elements); any such lattice is concretely representable as the up-set lattice of a poset (see Section 4 for further detail). For each additional basic operation f, it is then necessary to lift (the interpretation on A of) f up to A^σ, in a manner that is uniform over all algebras in $\mathcal{V}$. Suppose this has been achieved. The variety $\mathcal{V}$ is then said to be *canonical* if it is closed under the formation of canonical extensions. Within the variety $\mathcal{V}$ we have a subclass $\mathcal{C}$ of concrete algebras, namely those $C \in \mathcal{V}$ whose lattice reducts are in DL^+ and whose addtional operations are given by additional relational structure on the dual set of completely join-irreducible elements of the algebra. The import of canonicity is that $\mathcal{C}$ generates $\mathcal{V}$. In fact, $\mathcal{V} = \mathbb{IS}(\mathcal{C})$: the algebra A in $\mathcal{V}$ embeds into the concrete (complex) algebra A^σ, also in $\mathcal{V}$.

A discussion of the many sufficient conditions which are now known to ensure canonicity can be found in the introduction to [10]. However canonicity can fail, and it is a matter of intrinsic interest to discover how and why this happens. Assume we have a variety $\mathcal{V}$ of DLEs and that $A \in \mathcal{V}$. Each element of the canonical extension A^σ is both a join of meets and a meet of joins of elements drawn from the copy of the algebra A embedded in it. As a result, a map $f : B \to C$ between algebras B and C in $\mathcal{V}$ has two natural liftings, f^σ and f^π, both mapping B^σ into C^σ. These canonical extensions may be distinct. Normally, whether it is preferable to choose f^σ or f^π for a given basic operation f will depend on the way that f interacts with the lattice operations. Unsurprisingly, maps for which the σ- and π-extensions agree display particularly good behaviour. Such maps are called *smooth*. Any map which preserves join or meet, or which maps joins to meets or vice versa, is smooth ([10], Corollary 2.28). For example, any De Morgan negation is smooth. Hitherto, the phenomenon of non-canonicity has been most studied, and is best understood, in the context of varieties of DLEs whose non-lattice operations are unary, most notably in varieties of BAOs and, more generally, DMAs. There all operations are smooth and the only obstruction to canonicity comes from what is described as failure of compositionality: the compositions involved in forming term functions from basic operations does not always commute with the formation of the canonical extensions of the maps concerned.

We revealed in [13] the unwelcome fact that the variety of MV-algebras is not canonical, and such misbehaviour may also be expected of other varieties of DQAs with non-unary additional operations. What is especially interesting is that these varieties exhibit an unfamiliar type of non-canonicity and allow us to study this phenomenon in isolation. Failure of compositionality is not the issue in the DQA setting: the results in Section 3 indicate that compositionality, insofar as we require it, does not break down. Instead, non-canonicity happens when basic operations are not smooth and both σ- and π-extensions are necessary to capture the structure of the original algebra. Failure of smoothness of $\oplus$ (it fails for the implication too) is at the heart of the demonstration in [13] that the MV identity

$$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x,$$

known as MV6, is not preserved under a canonical extension based on either the σ- or π-extension of $\oplus$ alone. The problem is that the operations $\neg$ and $\oplus$ are nested in such a way as to force the appearance of both the σ- and π-extensions of $\oplus$ when we attempt to lift MV6 from an arbitrary MV-algebra to the canonical extension of its underlying lattice. A key feature of both MV-algebras and of ℓ-groups is the occurrence of operations which are residuated. Later in this paper we shall highlight the role that residuation plays, especially at the level of the canonical extensions of the abstract algebras from which we start. As we shall see, the extensions of the operations have sufficiently good properties to ensure that residuals exist on these concrete algebras, even when they do not exist for the original operations.

We now turn to correspondence. Consider a variety $\mathcal{V}$ of DLEs, not initially assumed to be canonical. Take any concrete algebra $C \in \mathcal{V}$. The algebraic operations of such a C may be restricted to the subset $X = J^\infty(C)$ of completely join-irreducible elements of C. The laws holding in $C \in \mathcal{V}$ impose 'algebraic' conditions on these restricted operations. Correspondence, as the term is traditionally understood in algebraic logic, is an algorithmic process whereby the algebraic conditions on concrete algebras are converted, via consideration of the restricted operations, into properties of associated relations on frames. This translation process is required to yield properties which are equivalent to the algebraic laws defining $\mathcal{V}$ and which are first-order definable. Such a translation may or may not be available (we return to this issue a little later).

When correspondence does work, we arrive at a dual equivalence, usually referred to as a discrete duality, between the subclass $\mathcal{C}$ of $\mathcal{V}$ and a category of relational frames. This allows one to keep the algebraic perspective while studying the variety via a dual category. Furthermore, the process is uniform, unlike the ad hoc methods originally applied in many particular cases. Its modular character also makes comparison in a hierarchy of varieties easy.

Let $\mathcal{V}$ be a variety of of DLEs and suppose that this class of abstract algebras models some logic, $\mathcal{L}$. Now make the assumptions that $\mathcal{V}$ is a canonical variety and that a discrete duality for the concrete algebras in $\mathcal{V}$ can be obtained by correspondence. In this situation we can refine the discrete duality by adding topology as an overlay. This results in a dual equivalence between the original variety $\mathcal{V}$ and suitable topological frames; here only the topological conditions fail to be first-order. As a consequence, canonicity and correspondence together lead to a complete (topologico-)relational semantics for the logic $\mathcal{L}$. (V. Sofronie-Stokkermans in [24] provides a useful recent overview of these ideas, as they apply to various different logics.) The success or failure of this enterprise, as developed through Sahlqvist theory, hinges on the syntactic shape of the algebraic laws defining $\mathcal{V}$, since this will determine both whether $\mathcal{V}$ is canonical and whether correspondence can be set up. For distributive modal algebras an in-depth analysis of these issues was undertaken in [11]. The theory developed there extends the well-known Sahlqvist theory for BAOs (see [1]). We note also the work of M. Gehrke and M. C. Palma [12], which develops topological dualities for various varieties of semi-De Morgan algebras, a certain class of DMAs. This work is an important forerunner of our study here, in that it presents a unified treatment of canonical extensions, Sahlqvist theory and topological duality. The algebras treated in [12] are DMAs which are canonical but only become Sahlqvist after a type-change has been performed.

Now let us consider what we may call distributive multimodal algebras (DMMAs): here each additional basic operation is, after flipping if necessary, an operator or a dual operator, but may be of any arity. The significance of such algebras for our study will be revealed shortly. Certainly DMMAs provide a common generalization of distributive lattices with operators, as studied in [8]), and the DMAs investigated in [11]. In [8] additional operations of any arity are allowed but no flipping is needed. It is well known that, since all the operations in [8] are already operators, every variety of algebras of this type is canonical. By contrast, the DMAs of [11] only have additional operations which are unary but one has to contend with the consequences of flipping. A large class of DMA laws is identified in [11] which are canonical and have first-order correspondents on frames. The Sahlqvist techniques employed in [11] do not depend directly on the restriction to operations of arity 1. However, a consequence of that restriction is that all the operations of DMAs are smooth, and this fact is relied on heavily in [11]. How much of the theory extends to DMMAs? The notion of canonical extension for a DMMA can be fixed, an appropriate choice of either the σ- or the π-extension being made for each operation. But even if all the basic operations happen to be smooth, canonicity may fail, just as in the unary case. Nevertheless a Sahlqvist-style theory of canonicity and correspondence can be developed, much like the one in [11]. Careful book-keeping is required: only the pertinent choices of σ and π extensions are permitted in the allowable left and right Sahlqvist terms. Such a theory would of course apply directly to DQAs as a special case. But, as we shall see, we can actually use these DMMA ideas to do even better for DQAs by a device we call type-doubling. (This type-change is quite different in character from that employed in [12].)

Take some variety $\mathcal{V}$ of DQAs. We know that canonicity in the traditional sense may fail for $\mathcal{V}$. We now outline how we are able to circumvent this obstacle. The central idea is both novel and conceptually simple: we replace each additional basic operation f by two operations, denoted f^L and f^R; the operation f^L is to be regarded as the quasioperator persona of f, so that its interpretation on the canonical extension A^σ will be the σ-extension of its interpretation on A; order dually, f^R will be viewed as a dual quasioperator, to be interpreted on A^σ via the π-extension. By replacing each non-lattice operation f by the pair of operations f^L and f^R as described above we associate with each algebra A in $\mathcal{V}$ a new algebra $\mathbb{A}$ of a 'doubled type'. For each term s in the original signature we define terms s^L and s^R in the new 'doubled' signature in an algorithmic way, using a signed generation tree for s. The details are given in Section 3. Then we are able to prove that an inequality $s \precsim t$ holding on any $A \in \mathcal{V}$ lifts to the corresponding inequality $s^L \precsim t^R$ interpreted on $\mathbb{A}^\sigma$. This assertion is the core part of our Generalized Canonicity Theorem, Theorem 3.6.

With a view to correspondence, we then interpret our results in a different way. We construct a certain variety $\mathcal{W}$ of DMMAs whose members have the signature of our doubled algebras and whose non-lattice operations occur in pairs f^L and f^R of the same arity, with f^L a quasioperator and f^R a dual quasioperator, and with the flipping to convert f^L to an operator and f^R to a dual operator being carried out in the same set of coordinates in each case. (This is made precise in Section 3.) The subvariety of $\mathcal{W}$ given by the laws $f^L \approx f^R$ contains precisely the doubled versions of the algebras in $\mathcal{V}$. In addition, $\mathcal{W}$ is a canonical variety of DMMAs; it is specified by laws of the form $s^L \precsim t^R$ and such laws can be shown to be Sahlqvist, of the simplest kind. Indeed, the term s^L can be viewed as a composition of 'multimodal diamonds' and the term t^R as a composition of 'multimodal boxes'. All such laws are guaranteed to translate into first-order definable properties of frames. Therefore we can be sure that a discrete duality is, at the least, available for $\mathcal{W}$.

We thereby pinpoint the laws $f^L \approx f^R$ as being the root cause of the difficulties posed by DQAs as regards correspondence. Let us examine this problem more closely with attention restricted to algebras A^σ (that is, consider pseudo-correspondence rather than correspondence; see [25] and note also [12], where the same restriction proved necessary and flipping is also crucially involved). We need to be able to recognize under what conditions two operations f^L and g^R interpreted on a canonical extension come from a common operation $f = g$ on A. In our companion paper [14] we reveal that the solution of this problem brings in topology. Not only that: topology is involved in a way that goes beyond its customary function, that of capturing how an algebra A sits inside A^σ.

We consider in [14] topologico-relational duality for the variety of all DQAs of a given type. Two formulations are given. The bi-relational duality associates with each additional basic operation h relations $R = R_{h^\sigma}$ and $S = S_{h^\pi}$ reflecting the two personae of h. Thanks to the good correspondence behaviour of inequalities of the form $s^L \precsim t^R$ in the doubled type, correspondents for these are available. Critically, correspondence conditions for the identities $h^L \approx h^R$ can be specified in terms of the relations R and S in such a way as to yield pseudo-correspondence. The uni-relational duality is available for varieties of DQAs satisfying an additional condition, akin to normality but significantly weaker, which holds in the particular varieties motivating this paper. In this duality the single relation $T = R \cap S$ is associated with h; pseudo-correspondence is again available. In this duality the doubling of the type, although essential to its derivation, is no longer visible in the end result.

We conclude this overview with a summary of what we accomplish in this paper. The focus is on algebraic aspects of duality for DQAs. We present the Generalized Canonicity Theorem as a purely algebraic result. We discuss how frames should be defined in our context. Working in the varieties $\mathcal{W}$, we carry through the first stage of the correspondence process for inequalities of the form $s^L \precsim t^R$, giving correspondents in terms of the restrictions to frames of the liftings of the algebraic operations. This translation may be viewed as providing duality in an algebraic guise. We do not include here the development of the topological part of our duality theory. This is presented in full in [14] and requires techniques quite different from those employed here. However we do outline at the end of Section 5 the form that this duality takes. This allows us to demonstrate to any sceptic that the algebraic correspondents given in our Algebraic Correspondence Theorem do convert into first-order properties of the dual relational structures, in either their bi-relational or their uni-relational form. The methodology for this is, as we have pointed out, Sahlqvist methodology, and not essentially new. However there are special features in our setting which make it worthwhile to describe explicitly how it works out.

2 Double quasioperators and the calculus of flipping

We shall usually use the same symbol to denote both a (bounded distributive) lattice, or a DLE, and its underlying set, making the type explicit only where this is necessary. As already noted, we shall need to consider, along with any bounded distributive lattice $(A; \vee, \wedge, 0, 1)$, also the dual lattice $A^\partial = (A; \wedge, \vee, 1, 0)$, obtained by reversing the underlying order. Sometimes we need also to tag the unflipped lattice A, and in this situation will denote it by A^1. Now let $\boldsymbol{\varepsilon} \in \{1, \partial\}^n$. Sometimes, for notational convenience, we write ε_i in place of $\boldsymbol{\varepsilon}(i)$. For brevity we shall write $A^{\boldsymbol{\varepsilon}}$ for $A^{\boldsymbol{\varepsilon}(1)} \times A^{\boldsymbol{\varepsilon}(2)} \times \cdots \times A^{\boldsymbol{\varepsilon}(n)}$. Expressed another way, the ordered set $A^{\boldsymbol{\varepsilon}}$ is the set of n-tuples of elements of A equipped with the order $\leqslant_{\boldsymbol{\varepsilon}}$ given by

$$(a_1, \dots, a_n) \leqslant_{\boldsymbol{\varepsilon}} (b_1, \dots, b_n) \iff \text{ for } i = 1, \dots, n, \quad \begin{cases} a_i \leqslant b_i & \text{if } \varepsilon_i = 1, \\ a_i \geqslant b_i & \text{if } \varepsilon_i = \partial. \end{cases}$$

We write $A^{\mathbf{n}}$ in place of $A^{\boldsymbol{\varepsilon}}$ when $\boldsymbol{\varepsilon}$ is the n-tuple $(1, \dots, 1)$.

A DL map $f : A \to B$ which sends joins to meets and meets to joins becomes both join- and meet-preserving when regarded as a map from A^∂ to B. More generally, consider any DL map $f : A^n \to B$ which is monotone in the sense that it is either order-preserving or order-reversing as a function of each of its coordinates. Then there exists $\boldsymbol{\varepsilon}$ such that f, now regarded as a map (which we still denote by f) from $A^{\boldsymbol{\varepsilon}} = A^{\boldsymbol{\varepsilon}(1)} \times \ldots \times A^{\boldsymbol{\varepsilon}(n)}$ to B, is order-preserving in all coordinates. We follow [11] in not designating by different symbols maps which are set-theoretically identical, but whose domains $A^{\mathbf{n}}$ and $A^{\boldsymbol{\varepsilon}}$ are differently ordered; see Remark 2.16 in [11] concerning such *order-variants*.

We say that an n-ary operation $f : A^n \to A$ on a DL A is

- an *operator*, resp. a *dual operator*, provided that, for each $i \in \{1, \dots, n\}$, the function obtained by fixing all arguments except the ith one in f preserves join, resp. meet;
- an $\boldsymbol{\varepsilon}$-*quasioperator*, resp. an $\boldsymbol{\varepsilon}$-*dual quasioperator*, if $f : A^{\boldsymbol{\varepsilon}} \to A$ is an operator, resp. a dual operator, and in this case we say that $\boldsymbol{\varepsilon}$ is the *monotonicity type* of f;
- a *quasioperator* if f is an $\boldsymbol{\varepsilon}$-quasioperator for some monontonicity type $\boldsymbol{\varepsilon}$;

- a *double operator* if f is both an operator and a dual operator and a *double quasioperator* if there exists $\boldsymbol{\varepsilon} \in \{1, \partial\}^n$ such that $f : A^{\boldsymbol{\varepsilon}} \to A$ is a double operator and in that case we say that f is a $\boldsymbol{\varepsilon}$-double quasioperator.

In case A is a complete lattice, we analogously define a *complete operator*, and so on, by requiring preservation of arbitrary non-empty joins (meets) instead of the binary joins (meets) above; see [10].

We define a *double quasioperator algebra* to be a DLE of the form $(A, \vee, \wedge, 0, 1, \{f_\lambda\}_{\lambda\in\Lambda})$ in which each operation is a double quasioperator. We call the sequence of monotonicity types of the additional operations of A, *viz.* $(\boldsymbol{\varepsilon}_\lambda)_{\lambda\in\Lambda}$, the monotonicity type of the algebra. Henceforth we shall suppress reference to the (fixed) index set Λ and write $\{f_\lambda\}$ rather than $\{f_\lambda\}_{\lambda\in\Lambda}$, etc.)

An additional basic operation in a DQA can be viewed as a double operator once certain coordinates in the domain have been flipped. We shall show more generally that any term of a DQA arises from a composition of double operators after suitable flipping. For non-atomic terms, both domain and codomain may be simultaneously involved in this flipping process. The way in which this happens is described in detail in [11], Remark 2.16. For our present purposes, what is important is the fact that flipping of the codomain of a map can affect the manner in which the map extends to the canonical extension of its domain, so that we must indicate such flipping explicitly. Accordingly, for a map $f : A \to B$ between DLs A and B, we denote by f^∂ the map with the same action as f but with both the domain and the codomain of f flipped, that is, $f^\partial : A^\partial \to B^\partial$.

The proposition below is conceptually simple, but heavy on notation in the general case. To provide motivation, let us illustrate what happens in the very simple case in which we have a term $f(g(x))$ interpreted on an algebra A, where f and g are unary $\boldsymbol{\varepsilon}$-double quasioperators each of monotonicity type $\boldsymbol{\varepsilon} = (\partial)$. (In other words, $f : A^\partial \to A$ and $g^\partial : (A^\partial)^\partial \to A^\partial$ act as dual endomorphisms.) Observe that

$$f \circ g^\partial : A = \left(A^\partial\right)^\partial \longrightarrow A^\partial \longrightarrow A$$

is a composition of double operators. Notice how the use of g^∂ ensures that A^∂ is passed to f to serve as its domain, in such a way that (an order-variant of) f will act as a double operator.

Proposition 2.1. *Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type and let $t(x_1, \dots, x_n)$ be a term of the corresponding similarity type in which each variable occurs at most once. Then for each double quasioperator algebra $(A; \vee, \wedge, 0, 1, \{f_\lambda\})$ of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$, the term operation t^A may be viewed as a composition of double operators, given an appropriate flipping of coordinates.*

Proof. The term algebra on n variables is generated by the variables as an algebra of the given type, and we prove the proposition by algebraic induction. The term function associated with a variable is a projection, which is both meet- and join-preserving. Now suppose $t_1, \dots, t_n$ are terms for which the proposition holds, and f is an n-ary basic operation symbol. Let A be a double quasioperator algebra of type $(\boldsymbol{\varepsilon}_\lambda)$. Then for each $j \in \{1, \dots, n\}$, there is $\zeta_j \in \{1, \partial\}^{m_j}$, where m_j is the number of variables in t_j, so that the term operation $t_j^A : A^{\zeta_j} \to A$ can be obtained as a composition of double operators. Also, there exists $\boldsymbol{\varepsilon} \in \{1, \partial\}^n$ so that $f^A : A^{\boldsymbol{\varepsilon}} \to A$ preserves both joins and meets in each coordinate. Now the term operation given by $f(t_1, \dots, t_n)$ on A can be viewed as a composition of double operators via the composition

$$
\begin{aligned}
&\left[\left(A^{\zeta_1}\right)^{\boldsymbol{\varepsilon}(1)} \times \ldots \times \left(A^{\zeta_n}\right)^{\boldsymbol{\varepsilon}(n)}\right] \\
&\quad \xrightarrow{\left((t_1^A)^{\boldsymbol{\varepsilon}(1)}\right)^*} \left[A^{\boldsymbol{\varepsilon}(1)} \times \left(A^{\zeta_2}\right)^{\boldsymbol{\varepsilon}(2)} \times \ldots \times \left(A^{\zeta_n}\right)^{\boldsymbol{\varepsilon}(n)}\right] \\
&\quad \xrightarrow{\left((t_2^A)^{\boldsymbol{\varepsilon}(2)}\right)^*} \ldots \; \ldots \; \ldots \\
&\quad \xrightarrow{\left((t_{n-1}^A)^{\boldsymbol{\varepsilon}(n-1)}\right)^*} \left[A^{\boldsymbol{\varepsilon}(1)} \times A^{\boldsymbol{\varepsilon}(2)} \times \ldots \times A^{\boldsymbol{\varepsilon}(n-1)} \times \left(A^{\zeta_n}\right)^{\boldsymbol{\varepsilon}(n)}\right] \\
&\quad \xrightarrow{\left((t_n^A)^{\boldsymbol{\varepsilon}(n)}\right)^*} \left[A^{\boldsymbol{\varepsilon}(1)} \times A^{\boldsymbol{\varepsilon}(2)} \times \ldots \times A^{\boldsymbol{\varepsilon}(n)}\right] \\
&\quad \xrightarrow{f^A} A.
\end{aligned}
$$

Here the symbol $()^*$ indicates that the map in the parentheses acts only in the appropriate coordinates while all other coordinates are left alone. $\square$

We now need to deal with terms in which repeated variables may occur. Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type and let $t(x_1, \ldots, x_n)$ be any term of the corresponding similarity type. Let $t'(y_1, \ldots, y_m)$ be the term obtained from t by replacing the variable occurrences in t by distinct variables. That is, $m \geqslant n$ and $t'(y_1, \ldots, y_m)$ is a term in which each variable occurs at most once. Observe that t can be reclaimed from t' via a suitable substitution map, η_t say. For each double quasioperator algebra $(A; \vee, \wedge, 0, 1, \{f_\lambda\})$ of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$, we have that $t^A = s^A \circ \delta$ where $\delta : A^{\mathbf{n}} \to A^{\mathbf{m}}$ is the diagonal map with $(\delta(a_1, \ldots, a_n))_j = a_i$ if and only if y_j replaces an occurrence of x_i; in other words, $(\delta(a_1, \ldots, a_n))_j = a_i$ if and only if $\eta_t(y_j) = x_i$. We have the following corollary of Proposition 2.1.

Proposition 2.2. *Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type and let $t(x_1, \ldots, x_n)$ be a term of the corresponding similarity type. For each double quasioperator algebra $(A, \vee, \wedge, 0, 1, \{f_\lambda\})$ of type $(\boldsymbol{\varepsilon}_\lambda)$, given an appropriate flipping of coordinates, the term operation t^A may be viewed as a composition made up of some diagonal map followed by double operators.*

To take a simple example, let f and g be binary $\boldsymbol{\varepsilon}$-double quasioperators each of monotonicity type $\boldsymbol{\varepsilon} = (\partial, 1)$. Interpreted on an algebra A, the term t with $t(x, y, z) = f(g(x, y), z)$ can be viewed as composition of double operators $t^A : (x, y, z) \mapsto f^A((g^\partial)^A(x, y), z)$ with $t^A : A \times A^\partial \times A \to A$. Even if f and g happen to be occurrences of the same operation, they need to be kept distinct, since a map and its flip generally behave differently under extensions. By contrast, the term $t(x, y, z) = f(x, g(y, z))$ gives rise on A simply to an order-variant of a double operator, with domain $A^\partial \times A^\partial \times A$; in this case neither f nor g needs to be flipped.

It is instructive to view DQA terms also in terms of their generation trees. Given a similarity type $(\boldsymbol{\varepsilon}_\lambda)$ and a term of that type, we assign a sign, either $+$ (plus) or $-$ (minus), to each node of its generation tree in the following way:

(1) Assign $+$ to the root node.

(2) If a node is $\vee$ or $\wedge$, assign the same sign to its successor nodes.

(3) If a node is f_λ for some λ, then for each $j \in \{1, \dots, n_\lambda\}$ assign the same sign to the jth successor provided $\varepsilon_j = 1$ and assign the opposite sign to the jth successor node provided $\varepsilon_j = \partial$.

For a term $t(x_1, \dots, x_n)$ and for $(A; \vee, \wedge, 0, 1, \{f_\lambda\})$ a double quasioperator algebra of some corresponding monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$, the composition of double operators obtained for the term operation t^A through Proposition 2.1 will have some occurrences of each basic operation symbol f_λ interpreted as the double operator $f_\lambda^A : A^{\boldsymbol{\varepsilon}_\lambda} \to A$ and it will have others interpreted as the dual function $\left(f_\lambda^A\right)^\partial : (A^{\boldsymbol{\varepsilon}_\lambda})^\partial \to A^\partial$. One can show that a given occurrence of f_λ will be interpreted as $f_\lambda^A : A^{\boldsymbol{\varepsilon}_\lambda} \to A$ provided that occurrence of f_λ carries a $+$ in the signed generation tree for t, and that a given occurrence of f_λ will be interpreted as $\left(f_\lambda^A\right)^\partial : (A^{\boldsymbol{\varepsilon}_\lambda})^\partial \to A^\partial$ provided that occurrence of f_λ carries a $-$ in the signed generation tree for t.

3 Generalized canonicity

We follow [10] in the way in which we define the canonical extension A^σ of the DL reduct $(A; \vee, \wedge, 0, 1)$ of a DQA and the extensions of a monotone operation f:

$$\begin{aligned}
f^\sigma(p) &= \bigwedge\{\, f(a) \ : \ a \in A \text{ and } a \geqslant p \,\} \text{ for all } p \in K,\\
f^\pi(q) &= \bigvee\{\, f(a) \ : \ a \in A \text{ and } a \leqslant q \,\} \text{ for all } q \in O;\\
f^\sigma(\alpha) &= \bigvee\{\, f^\sigma(p) \ : \ p \in K \text{ and } p \leqslant \alpha \,\} \text{ for all } \alpha \in A^\sigma,\\
f^\pi(\alpha) &= \bigwedge\{\, f^\pi(q) \ : q \in O \text{ and } \alpha \leqslant q \,\} \text{ for all } \alpha \in A^\sigma.
\end{aligned}$$

Here, as usual, K and O denote, respectively, the sets of closed and open elements of A^σ, that is, respectively meets of and joins of elements drawn from (the image of) A in A^σ. We note that both σ- and π-extensions of the underlying lattice operations are the corresponding lattice operations on the canonical extension. We also need the elementary facts about the extensions collected together below; for a fuller discussion see Section 2 of [10]. To see why (2) and (3) hold note that the maps on each side of the equation are simply order-variants of each other and so set-theoretically equal.

Lemma 3.1. *Let $f : A^n \to A$ be a monotone map of monotonicity type $\boldsymbol{\varepsilon}$. Then*

(1) $(A^{\boldsymbol{\varepsilon}})^\sigma = (A^\sigma)^{\boldsymbol{\varepsilon}}$;

(2) $(f : A^{\boldsymbol{\varepsilon}} \to A)^\sigma = (f : A^{\mathbf{n}} \to A)^\sigma$;

(3) $\left(f^\partial : (A^{\boldsymbol{\varepsilon}})^\partial \to A^\partial\right)^\sigma = (f : A^{\mathbf{n}} \to A)^\pi$, *and likewise with σ and π interchanged.*

The extensions of double operators, and more generally of $\boldsymbol{\varepsilon}$-double operators, are central to our investigations. We record their properties below.

Proposition 3.2. *Suppose that $h : A^n \to A$ is a double operator. Then*

(1) *h^σ is a complete operator and h^π is a complete dual operator;*

(2) h^σ *is a (not necessarily complete) dual operator and* h^π *is a (not necessarily complete) operator.*

Analogous statements can be made concerning the extensions of a $\boldsymbol{\varepsilon}$*-double operator.*

Proof. (1) is well known; see for example Theorem 2.20 of [10]. For (2), the claim is that the equations stating that h is a dual operator and an operator are σ- and π-canonical. The validity of the claim follows from Theorem 4.6 in [8]. $\square$

We remark that it is not generally the case that the σ-extension of a double operator h is a complete dual operator. An example is provided by our analysis in [13] of the extension of double operator $\oplus : A \times A \to A$ when A is the Chang chain. Here $A^\sigma \setminus A$ consists of two points $\{x, y\}$; here x covers y in the order of A^σ and we have $x \in J^\infty(A^\sigma)$ and $y \in M^\infty(A^\sigma)$. We showed in [13] that $f^\sigma(x, y) \neq f^\pi(x, y)$, where f denotes $\oplus$. From general theory (see for example [10], Theorem 2.20), we know that $f^\sigma = f^\pi$ on the open elements of $(A^\sigma)^2$ and in particular on every element (a, y) with $a \in A$ and $a > x$. The meet of such elements a is x. If f^σ were to preserve arbitrary non-empty meets in the first coordinate, we would have $f^\sigma(x, y) = f^\pi(x, y)$, which is untrue.

We have already noted that the extensions h^σ and h^π of a double operator (or more generally an $\boldsymbol{\varepsilon}$-double operator) h on A in general do not coincide. When this occurs neither h^σ nor h^π alone will fully reflect, in terms of the preservation of infinitary operations, the preservation properties of the original h. It is then necessary on the canonical extension to consider *both* h^σ and h^π.

We now introduce the algebraic formalism of 'type-doubling' which allows us to accommodate both extensions of each non-lattice basic operation. Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type and $(2, 2, 0, 0, \{n_\lambda\})$ the corresponding similarity type. Let $\mathcal{V}_\Lambda$ be the variety of all DQAs $A = (A; \vee, \wedge, 0, 1, \{f_\lambda\})$ of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$. Now let $\mathcal{V}_{\mathbb{A}}$ be the variety of all algebras $(A; \vee, \wedge, 0, 1, \{f_\lambda^L\}, \{f_\lambda^R\})$ of similarity type $(2, 2, 0, 0, \{n_\lambda\}, \{n_\lambda\})$ with the property that f_λ^L is an $\boldsymbol{\varepsilon}_\lambda$-operator and f_λ^R is an $\boldsymbol{\varepsilon}_\lambda$-dual operator for each λ. Here $\mathcal{V}_{\mathbb{A}}$ is a variety of DMMAs, and consists of all algebras of the given type; in modal terminology, each f_λ^L (f_λ^R) is a $\boldsymbol{\varepsilon}$-$\Diamond$ ($\boldsymbol{\varepsilon}$-$\Box$), not necessarily normal.

Let $A = (A; \vee, \wedge, 0, 1, \{f_\lambda\}) \in \mathcal{V}_\lambda$. We define the *(generalized) canonical extension* of A to be the algebra $\mathbb{A}^\sigma = (A^\sigma; \vee, \wedge, 0, 1, \{f_\lambda^\sigma\}, \{f_\lambda^\pi\})$. Note that $\mathbb{A}^\sigma$ is an algebra of the doubled type, rather than being of the same type as the original algebra A.

Before we can present the main result of this paper, asserting that all varieties of double quasioperator algebras are canonical in a generalized sense, we need to explain how to carry the equational properties of the original double quasioperator algebras into the setting with the doubled type. To this end we make the following definition. Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type. For each term t in the associated DQA similarity type we define terms t^L and t^R in the doubled type by induction over the complexity of the term.

(1) For each variable x, define $x^L = x^R = x$.

(2) Define $\vee^L = \vee^R = \vee$ and $\wedge^L = \wedge^R = \wedge$. Also define $0^L = 0^R = 0$ and $1^L = 1^R = 1$.

(3) Let $\lambda \in \Lambda$ and suppose $t_1, \ldots, t_{n_\lambda}$ are terms t for which both t^L and t^R have been defined. Then, for $t = f_\lambda(t_1, \ldots, t_{n_\lambda})$, define

$$t^L = f_\lambda^L(t_1^{X_1}, \ldots, t_{n_\lambda}^{X_{n_\lambda}}) \quad \text{where, for } 1 \leqslant j \leqslant n_\lambda, \quad \begin{cases} X_j = L & \text{if } \boldsymbol{\varepsilon}_\lambda(j) = 1, \\ X_j = R & \text{if } \boldsymbol{\varepsilon}_\lambda(j) = \partial \end{cases}$$

and

$$t^R = f_\lambda^R(t_1^{X_1}, \ldots, t_{n_\lambda}^{X_{n_\lambda}}), \qquad \text{where, for } 1 \leqslant j \leqslant n_\lambda, \quad \begin{cases} X_j = R & \text{if } \boldsymbol{\varepsilon}_\lambda(j) = 1, \\ X_j = L & \text{if } \boldsymbol{\varepsilon}_\lambda(j) = \partial. \end{cases}$$

The following lemma is easily proved by induction over the complexity of the term t.

Lemma 3.3. *Let $(\boldsymbol{\varepsilon}_\lambda)$ be a monotonicity type and let $t(x_1, \ldots, x_n)$ be a term of the corresponding similarity type. Then*

(1) *a given occurrence of a basic operation f_λ in t will be replaced by f_λ^L, resp. f_λ^R, in t^L provided that occurrence of f_λ carries a $+$, resp. a $-$, in the signed generation tree for t;*

(2) *a given occurrence of f_λ in t will be replaced by f_λ^L, resp. f_λ^R, in t^R provided that occurrence of f_λ carries a $-$, resp. a $+$, in the signed generation tree for t.*

We note that when we combine the above lemma with our definition of the canonical extension $\mathbb{A}^\sigma$ of a DQA $\mathbb{A}$ in the doubled type we obtain, inductively, our intended interpretations of t^L and of t^R on $\mathbb{A}^\sigma$. Of course, the passage from t to t^L (or t^R) will not in general result in different occurrences in t of the same basic operation being replaced by the same choice of extension in the corresponding term t^L; which extension is chosen at any node of the appropriate signed generation tree for t is dictated by the sign attached to that node. For an illustration see the example following Corollary 3.7.

We may view terms of the form s^L and t^R as DMMA terms. As such, they may be fitted into a Sahlqvist framework, generalizing that developed in [11]. A DMMA term whose signed generation tree has the property that all nodes labelled with $+$ are nodes of quasioperators ($\vee$, $\wedge$, or $\boldsymbol{\varepsilon}$-$\Diamond$s) and all nodes labelled with $-$ are nodes of dual quasioperators ($\vee$, $\wedge$, or $\boldsymbol{\varepsilon}$-$\Box$s) may be called *choice terms*, cf. Definition 3.1 in [11]). These are particularly simple left-Sahlqvist terms. DMMA terms with the dual property may then be called *dual choice terms* and these are particularly simple right-Sahlqvist terms. In this terminology the above lemma simply says that the translation t^L of a term t for $\mathcal{V}_\Lambda$ is a choice term for the DMMA variety $\mathcal{V}_{\mathbb{A}}$ and that t^R is a dual choice term for that variety.

We express the semantic consequence of our construction in the following proposition. This is easily seen to be a consequence of Lemma 3.3, of the arguments used to derive Proposition 2.2 and of Proposition 3.2.

Proposition 3.4. *Let $(A, \vee, \wedge, 0, 1, \{f_\lambda\})$ be a double quasioperator algebra of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$, let s and t be terms of the corresponding similarity type, and let $C = \mathbb{A}^\sigma$. Then there exists a flipping of coordinates such that, after flipping, the term s^L, when interpreted on C, is a composition in which some diagonal map is followed by complete operators. Likewise the interpretation of t^R on C can, after flipping, be expressed as a composition in which some diagonal map is followed by complete dual operators. (Here the flippings to be used are uniquely determined by the signed generation trees for the terms s and t, respectively.)*

In order to prove one of the implications in our generalized canonicity theorem, we will need a result about compositions of extensions of maps. Here we give the necessary statement as a lemma and explain how it follows from results in [10]. We refer the reader to that paper for the definitions of the topologies involved.

Lemma 3.5. *Let A, B and C be DLs and let $f : A \to B$ and $g : B \to C$ be arbitrary maps. Then*

(1) *if g is an operator then $g^\sigma f^\sigma \leqslant (gf)^\sigma$;*

(2) *if g is a dual operator then $(gf)^\pi \leqslant g^\pi f^\pi$.*

Proof. For every DL map $f : A \to B$ the extension f^σ is $(\sigma, \iota^\uparrow)$-continuous (see [10], Theorem 2.24). Also, if g is an operator then the extension g^σ is $(\iota^\uparrow, \iota^\uparrow)$-continuous (see [10], Theorem 2.24(ii)). Thus we have that the composition $g^\sigma f^\sigma$ is $(\sigma, \iota^\uparrow)$-continuous. Now by [10], Theorem 2.30(i) we have that $g^\sigma f^\sigma \leqslant (gf)^\sigma$.

Statements about the π-extensions are dual. For every DL map $f : A \to B$, the extension f^π is $(\sigma, \iota^\downarrow)$-continuous, and for a dual operator g we have that g^π is $(\iota^\downarrow, \iota^\downarrow)$-continuous. This implies that $g^\pi f^\pi$ is $(\sigma, \iota^\downarrow)$-continuous, and thus $(gf)^\pi \leqslant g^\pi f^\pi$. □

We now state the main canonicity result of this paper.

Theorem 3.6. (Generalized Canonicity Theorem) *Let $A = (A; \vee, \wedge, 0, 1, \{f_\lambda\})$ be a double quasioperator algebra of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$ and let $s \approx t$ be an identity of the corresponding similarity type. Then the following statements are equivalent:*

(1) *$s \approx t$ holds in $A = (A; \vee, \wedge, 0, 1, \{f_\lambda\})$;*

(2) *$s^L \precsim t^R$ and $t^L \precsim s^R$ hold in $\mathbb{A} = (A; \vee, \wedge, 0, 1, \{f_\lambda\}, \{f_\lambda\})$;*

(3) *$s^L \precsim t^R$ and $t^L \precsim s^R$ hold in $\mathbb{A}^\sigma = (A^\sigma; \vee, \wedge, 0, 1, \{f_\lambda^\sigma\}, \{f_\lambda^\pi\})$.*

Proof. The fact that statements (1) and (2) are equivalent is completely trivial since $s^A = t^A$ holding is equivalent to $s^A \leqslant t^A$ and $t^A \leqslant s^A$, holding for any DQA A, and both f_λ^L and f_λ^R are interpreted as f_λ in $\mathbb{A}$. The fact that statement (3) implies statement (2) is also clear as both f_λ^σ and f_λ^π restrict to f_λ on A.

Finally, for the implication (2) $\implies$ (3) we use Lemma 3.5 in conjunction with Proposition 2.2. That is, by Proposition 2.2, we have that $\left(s^L\right)^{\mathbb{A}}$ can be viewed as the composition of a diagonal map followed by operators. Now, as remarked above, any occurrence of a basic operation f_λ in s that is replaced by f_λ^L occurs as $f_\lambda^A : A^{\boldsymbol{\varepsilon}_\lambda} \to A$ in the composition, and any occurrence of a basic operation f_λ in s that is replaced by f_λ^R occurs as $\left(f_\lambda^A\right)^\partial : (A^{\boldsymbol{\varepsilon}_\lambda})^\partial \to A^\partial$ in the composition. Finally $\left(f_\lambda^A : A^{\boldsymbol{\varepsilon}_\lambda} \to A\right)^\sigma = \left(f_\lambda^A : A^{\mathbf{n}_\lambda} \to A\right)^\sigma$ and $\left(\left(f_\lambda^A\right)^\partial : (A^{\boldsymbol{\varepsilon}_\lambda})^\partial \to A^\partial\right)^\sigma = \left(f_\lambda^A : A^{\mathbf{n}_\lambda} \to A\right)^\pi$. (Here we recall from Section 2 that we write $A^{\mathbf{n}}$ in place of $A^{\boldsymbol{\varepsilon}}$ when $\boldsymbol{\varepsilon}$ is the n-tuple $(1, \dots, 1)$, so that $\mathbf{n}_\lambda$ denotes the n_λ-ary monotonicity type $(1, \dots, 1)$.) We also know that $f_1^\sigma \leqslant f_1^\pi \leqslant f_2^\pi$ for any DL maps f_1 and f_2 with $f_1 \leqslant f_2$. By Lemma 3.5 it follows, by induction on the complexity of the terms on each side of the given inequality, that applying σ to the individual maps in this composition yields a lesser result than applying σ to the composition and the result follows. □

For operations f_λ which are smooth, in particular for unary double quasioperators, we do not need to distinguish between the two manifestations f_λ^L and f_λ^R of f_λ in the doubled type. In such cases we can, and shall, suppress the superscripts.

The following corollary to the Generalized Canonicity Theorem is worth recording. This result has not been explicitly stated before but is in fact a special case of Theorem 25 of [9] or of results in Section 3 of [11].

Corollary 3.7. *Let $\mathcal{V}$ be a variety of DQAs each of whose additional basic operations is smooth. Then $\mathcal{V}$ is canonical (in the traditional sense). In particular any variety of DQAs each of whose additional basic operations is unary is canonical.*

Proof. For the first assertion it is enough to note that a trivial induction shows that, on a canonical extension, s^L and s^R are interpreted in the same way.

The final assertion follows from the first and the fact that unary double quasioperators are smooth (see for example [10], Corollary 2.28). $\square$

We now present an illustrative example, namely the non-canonical MV identity

$$\neg(\neg x \oplus y) \oplus y \approx \neg(\neg y \oplus x) \oplus x;$$

$\neg$ is unary, of monotonicity type (∂) and $\oplus$ is a double operator. For the term $s(x, y, z) = \neg(\neg x \oplus y) \oplus z$, the signed generation tree with initial condition $+$ is shown in Fig. 1.

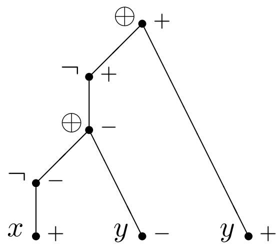

Fig. 1

We have

$$s^L(x, y, y) = \neg(\neg x \oplus^R y) \oplus^L y \text{ and } s^R(y, x, x) = \neg(\neg x \oplus^L y) \oplus^R y.$$

It follows that $s(x, y, y) \approx s(y, x, x)$ holds in an MV-algebra A if and only if

$$\neg(\neg x \oplus^R y) \oplus^L y \precsim \neg(\neg y \oplus^L x) \oplus^R x$$

holds in the algebra $\mathbb{A}^\sigma$, where $\oplus^L$ and $\oplus^R$ are interpreted, respectively, as $\oplus^\sigma$ and $\oplus^\pi$. Of course, in this case $s^L \precsim t^R$ and $t^L \precsim s^R$ both yield the same condition.

In Section 1 we asserted that it is possible to extend canonicity results for distributive modal algebras given in [11] to algebras having non-lattice operations of arity > 1. We now elaborate on this, showing how our Generalized Canonicity Theorem can be fitted into this framework.

Assume given a variety $\mathcal{V}$ of DQAs of similarity type $(2, 2, 0, 0, \{n_\lambda\})$ and monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$. As described above, $\mathcal{V}$ is a subvariety of $\mathcal{V}_*$, the variety of all DQAs of monotonicity type $(\boldsymbol{\varepsilon}_\lambda)$. Define as above the associated variety $\mathcal{V}_{\mathbb{A}}$ of all DMMAs of the doubled type. Now let $\mathcal{W}$ be the subvariety of $\mathcal{V}_{\mathbb{A}}$ specified by the set of inequalities

$$\{\, s^L \precsim t^R \text{ and } t^L \precsim s^R : s \approx t \text{ holds in } \mathcal{V}\,\}.$$

For every algebra in $\mathcal{V}_{\mathbb{A}}$, and thus for every algebra in $\mathcal{W}$, we may define the (non-generalized) canonical extension by taking the σ-extension of basic operations with superscript L and the π-extension of basic operations with superscript R. Then we may see the crux of our Generalized Canonicity Theorem as the following DMMA canonicity result, in which canonicity has its traditional meaning.

Theorem 3.8. *The variety $\mathcal{W}$, as defined above, is canonical.*

The proof is simply an extension to the non-unary case of a restricted version of the Sahlqvist arguments given in [11]. Alternatively (and equivalently) we can prove the result just the same way as we did the implication (2) $\implies$ (3) in the Generalized Canonicity Theorem. The subvariety of $\mathcal{W}$ specified by the additional laws $f^L_\lambda \approx f^R_\lambda$, for each λ, can be identified with the original variety $\mathcal{V}$. The Generalized Canonicity Theorem is then simply the specialization of Theorem 3.8 to the algebras satisfying the laws $f^L_\lambda \approx f^R_\lambda$.

4 Canonical extensions and residuation

Many of the operations arising in algebraic logic, and in particular many of the double quasi-operators of interest to us, are residuated binary maps. For such maps, canonical extensions work in a special way. Residuation arises in many different contexts and, like other ubiquitous mathematical notions, a variety of notation is employed. Consider posets P, Q, R and suppose that operations $\circ$ ('fusion') and $\rightarrow$ and $\leftarrow$ are linked by, for all $p \in P$ $q \in Q$ and $r \in R$, by

$$\begin{aligned} p \circ q \leqslant r &\iff q \leqslant p \rightarrow r \\ &\iff p \leqslant r \leftarrow q \end{aligned}$$

Then $\rightarrow$ is said to be the *right residual* of $\circ$ and $\leftarrow$ the *left residual.* These same conditions are frequently stated in the alternative notation:

$$p \cdot q \leqslant r \iff q \leqslant p \backslash r \iff p \leqslant r/q.$$

In notation which suits our present purposes we say that $f : P \times Q \rightarrow R$ is residuated if there exist maps $f^\sharp_1$ and $f^\sharp_2$ such that

$$\begin{aligned} f(p,q) \leqslant r &\iff q \leqslant f^\sharp_1(p,r) \\ &\iff p \leqslant f^\sharp_2(r,q). \end{aligned}$$

With p held fixed, we see that the relationship between q and r is exactly that given by a Galois connection in covariant form (that is, with the maps order-preserving), and likewise with q held fixed: $f^\sharp_1(p,-)$ is the upper adjoint of $f(p,-)$ and $f^\sharp_2(-,q)$ is the upper adjoint of $f(-,q)$. Here we refer to the first of these as the right adjunction and the second as the left adjunction. The subscript denotes the coordinate in which the passive variable occurs. Observe that we also have

$$q \leqslant f^\sharp_1(p,r) \iff p \leqslant f^\sharp_2(r,q).$$

That is, there is a contravariant adjunction between $f_1(-,r)$ and $f_2(r,-)$, for each fixed r.

Dually, it may happen that a map $g : P \times Q \rightarrow R$ possesses lower adjoints

$$\begin{aligned} r \leqslant g(p,q) &\iff g^\flat_1(p,r) \leqslant q \\ &\iff g^\flat_2(r,q) \leqslant p. \end{aligned}$$

Again, by fixing each variable in turn, we obtain three intertwined Galois connections, two covariant and one contravariant.

The basic facts about residuated maps have been rediscovered many times and are now folklore. We have been unable to find a single source for all the facts we need and so record these in a proposition. All the assertions follow from the observations above and standard properties of Galois connections; these properties can be found, for example, in [6] or in [4], Chapter 7.

Proposition 4.1. *Assume that P, Q and R are complete lattices.*

(1) *Assume that $f : P \times Q \to R$ has upper adjoints, to be regarded as maps $f_1^\sharp : P^\partial \times R \to Q$ and $f_2^\sharp : R \times Q^\partial \to P$. Then*

  (i) *each of f, $f_1^\sharp$ and $f_2^\sharp$ is order-preserving in each coordinate;*

  (ii) *f preserves arbitrary joins in each coordinate and each of $f_1^\sharp$ and $f_2^\sharp$ preserves arbitrary meets in each coordinate.*

(2) (i) *Assume that $f : P \times Q \to R$ preserves arbitrary joins in each coordinate. Then there exist upper adjoints $f_1^\sharp$ and $f_2^\sharp$.*

  (ii) *Assume either (a) $k_1 : P^\partial \times R \to Q$ or (b) $k_2 : R \times Q^\partial \to P$ is a map which preserves arbitrary meets in each coordinate. Then there exists $f : P \times Q \to R$ for which $f_1^\sharp$ and $f_2^\sharp$ both exist and are such that in case (a) $k_1 = f_1^\sharp$ and $f_2^\sharp = \left(k_1^\flat\right)^\partial$ and likewise in case (b).*

Dual statements hold for lower adjoints.

With the necessary general machinery in place we now return to canonical extensions of DQAs and recall some structural facts concerning the lattice reducts of algebras A^σ. We note that the distributive lattices that we consider are assumed to have universal bounds, 0 and 1.

A distributive lattice C is called a DL$^+$ if it satisfies one and consequentially all of the following equivalent conditions:

(1) C is doubly algebraic (that is, both C and its order dual are algebraic);

(2) C is complete, completely distributive and join-generated by the set $J^\infty(C)$ of all completely join-irreducible (equivalently, completely join-prime) elements of C;

(3) C is isomorphic to the up-set lattice of some partially ordered set.

(The equivalence of these conditions is well known; see for example Section 2 of [10], [16] or [4], Theorem 10.29.) Since condition (1) is order-theoretically self-dual, the join-generation in (2) can be replaced by the dual condition, *viz.* that C be meet-generated by the set $M^\infty(C)$ of all completely meet-irreducible (equivalently, completely meet-prime) elements of C. For any DL A, the canonical extension $C = A^\sigma$ is a DL$^+$. Conversely, a DL$^+$ is isomorphic to some A^σ only if it is isomorphic to the up-set lattice of a poset which is representable, *alias* spectral.

As we indicate in Section 5, the sets $J^\infty(C)$ and $M^\infty(C)$ play a central role in the development of frame semantics for logics associated with DQAs and also of dualities for varieties of DQAs . Since C will always be fixed, we shall henceforth usually write $J^\infty(C)$ as J^∞ and $M^\infty(C)$ as M^∞. Recall that the definitions imply that $0 \notin J^\infty$ and $1 \notin M^\infty$. For technical reasons which will emerge below, we shall frequently need also to consider $J_0^\infty := J^\infty \cup \{0\}$ and $M_1^\infty := M^\infty \cup \{1\}$.

The next result tells us that adjunctions can be lifted up to canonical extensions. The result is proved in [5], Proposition 3.6. It was first obtained, in the distributive case, by B. Jónsson (private communication).

Proposition 4.2. *Let P, Q and R be posets, let $h : P \times Q \to R$ be order-preserving and let $f = h^\sigma$. Assume that h possesses a right upper adjoint $h_1^\sharp$. Then f also possesses a right upper adjoint, given by $(h_1^\sharp)^\pi$, and a similar statement holds for the left upper adjoint. Dual statements for the lower adjoints hold as well.*

As an example we note that in an MV-algebra A we have

$$b \odot a = a \odot b \leqslant c \Longleftrightarrow a \leqslant b \to c.$$

The adjunction between $\odot$ and $\to$ lifts to give

$$\beta \odot^\sigma \alpha \leqslant \gamma \Longleftrightarrow \alpha \leqslant \beta \to^\pi \gamma$$

on A^σ.

Proposition 4.2 asserts that, when a binary operation on a DL A possesses a right or left upper (lower) adjoint, then this property lifts to its σ-extension (π-extension) on the corresponding concrete algebra A^σ. But more can be said. Proposition 4.1(2) tells us that passage to the canonical extension creates adjoints for operations which were not already residuated but whose extensions have the right preservation properties for adjoints to exist.

This is an opportune point at which to draw attention to the important distinction between two closely related conditions. A binary map on a complete lattice is residuated if and only if, in each coordinate, it preserves all joins. On the other hand, a binary complete operator f is required, coordinatewise, to preserve arbitrary non-empty joins. In any given coordinate, f preserves the empty join if and only if f sends to 0 any n-tuple having 0 in that coordinate. Thus, for example, we cannot assert that a binary complete operator $f : C \to C$ has a right upper adjoint unless we also know that $f(\alpha, 0) = 0$ for all $\alpha \in C$. A consequence of these observations is that it is impossible for $f_1^\sharp$ and $f_2^\flat$, or $f_2^\sharp$ and $f_1^\flat$, to exist simultaneously.

We now turn to adjoints arising from extensions of operations on a DL A to its canonical extension $C = A^\sigma$. The following lemma treats the unary case. It is a very minor refinement of Lemma 3.2 in [7].

Lemma 4.3. *Let $h : A \to A$ preserve $\vee$ and $\wedge$, so that $f = h^\sigma = h^\pi : C \to C$ preserves arbitrary non-empty joins and meets, where $C = A^\sigma$.*

(1) *Assume that $h(0) = 0$. Then f possesses an upper adjoint $f^\sharp$, which maps M_1^∞ into M_1^∞.*

(2) *Assume that $h(1) = 1$. Then f possesses a lower adjoint $f^\flat$, which maps J_0^∞ into J_0^∞.*

Proof. We prove (1). Let $m \in M_1^\infty$. If $m = 1$ then $f^\sharp(m) = 1$ too. And, for any $m \in M_1^\infty$, there is nothing to prove if $f^\sharp(m) = 1$. So assume $m \in M^\infty$ and $f^\sharp(m) \neq 1$. Then we may write $f^\sharp(m) = \bigwedge_{i \in I} n_i$, where $I \neq \emptyset$ and each $n_i \in M^\infty$. Then $f(\bigwedge_{i \in I} n_i) \leqslant m$. This implies $\bigwedge_{i \in I} f(n_i) \leqslant m$, from which we deduce that there exists $j \in I$ such that $f(n_j) \leqslant m$. Therefore $n_j \leqslant f^\sharp(m)$ and it follows easily that $f^\sharp(m) = n_j \in M$. The second assertion is proved dually. $\Box$

We remark in passing that if $h : A \to A$ is a homomorphism preserving 0 and 1, then $f = h^\sigma = h^\pi$ is a complete homomorphism, and that its upper adjoint $f^\sharp$ restricts to an order-preserving map from M^∞ into M^∞ (the addition of 1 to this set is not needed in this special case) and $f^\flat$ restricts to an order-preserving map from J^∞ into J^∞. Each of these restricted maps "is" the map dual to the DL homomorphism h. The two different manifestations arise from two different, but equivalent, ways of setting up duality for DLs. The ideas here may be seen as central to the canonical extensions approach to duality. We seek to work with operations whose extensions preserve or reverse arbitrary joins or meets in each coordinate. This is equivalent to the extensions possessing certain adjoints, and it is these adjoints which can be used to capture the operations dually. It is regrettable, but unavoidable, that some adjoints we would like to exist do not exist, because empty joins and/or meets do not behave in the correct way.

In the DQA setting there is one instance in which adjoints do exist and behave admirably well. The statement given in the theorem below is one of four that can be made about upper and lower adjoints, either right or left. It may be seen as a binary analogue of Lemma 4.3(1). However, unlike Proposition 4.1, which is proved by reduction to the unary case by fixing each variable in turn, the proof of Theorem 4.4 is more complicated since if we keep one coordinate fixed, this has to be assumed to be clopen.

Theorem 4.4. *Let $h : A \times A \to A$ be a double operator such that $h(a, 0) = 0$ for all $a \in A$, and let $C = A^\sigma$. The complete operator $f = h^\sigma : C \times C \to C$ preserves all joins in the second coordinate; let $f_1^\sharp : C^\partial \times C \to C$ be the right upper adjoint of f, so that*

$$f(\alpha, \beta) \leqslant \gamma \iff \beta \leqslant f_1^\sharp(\alpha, \gamma) \quad (\alpha, \beta, \gamma \in C).$$

Then $f_1^\sharp$ maps $J_0^\infty \times M_1^\infty$ into M_1^∞.

Proof. Denote $f_1^\sharp$ by k. We want to show that k maps $J_0^\infty \times M_1^\infty$ into M_1^∞ but shall in fact show a little more, *viz.* that k maps $K \times M_1^\infty$ into M_1^∞, where, as usual, K denotes the set of closed elements.

For $a \in A$ we may apply Lemma 4.3 with h as h_a defined by $h_a(b) := h(a, b)$, for $b \in A$. The lemma tells us that $f_a := h_a^\sigma$ has an upper adjoint $f_a^\sharp$ which maps M_1^∞ into M_1^∞. Furthermore, since h preserves $\vee$ and 0 in the second coordinate, it follows that h^σ preserves all joins in the second coordinate ([10], Theorem 2.20). We can now assert that $h^\sigma = f$ has a right upper adjoint $f_1^\sharp$. It is then clear by uniqueness of adjoints that for any $a \in A$ and any $c \in C$ we have $f_1^\sharp(a, c) = f_a^\sigma(c)$. We conclude that $f_1^\sharp$ maps $A \times M_1^\infty$ into M_1^∞.

Now suppose that $\alpha \in K$ and, as before, that $m \in M_1^\infty$ (m fixed). Let

$$Q := \{\, k(a, m) \ : \ a \geqslant \alpha \,\}.$$

From what has already been proved, $Q \subseteq M_1^\infty$. Notice that Q is up-directed since for $a_1, a_2 \in A$ we have $k(a_1 \wedge a_2, m) \geqslant k(a_i, m)$ $(i = 1, 2)$. Since M_1^∞ is a representable poset, $\bigvee_{M_1^\infty} Q$ exists in M_1^∞, and is necessarily also the supremum of Q in C, since it is the join of the clopen elements below it. We claim that $\bigvee Q = k(\alpha, m)$. Because k is order-reversing in the first coordinate, $k(\alpha, m) \geqslant \bigvee Q$. To prove the reverse inequality it suffices to show that

$$\forall \beta \in K \big(\beta \leqslant k(\alpha, m) \Longrightarrow \beta \leqslant \bigvee Q\big).$$

But $\beta \leqslant k(\alpha, m)$ if and only if $f(\alpha, \beta) \leqslant m$. Since $f = h^\sigma$ we have

$$\bigwedge\{\, f(a,b) \ :\ \alpha \leqslant a \in A \ \&\ \beta \leqslant b \in A \,\} = f(\alpha, \beta) \leqslant m.$$

By compactness there exist $a_0 \in A$ and $b_0 \in B$ such that $f(a_0, b_0) \leqslant m$. Now we can use residuation to switch back to k: we get

$$\beta \leqslant b_0 \leqslant k(a_0, m) \leqslant \bigvee Q.$$

This proves our claim. □

5 The algebraic component of duality

Our objective in this section is to set up suitable frames for DQAs and to explore the extent to which correspondence, or at least pseudo-correspondence, is available.

By way of introduction let us recall how frames are defined in the case of a variety $\mathcal{V}$ of DLEs with a single additional basic operation h which is a normal operator. Then $f = h^\sigma$ is a complete operator such that $f(0) = 0$. With any concrete algebra $C \in \mathcal{V}$ we then associate a dual frame X, *viz.* the subposet of C consisting of its completely join-irreducible elements. (If h does not preserve 0, then J^∞ has to be replaced by J^∞_0; *cf.* our discussion in Section 4.) In addition, by restriction of its extension to the frame, the n-ary complete operator f on C determines, and is determined by, an $(n+1)$-ary relation

$$R_f = \{\, (p, x_1, \ldots, x_n) \in X^{n+1} \ :\ p \leqslant f(x_1, \ldots, x_n) \,\}$$

on the frame. If A is any algebra in $\mathcal{V}$ then its associated frame is defined to be $(J^\infty(A^\sigma); \leqslant, R_{h^\sigma})$. Similar considerations apply, order dually, if h is a dual operator. These ideas originate in Kripke's relational semantics and, in the form described here, in Goldblatt's paper [18]. They extend, *mutatis mutandis*, to DMAs (see [11]) and also to DMMAs; having quasioperators, rather than operators, does not in itself present an obstacle, since by flipping we can convert quasioperators to operators, and likewise dually. Defining frames for DQAs is less straightforward and we defer explaining how this can be done until after we have indicated what use we shall wish to make of these frames.

Given a variety $\mathcal{V}$ of DLEs for which relational frames have been defined, the next goal should be to capture on the frames, via first-order properties, the laws characterizing the variety. We can pursue this goal in two stages:

(D1) an automated translation of the algebraic laws into equivalent properties expressed in terms of the restrictions to the frames of the operations on concrete algebras;

(D2) conversion of the properties so obtained into first-order properties stated in the appropriate language of frames, *qua* relational structures.

When (D1) and (D2) can be executed, they yield a discrete duality.

Stage (D1) of the process may be seen as purely algebraic in nature, and we shall refer to it as *algebraic correspondence*. It is only at the second stage that the transition is made from an algebraic perspective to a relational one. Thus (D1) can be viewed as the algebraic component of discrete duality. We note, however, that 'first-order' is a key requirement in (D2). We cannot legitimately present (D1) as a formulation of duality in an algebraic guise unless we have confirmation that it does lead on to (D2), that is, to correspondence in the accepted sense, or at least to pseudo-correspondence.

Let $\mathcal{V}$ be a variety of DQAs, and let $\mathcal{W}$ be the associated variety of DMMAs as defined in Section 3. As we have already suggested, we can then view the laws defining $\mathcal{V}$, in the doubled type, as being of two kinds:

(1) inequalities $s^L \precsim t^R$, where $s \precsim t$ holds in $\mathcal{V}$;

(2) laws $h^L \approx h^R$, for h an additional basic operation.

The laws of type (1) can be treated by Sahlqvist methods as these extend to DMMAs. Stating this another way, (D1) and (D2) go through for $\mathcal{W}$. In some respects the situation is more complicated than it would be in the case of DMAs. This is due to the possible non-smoothness of the additional basic operations which introduces features not seen with DMAs. On the other hand, we do benefit from the special form of the laws of type (1). Consequently it is worthwhile to see precisely how (D1), the first, algebraic, stage of the translation, works out in our setting. This we do in the Algebraic Correspondence Theorem, Theorem 5.2. General considerations, extending the ideas of [11] to DMMAs, will ensure that (D2), the transition from algebraic to relational first-order conditions on suitably defined frames, can also be carried out. We say no more about this for now, since we must also consider how laws of type (2) are to be treated.

At this point we shift the focus from the class $\mathcal{C}$ of all concrete algebras in $\mathcal{V}$ to $\mathcal{V}$ itself and to the subclass of $\mathcal{C}$ consisting of algebras A^σ. Here topology comes on stage. Leaving aside additional operations for a moment, we observe that the role of topology can be seen as a means to encode how the lattice reduct of an abstract algebra A sits inside its canonical extension. It might have been expected that, by analogy with many other classes of DLEs, it would be possible to obtain a discrete duality applicable to all concrete algebras in the variety $\mathcal{V}$ and also a topological form applicable to $\mathcal{V}$ itself. This does not happen for DQA varieties in general—a phenomenon which can be seen as a by-product of the form of non-canonicity that such varieties may exhibit. Expressed another way, we cannot carry through (D1) and (D2) for laws of type (2). The crux of the matter is that we cannot take account of the fact that both h^L and h^R on a canonical extension A^σ come from the same function h on the original algebra A without involving the way A sits in A^σ. For this, topology has to be exploited in a new and distinctly subtle way. In recompense, the very strong preservation properties of double quasioperators do allow us to associate with each double quasioperator h in the original type a pair of relations $R = R_{h^\sigma}$ and $S = S_{h^\pi}$ on (topologized) frames, linked in a first-order way; from these h can be recovered. We stress that this duality is for the variety $\mathcal{V}$ of all DQAs of the original, undoubled, type; the identities of type (2) are accounted for in it. If, in addition, we assume that $h(0_{A^\epsilon}) = 0$ and $h(1_{A^\epsilon}) = 1$ for any $A \in \mathcal{V}$ then a *single* relation T suffices to capture h. Here $T = R \cap S$ and we can encode the condition which links R and S as a condition on T. All this is explained in full in [14]. Finally, for subvarieties of $\mathcal{V}$, we may consider stage (D2) of the correspondence process for laws of type (1). This converts the algebraic conditions from stage (D1) supplied by Theorem 5.2 into conditions involving the relations $R \cap S$ associated with the double quasioperator operations. Since these conditions are indeed first-order, we can truly claim Theorem 5.2 to be the algebraic component of duality, in either its bi-relational form using linked pairs of relations R, S or in its uni-relational form when this is available.

The rest of this section is structured as follows. We first introduce the class of frames we wish to use for DQAs. We then discuss (D1), as it operates for inequalities $s^L \precsim t^R$, and present the Algebraic Correspondence Theorem. This may be seen as the end of the purely algebraic part of our analysis. But we then describe the topological duality from [14] in just enough detail to enable us to show how (D2) works out.

We now introduce the frames which will serve as the underlying posets of dual structures for DQAs. Consider a variety $\mathcal{V}$ of DQAs and let C be a concrete algebra in $\mathcal{V}$. We wish to identify a subposet of C to serve as (the underlying poset of) a dual frame for C, and, if $C = A^\sigma$ for some $A \in \mathcal{V}$, for A. For operations on a DLE which are operators, their σ-extensions are complete operators and the natural way to define the dual frame is to take X to be the set J_0^∞. Likewise, in a case where the π-extension has good properties, we would take X to be the set M_1^∞. But as a result of needing to keep both σ- and π-extensions of operations in play simultaneously, we cannot expect to obtain a suitable dual frame solely by restricting either to J_0^∞ or to M_1^∞. Since, in the distributive case that concerns us here, the posets J^∞ and M^∞ are order-isomorphic, this difficulty can however be overcome relatively easily. We may use the usual isomorphism to move between J^∞ and M^∞. This is given by

$$\begin{aligned}\kappa \,:\, J^\infty(C) &\to M^\infty(C),\\ p &\mapsto \bigvee \{\, a \in C \;:\; p \nleqslant a \,\}.\end{aligned}$$

It might seem that this should allow us to identify J^∞ and M^∞. But suppose we were to do this. We have already seen when dealing, as we must, with both σ- and π-extensions of double operators, that we may need to replace J^∞ by J_0^∞ and M^∞ by M_1^∞ (see Lemma 4.3). Certainly J_0^∞ and M_1^∞ are not isomorphic, since in the order inherited from C, we have 0 at the bottom and 1 at the top. So identifying J^∞ and M^∞ and adjoining just one extra point cannot solve the problems caused by the need to be able to form an empty join of completely join-irreducibles and an empty meet of completely meet-irreducibles. But working with $J^\infty \cup \{0,1\}$, and thinking of M^∞ as sitting within this as a copy of J^∞, is not admissible either: $0 \notin J^\infty$ but we may have $0 \in M^\infty$; in this situation 0 would have to play two different roles at the same time. We therefore must keep separate 0 *qua* empty join and 0 as (possibly) an element of M^∞, and likewise for 1.

We are at last ready to introduce the structures which will serve as dual frames of DQAs. Let C be the canonical extension of a DQA (or more generally any concrete DQA) and let $X = J^\infty(C)$, with order induced from C. We take additional elements $\star$ and $\#$ not in X and define $X^\Diamond = \{\star\} \oplus X \oplus \{\#\}$; here $\oplus$ denotes order sum. The notation $X^\Diamond$ serves as a reminder that our frames are pointed posets, that is, posets having universal bounds. In defining $X^\Diamond$ we appear to have given undue precedence to J^∞ over M^∞. To restore a proper balance, as befits a treatment of quasioperators and dual quasioperators, and also to handle the adjoined points $\star$ and $\#$, we introduce two 'interpretations' $I_j : x \mapsto \underline{x}$ and $I_m : x \mapsto \overline{x}$ each mapping $X^\Diamond$ into C. These are defined as follows:

$$\underline{x} = \begin{cases} x & \text{if } x \in J^\infty,\\ 0 & \text{if } x = \star,\\ 1 & \text{if } x = \# \end{cases} \qquad \text{and} \qquad \overline{x} = \begin{cases} \kappa(x) & \text{if } x \in J^\infty,\\ 0 & \text{if } x = \star,\\ 1 & \text{if } x = \#. \end{cases}$$

Both interpretation maps are order-preserving. Either may fail to be an order-embedding, but only because $|I_j^{-1}(1)| = 2$ whenever 1 is completely join-irreducible, and, likewise, we may have

$|I_m^{-1}(0)| = 2$. To avoid notation becoming unduly cumbersome we shall henceforth, for a variable carrying a superscript and/or subscript, indicate the intended interpretation by under- or overlining only the variable's alphabetic label. To deal with flipping, we shal also use the fact that

$$J_0^\infty(C^\partial) = (M_1^\infty(C))^\partial \quad \text{and} \quad M_1^\infty(C^\partial) = (J_0^\infty(C))^\partial.$$

This tells us, of course, that flipping swaps completely meet-irreducibles and completely join-irreducibles. (We remark that it would have been possible, and quite natural, to keep both J^∞ and M^∞, and the linking map κ, explicitly in play when defining our frames. This is necessary in the non-distributive case; see [5] and [7]. However, for several reasons, we have elected not to formulate our theory for DQAs in this way. First of all, to do so would have resulted in structures which involve avoidable redundancy. Secondly, we wish to maintain a close connection with duality as it is normally formulated when the underlying lattices are distributive. Finally, our choice of frames keeps in the foreground the highly special features of DQAs and allows us to exploit this to the full.)

To indicate how we shall work with our frames we present an illustrative example. Suppose that we have a ternary complete $\boldsymbol{\varepsilon}$-operator f of monotonicity type $(1, 1, \partial)$ on a DL^+ C. Let $\mathbf{y} = (y_1, y_2, y_3) \in (X^\Diamond)^3$. Then the appropriate way to interpret $\mathbf{y}$ in C^3 is to map it to $(\underline{y}_1, \underline{y}_2, \overline{y}_3)$. Then for any $(\alpha_1, \alpha_2, \alpha_3) \in C^3$, we have

$$\begin{aligned} f(\alpha_1, \alpha_2, \alpha_3) &= f(\textstyle\bigvee_{\alpha_1 \geqslant \underline{y}_1 \in J_0^\infty} \underline{y}_1, \bigvee_{\alpha_2 \geqslant \underline{y}_2 \in J_0^\infty} \underline{y}_2, \bigwedge_{\alpha_3 \leqslant \overline{y}_3 \in M_1^\infty} \overline{y}_3) \\ &= \bigvee \left\{ f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \ : \ \alpha_1 \geqslant \underline{y}_1 \in J_0^\infty, \ \alpha_2 \geqslant \underline{y}_2 \in J_0^\infty, \ \alpha_3 \leqslant \overline{y}_3 \in M_1^\infty \right\}. \end{aligned}$$

This shows how we should interpret input variables in order to compute $f(\alpha_1, \alpha_2, \alpha_3)$ in C. To complete the argument that it is possible to capture f without going outside the frame we use the fact that, for any $p \in J^\infty$,

$$\begin{aligned} &p \leqslant f(\alpha_1, \alpha_2, \alpha_3) \iff \\ &\qquad \exists y_1 \exists y_2 \exists y_3 \Big(\big(\alpha_1 \geqslant \underline{y}_1 \in J_0^\infty, \ \alpha_2 \geqslant \underline{y}_2 \in J_0^\infty, \ \alpha_3 \leqslant \overline{y}_3 \in M_1^\infty\big) \ \ \& \ \ p \leqslant f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \Big). \end{aligned}$$

This shows that, as we would expect, f is completely determined by the relation

$$R_f = \{ (p, y_1, y_2, y_3) \in X \times (X^\Diamond)^3 \ : \ p \leqslant f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \}.$$

In the same way we can use the interpretation $(\overline{y}_1, \overline{y}_2, \underline{y}_3)$ of a typical element (y_1, y_2, y_3) of $(X^\Diamond)^3$ to capture through our dual frame the values of a complete $\boldsymbol{\varepsilon}$-dual operator g of type $(1, 1, \partial)$. For a general n-ary complete quasioperator we use, in each coordinate, the interpretation which matches up with the preservation property in that coordinate, so that we are then able to exploit that preservation property.

We now revert to the general situation. To establish the algebraic correspondence asserted in Theorem 5.2 one needs to invoke the interpolation property given in Proposition 5.1(1) below. For convenience we state also the stronger interpolation result (2) which we shall need later. This latter should be seen as a topological phenomenon and is required only when we move beyond algebraic correspondence. Accordingly we do not include its proof here. The result first appeared in the distributive (rather than Boolean) setting, albeit without flipping, in [8], Corollary 3.9; a more self-contained proof can be found in [14]. To state the result we need to introduce some notation to handle flippping. Given a monotonicity type $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n)$ and let $\mathbf{x} = (x_1, \ldots, x_n) \in (X^{\diamond})^n$. Then we shall define $\mathbf{x}_{\underline{\boldsymbol{\varepsilon}}}$ and $\mathbf{x}_{\overline{\boldsymbol{\varepsilon}}}$ as follows: for $i = 1, \ldots, n$, let

$$(\mathbf{x}_{\underline{\boldsymbol{\varepsilon}}})_i = \begin{cases} \underline{x}_i & \text{if } \varepsilon_i = 1, \\ \overline{x}_i & \text{if } \varepsilon_i = \partial \end{cases} \quad \text{and} \quad (\mathbf{x}_{\overline{\boldsymbol{\varepsilon}}})_i = \begin{cases} \overline{x}_i & \text{if } \varepsilon_i = 1, \\ \underline{x}_i & \text{if } \varepsilon_i = \partial. \end{cases}$$

In a similar way, we define

$$J^{\boldsymbol{\varepsilon}} = Y_1 \times \ldots \times Y_n, \quad \text{where } Y_i = \begin{cases} J_0^\infty & \text{if } \varepsilon_i = 1, \\ M_1^\infty & \text{if } \varepsilon_i = \partial, \end{cases}$$

$$M^{\boldsymbol{\varepsilon}} = Z_1 \times \ldots \times Z_n, \quad \text{where } Z_i = \begin{cases} M_1^\infty & \text{if } \varepsilon_i = 1, \\ J_0^\infty & \text{if } \varepsilon_i = \partial. \end{cases}$$

Proposition 5.1. *Let $\boldsymbol{\varepsilon}$ be an n-ary monotonicity type and let C be a DL^+.*

(1) (a) *Let $f : C^n \to C$ be a complete $\boldsymbol{\varepsilon}$-operator. For $p \in J^\infty$ and $\mathbf{u} \in C^n$ with $p \leqslant f(\mathbf{u})$, there exists $\mathbf{y} \in J^{\boldsymbol{\varepsilon}}$ such that $\mathbf{y} \leqslant_{\boldsymbol{\varepsilon}} \mathbf{u}$ and $p \leqslant f(\mathbf{y}_{\underline{\boldsymbol{\varepsilon}}})$.*

(b) *Let $g : C^n \to C$ be a complete $\boldsymbol{\varepsilon}$-dual operator. For $p \in J^\infty$ and $\mathbf{v} \in C^n$ with $\kappa(p) \geqslant g(\mathbf{v})$, there exists $\mathbf{z} \in M^{\boldsymbol{\varepsilon}}$ such that $\mathbf{z} \geqslant_{\boldsymbol{\varepsilon}} \mathbf{v}$ and $\kappa(p) \geqslant g(\mathbf{z}_{\overline{\boldsymbol{\varepsilon}}})$.*

(2) *Assume additionally that $C = A^\sigma$ and that h is an n-ary $\boldsymbol{\varepsilon}$-operator. In case $f = h^\sigma$ in (1) then $\mathbf{y}$ can be taken to be minimal in $C^{\boldsymbol{\varepsilon}}$ with respect to $p \leqslant f(-)$ and in case $g = h^\pi$ in (2) then $\mathbf{z}$ can be taken to be maximal in $C^{\boldsymbol{\varepsilon}}$ with respect to $\kappa(p) \geqslant g(-)$.*

Proof. (1) Consider $p \leqslant f(\mathbf{u})$ as above, with $\mathbf{u} = (u_1, \ldots, u_n)$ and let $i \in 1, \ldots, n$ with $\varepsilon_i = 1$. Then $u_i = \bigvee\{\, q \in J_0^\infty : q \leqslant u_i \,\}$ and this join is non-empty for all u_i. So we obtain

$$\begin{aligned} p &\leqslant f(u_1, \ldots, u_{i-1}, \textstyle\bigvee\{\, q \in J_0^\infty : q \leqslant u_i \,\}, u_{i+1}, \ldots, u_n) \\ &= \bigvee\{\, f(u_1, \ldots, u_{i-1}, q, u_{i+1}, \ldots, u_n) : q \in J_0^\infty(C) \text{ and } q \leqslant u_i \,\}. \end{aligned}$$

Since p is completely join-irreducible, it follows that $p \leqslant f(u_1, \ldots, u_{i-1}, q, u_{i+1}, \ldots, u_n)$ for some $q \in J_0^\infty$ with $q \leqslant u_i$. Now if $\varepsilon_i = \partial$, then by this exact same argument, we get $p \leqslant f(u_1, \ldots, u_{i-1}, m, u_{i+1}, \ldots, u_n)$ for some $m \in J_0^\infty(C^\partial)$ with $m \leqslant_\partial u_i$. But $J_0^\infty(C^\partial) = (M_1^\infty)^\partial$, so we have $p \leqslant f(u_1, \ldots, u_{i-1}, m, u_{i+1}, \ldots, u_n)$ for some $m \in M_1^\infty$ with $m \geqslant u_i$. So in either (1) or (2) we can satisfy the required conclusion in any one coordinate, and so the result follows. $\square$

We now consider the promised algebraic correspondence: the passage from the laws of $\mathcal{V}$, via the Generalized Canonicity Theorem, to equivalent algebraic conditions on frames. As we have already indicated in Section 3, s^L and t^R are, respectively, a choice term and a dual choice term for the appropriate DMMA variety. The way the canonical extension was defined then ensures that a DMMA law of the form $s^L \precsim t^R$ is Sahlqvist, in the simplest way possible. Therefore, at the level of the doubled type, correspondence for these particular laws is guaranteed to work.

For the benefit of readers not familiar with correspondence we now present an illustrative example since the ideas are simpler than the notation-heavy general statement of Theorem 5.2 might suggest. The manipulations involved are of a kind which will be entirely familiar to those well versed in Sahlqvist theory. As we saw in Section 3, the MV-algebra identity MV6 holds on an algebra A of the original type if and only if the inequality

$$\mathfrak{s} = \neg(\neg\alpha \oplus^{\pi} \beta) \oplus^{\sigma} \beta \precsim \mathfrak{t} = \neg(\neg\beta \oplus^{\sigma} \alpha) \oplus^{\pi} \alpha$$

holds on the concrete algebra $C = \mathbb{A}^{\sigma}$ in the doubled type. We first want to re-state this as a condition where the input variables are (interpretations of) elements from the frame. Here (see the generation tree in Section 3) the repeated variables on each side occur both positively and negatively. Accordingly we must consider, in the same manner as in Section 2, corresponding terms without recurrent variables:

$$\mathfrak{s}'(y_1^+, y_2^+, y_3^-) = \neg(\neg y_1^+ \oplus y_3^-) \oplus y_2^+ \quad \text{and} \quad \mathfrak{t}'(z_1^+, z_2^+, z_3^-) = \neg(\neg z_1^+ \oplus z_3^-) \oplus z_2^+$$

with maps $\rho_{\mathfrak{s}}(y_1^+) = \alpha$, $\rho_{\mathfrak{s}}(y_2^+) = \beta$, $\rho_{\mathfrak{s}}(y_3^-) = \beta$ and $\rho_{\mathfrak{t}}(z_1^+) = \beta$, $\rho_{\mathfrak{t}}(z_2^+) = \alpha$, $\rho_{\mathfrak{t}}(z_3^-) = \alpha$ indicating the substitutions which must be made to get $\mathfrak{s}$ from $\mathfrak{s}'$ and $\mathfrak{t}$ from $\mathfrak{t}'$. Here variables occurring positively (negatively) are tagged with $+$ ($-$) and 'positive' variables have been listed first.

For the term $\mathfrak{s}'$ we now want to approximate from below by join-irreducibles those variables which in the generation tree have sign $+$ and from above by meet-irreducibles those variables with sign $-$, and the other way round for $\mathfrak{t}'$. In what follows, all of $y_1, y_2, y_3, z_1, z_2, z_3$ are drawn from $X^{\diamond}$.

We may regard $\mathfrak{s}'$ and $\mathfrak{t}'$ as defining maps

$$f, g : C \times C \times C^{\partial} \to C$$

with f a complete operator and g a complete dual operator (see Proposition 3.4). Hence we have

$$\begin{aligned} f(\alpha, \beta, \beta) &= \bigvee \{\, f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \ : \ \underline{y}_1 \leqslant \alpha, \ \underline{y}_2 \leqslant \beta \leqslant \overline{y}_3 \,\}, \\ g(\beta, \alpha, \alpha) &= \bigwedge \{\, g(\overline{z}_1, \overline{z}_2, \underline{z}_3) \ : \ \beta \leqslant \overline{z}_1, \ \underline{z}_3 \leqslant \alpha \leqslant \overline{z}_2 \,\}. \end{aligned}$$

Consequently

$$\begin{aligned}
&\forall\alpha\forall\beta\,(f(\alpha,\beta,\beta) \leqslant g(\beta,\alpha,\alpha)) \\
&\iff \forall\alpha\forall\beta\,\Big(\bigvee\{\, f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \,:\, (\underline{y}_1 \leqslant \alpha)\,\&\,(\underline{y}_2 \leqslant \beta \leqslant \overline{y}_3)\,\} \\
&\qquad\qquad \leqslant \bigwedge\{\, g(\overline{z}_1, \overline{z}_2, \underline{z}_3) \,:\, (\beta \leqslant \overline{z}_1)\,\&\,(\underline{z}_3 \leqslant \alpha \leqslant \overline{z}_2)\,\}\Big) \\
&\iff \forall\alpha\forall\beta\,\Big(\forall y_1\forall y_2\forall y_3\forall z_1\forall z_2\forall z_3\,\Big[\big((\underline{y}_1 \leqslant \alpha)\,\&\,(\underline{z}_3 \leqslant \alpha \leqslant \overline{z}_2)\,\&\,(\underline{y}_2 \leqslant \beta \leqslant \overline{y}_3)\,\& \\
&\qquad\qquad (\beta \leqslant \overline{z}_1)\big) \implies f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \leqslant g(\overline{z}_1, \overline{z}_2, \underline{z}_3)\Big]\Big) \\
&\iff \forall y_1\forall y_2\forall y_3\forall z_1\forall z_2\forall z_3\,\Big(\forall\alpha\forall\beta\,\Big[\big((\underline{y}_1 \leqslant \alpha)\ \&\ (\underline{z}_3 \leqslant \alpha \leqslant \overline{z}_2)\,\&\,(\underline{y}_2 \leqslant \beta \leqslant \overline{y}_3)\,\& \\
&\qquad\qquad (\beta \leqslant \overline{z}_1)\big) \implies f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \leqslant g(\overline{z}_1, \overline{z}_2, \underline{z}_3)\Big]\Big) \\
&\iff \forall y_1\forall y_2\forall y_3\forall z_1\forall z_2\forall z_3\,\Big[\exists\alpha\exists\beta\,\big((\underline{y}_1 \vee \underline{z}_3 \leqslant \alpha \leqslant \overline{z}_2)\,\&\,(\underline{y}_2 \leqslant \beta \leqslant \overline{z}_1 \wedge \overline{y}_3)\big) \\
&\qquad\qquad \implies f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \leqslant g(\overline{z}_1, \overline{z}_2, \underline{z}_3)\Big] \\
&\iff \forall y_1\forall y_2\forall y_3\forall z_1\forall z_2\forall z_3\,\Big[\big((\underline{y}_1 \vee \underline{z}_3 \leqslant \overline{z}_2)\,\&\,(\underline{y}_2 \leqslant \overline{z}_1 \wedge \overline{y}_3)\big) \implies f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \leqslant g(\overline{z}_1, \overline{z}_2, \underline{z}_3)\Big].
\end{aligned}$$

We now turn to the general case. As in the example above, the scenario that is of principal interest to us is that in which the concrete algebras occurring are canonical extensions and the choice term $\mathfrak{s}$ and the dual choice term $\mathfrak{t}$ are restricted to being, respectively, s^L and t^R, where s and t are terms in the original DQA type. However, as stated, Theorem 5.2 serves to emphasize that correspondence, as it applies in the special situation, stems from a discrete duality, in the Sahlqvist style, for algebras of the doubled type.

Consider the class $\mathcal{C}_{\mathbb{A}}$ of concrete algebras in $\mathcal{V}_{\mathbb{A}}$. That is, $\mathcal{C}_{\mathbb{A}}$ consists of those algebras $(A, \wedge, \vee, 0, 1, \{f^L_\lambda\}, \{f^R_\lambda\})$ in $\mathcal{V}_{\mathbb{A}}$ for which the bounded lattice reduct is a DL^+ and f^L_λ is a complete $\boldsymbol{\varepsilon}_\lambda$-operator and f^R_λ is a complete $\boldsymbol{\varepsilon}_\lambda$-dual operator for each λ. In particular, consider the class $\mathcal{C}$ of concrete algebras in $\mathcal{W}$, where $\mathcal{W}$ arises from a variety $\mathcal{V}$ of DQAs by the process described in Section 3. Then for each $A \in \mathcal{V}$, the canonical extension $C = \mathbb{A}^\sigma$ belongs to $\mathcal{C}$.

Let $\mathfrak{s}$ be a choice term and $\mathfrak{t}$ a dual choice term. Let $\mathfrak{s}'(y_1^+, \dots, y_j^+, y_{j+1}^-, \dots, y_k^-)$ be the term with no recurrent variables corresponding to $\mathfrak{s}$. Here we assume the variables with superscript $+$ occur positively in $\mathfrak{s}'$ and the variables with superscript $-$ occur negatively in $\mathfrak{s}'$. Furthermore we let ρ_s denote the map from $\{y_1^+, \dots, y_j^+, y_{j+1}^-, \dots, y_k^-\}$ to $\{\alpha_1, \dots, \alpha_n\}$ giving the substitutions that must be made to get $\mathfrak{s}$ from $\mathfrak{s}'$. Similarly for the term $\mathfrak{t}$ we define $\mathfrak{t}'(z_1^+, \dots, z_\ell^+, z_{\ell+1}^-, \dots, z_m^-)$ and the map ρ_t.

If $\mathfrak{s}$ is a choice term then the interpretation of $\mathfrak{s}'$ on any $C \in \mathcal{C}_{\mathbb{A}}$ is, after flipping as dictated by its signed generation tree, a composition of complete operators. Similarly for a dual choice term $\mathfrak{t}$, after the appropriate flipping, the interpretation of $\mathfrak{t}'$ is a composition of complete dual operators. When $\mathfrak{s}$ is a choice term, the variables labelled $+$ in $\mathfrak{s}'$ are therefore those occurring in the unflipped coordinates and which will be interpreted on any $C \in \mathcal{C}_{\mathbb{A}}$ as elements of $J_0^\infty(C)$. Likewise when $\mathfrak{t}$ is an dual choice term, the variables labelled $+$ in $\mathfrak{t}'$ are those in the unflipped coordinates and will be interpreted as elements of $M_1^\infty(C)$.

We can now present our general correspondence result. We omit the proof, since this demands no techniques not seen in the example above. In the remainder of the paper we shall refer to inequalities $\mathfrak{s} \precsim \mathfrak{t}$ where $\mathfrak{s}$ is a choice term and $\mathfrak{t}$ is a dual choice term as *left-right inequalities.*

Theorem 5.2. (Algebraic Correspondence Theorem for DQAs) *Let C be a concrete DQA of the doubled type associated with some given class $\mathcal{V}$ consisting of all DQAs of a given type. Let $X^\Diamond$ be the frame associated with C. Let $\mathfrak{s}$ and $\mathfrak{t}$ be respectively a choice term and a a dual choice term and let $\mathfrak{s}', \mathfrak{t}'$ be as above. Then $\mathfrak{s} \leqslant \mathfrak{t}$ holds in C if and only if*

$$\Big(\forall y_1^+, \dots, y_j^+, y_{j+1}^-, \dots, y_k^-, z_1^+, \dots, z_\ell^+, z_{\ell+1}^-, \dots, z_m^- \text{ all in } X^\Diamond\Big)$$

$$\Bigg[\Big(\Big(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_1} \underline{y}_i^+\Big) \vee \Big(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_1} \underline{z}_i^-\Big)\Big) \leqslant \Big(\Big(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_1} \overline{y}_i^-\Big) \wedge \Big(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_1} \overline{z}_i^+\Big)\Big)$$

$$\& \ \dots \ \&$$

$$\Big(\Big(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_n} \underline{y}_i^+\Big) \vee \Big(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_n} \underline{z}_i^-\Big)\Big) \leqslant \Big(\Big(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_n} \overline{y}_i^-\Big) \wedge \Big(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_n} \overline{z}_i^+\Big)\Big)\Bigg]$$

$$\implies \mathfrak{s}'(\underline{y}_1^+, \dots, \underline{y}_j^+, \overline{y}_{j+1}^-, \dots, \overline{y}_k^-) \leqslant \mathfrak{t}'(\overline{z}_1^+, \dots, \overline{z}_\ell^+, \underline{z}_{\ell+1}^-, \dots, \underline{z}_m^-).$$

This completes our purely algebraic analysis.

Finally, as promised, we indicate in outline how algebraic correspondence for varieties of DQAs leads on to pseudo-correspondence. Once again we preface generalities with an example, in this case returning to that presented before Theorem 5.2. Let us look more closely at the final consequent $f(\underline{y}_1, \underline{y}_2, \overline{y}_3) \leqslant g(\overline{z}_1, \overline{z}_2, \underline{z}_3)$. This is the condition

$$\neg(\neg\underline{y}_1^{+} \oplus^{\pi} \overline{y}_3^{-}) \oplus^{\sigma} \underline{y}_2^{+} \leqslant \neg(\neg\overline{z}_1^{+} \oplus^{\sigma} \underline{z}_3^{-}) \oplus^{\pi} \overline{z}_2^{+}.$$

If, for example, it were the case that $(\neg\underline{y}_1^{+} \oplus^{\pi} \overline{y}_3^{-}) \notin M_1^{\infty}$ for some $y_1, y_3 \in X^{\Diamond}$, then we would not have arrived at a condition purely on frames and would have to repeat the approximation process on the offending subterm. It turns out that this is not a problem in this particular case. Theorem 4.4 implies that $(\neg\underline{y}_1^{+} \oplus^{\pi} \overline{y}_3^{-}) \in M_1^{\infty}$; to see this, note that in the original type $\neg a \oplus b = a \to b$ is the right residual of $a \odot b$ and then apply Proposition 4.2. Then $\neg(\neg\underline{y}_1^{+} \oplus^{\pi} \overline{y}_3^{-}) \in J_0^{\infty}$. Similarly, $\neg(\neg\overline{z}_1^{+} \oplus^{\sigma} \underline{z}_3^{-}) \in M_1^{\infty}$. We have here an indication of the potential benefits for dual representations of having operations which are residuals. In this special situation we may arrive at a translation of our left-right inequalities to conditions on frames obtained just by restricting the variables to take values in the frame and interpreting these values in J_0^{∞} or in M_1^{∞} as appropriate.

In general we will need to take account of the fact that the values of basic operations even on such inputs will not lie in the frame and further work is then necessary to translate a given left-right inequality into a frame condition. We first briefly describe the strategy for unravelling left-right inequalities, to indicate that we can arrive at conditions wholly expressible in terms of frame elements. Once we have given this outline we shall look at the problem in more detail in order to show that, on the frames associated with concrete algebras, the conditions can be cast as first-order statements in the language of frames, where now the frames are regarded as being equipped with relational structure.

So let us give an overview of the unravelling process for a left-right inequality in C. We wish, using induction on the complexity of the terms $\mathfrak{s}$ and $\mathfrak{t}$, to replace the inequality by an equivalent statement involving frame elements alone. The base case of this induction simply requires us to recast in such a way an inequality of the form $f(\mathbf{u}) \leqslant g(\mathbf{v})$, where f is an n-ary complete $\boldsymbol{\varepsilon}$-operator and $\mathbf{u}$ the interpretation of an element of $(X^{\Diamond})^n$ appropriate to such an f; dually for g (taken to be m-ary) and $\mathbf{v}$. The inductive step requires recasting of a similar inequality, but now with $\mathbf{u}$ and $\mathbf{v}$ general elements of C. Note that

$$\begin{aligned} f(\mathbf{u}) \leqslant g(\mathbf{v}) &\iff \forall p \in X\,(p \leqslant f(\mathbf{u}) \implies p \leqslant g(\mathbf{v})) \\ &\iff \forall p \in X\,(p \leqslant f(\mathbf{u}) \implies \kappa(p) \ngeqslant g(\mathbf{v}))\,. \end{aligned}$$

Proposition 5.1(1) and its contrapositive are exactly what we need to replace elements $\mathbf{u}$ and $\mathbf{v}$ of C by interpretations of elements of our frame. This reduces the inductive step in the unravelling of a left-right inequality to that needed in the base case.

Now we switch to a relational perspective on the dual side, and address the issue of first-order definability. Assume we have a concrete DQA C of the form $C = \mathbb{A}^{\sigma}$ and let $X^{\Diamond}$ be the associated frame. This frame has an order relation, $\leqslant$. It also carries, for each n-ary $\boldsymbol{\varepsilon}$-double quasioperator h in the algebraic type of A, an $(n+1)$-ary relation $T \subseteq X \times (X^{\Diamond})^n$, related to h as follows:

$$pT\mathbf{x} \iff pR\mathbf{x} \text{ and } pS\mathbf{x},$$

where

$$pR\mathbf{x} \iff p \leqslant f(\mathbf{x}_{\underline{\varepsilon}}), \quad \text{where } f = h^{\sigma},$$
$$pS\mathbf{x} \iff \kappa(p) \geqslant g(\mathbf{x}_{\overline{\varepsilon}}), \quad \text{where } g = h^{\pi}.$$

We also note a key theorem from [14], available in the topological setting in which we are assuming we are working. This theorem asserts that provided that $f(0_{C^{\varepsilon}}) = 0$ and $g(1_{C^{\epsilon}}) = 1$ then

$$pR_{\min} = pT_{\min} \quad \text{and} \quad pS_{\max} = pT_{\max};$$

here the order on the relational images pR, pS and pT is the product order on $(X^{\Diamond})^n$ derived from $\leqslant_{\varepsilon}$ and, for any subset Q of $(X^{\Diamond})^n$, we denote by $Q_{\min}$ and $Q_{\max}$ the minimal and maximal points of Q, respectively. The significance of the minimal/maximal interpolants supplied by Proposition 5.1(2) should now be apparent. The theorem implies, of course, that we can reconstruct R and S from T.

We are ready to examine the statement in Theorem 5.2 in more detail. First consider a statement in the antecedent of the condition in Theoren 5.2, *viz.*

$$\Big(\Big(\bigvee_{\rho_{\mathfrak{s}}(y_i^+)=\alpha_1} \underline{y}_i^+\Big) \vee \Big(\bigvee_{\rho_{\mathfrak{t}}(z_i^-)=\alpha_1} \underline{z}_i^-\Big)\Big) \leqslant \Big(\Big(\bigwedge_{\rho_{\mathfrak{s}}(y_i^-)=\alpha_1} \overline{y}_i^-\Big) \wedge \Big(\bigwedge_{\rho_{\mathfrak{t}}(z_i^+)=\alpha_1} \overline{z}_i^+\Big)\Big).$$

A technical point now arises. As the statement of Proposition 5.1 indicates, vectors $\mathbf{y}$ to be interpreted as $\mathbf{y}_{\underline{\varepsilon}}$ may be assumed to lie in J^{ε} rather than in $(X^{\Diamond})^n$, and likewise for the order-dual situation. Accordingly, it is permissible to assume that positively tagged y variables and negatively tagged z variables do not take value $\#$ and that negatively tagged y variables and positively tagged z variables do not take value $\star$. This restriction is necessary for the claims below to be valid. The above inequality in C is equivalent to the conjunction of all the following statements:

(i) $y_{i_2}^- \nleqslant y_{i_1}^+$ whenever $1 \leqslant i_1 \leqslant j$, $\rho_{\mathfrak{s}}(y_{i_1}^+) = x_1$, $j+1 \leqslant i_2 \leqslant k$, and $\rho_{\mathfrak{s}}(y_{i_2}^-) = x_1$;

(ii) $z_{i_2}^+ \nleqslant y_{i_1}^+$ whenever $1 \leqslant i_1 \leqslant j$, $\rho_{\mathfrak{s}}(y_{i_1}^+) = x_1$, $l+1 \leqslant i_2 \leqslant m$, and $\rho_{\mathfrak{t}}(z_{i_2}^+) = x_1$;

(iii) $y_{i_2}^- \nleqslant z_{i_1}^-$, whenever $1 \leqslant i_1 \leqslant l$, $\rho_{\mathfrak{t}}(z_{i_1}^-) = x_1$, $j+1 \leqslant i_2 \leqslant k$, and $\rho_{\mathfrak{s}}(y_{i_2}^-) = x_1$;

(iv) $z_{i_2}^+ \nleqslant z_{i_1}^-$, whenever $1 \leqslant i_1 \leqslant l$, $\rho_{\mathfrak{t}}(z_{i_1}^-) = x_1$, $l+1 \leqslant i_2 \leqslant m$, and $\rho_{\mathfrak{t}}(z_{i_2}^+) = x_1$.

Each of these statements individually involves only the order structure of $X^{\Diamond}$ and is therefore a first-order statement about the frame. Consequently the conjunction of the statements is also first-order.

Now consider the conclusion of the implication, *viz.*

$$\mathfrak{s}'(\underline{y}_1^+, \dots, \underline{y}_j^+, \overline{y}_{j+1}^-, \dots, \overline{y}_k^-) \leqslant \mathfrak{t}'(\overline{z}_1^+, \dots, \overline{z}_\ell^+, \underline{z}_{\ell+1}^-, \dots, \underline{z}_m^-).$$

This is equivalent to the statement:

$$(\forall p \in X)\big(p \leqslant \mathfrak{s}'(\underline{y}_1^+, \dots, \underline{y}_j^+, \overline{y}_{j+1}^-, \dots, \overline{y}_k^-) \implies p \leqslant \mathfrak{t}'(\overline{z}_1^+, \dots, \overline{z}_\ell^+, \underline{z}_{\ell+1}^-, \dots, \underline{z}_m^-)\big).$$

Consider the statement

$$p \leqslant \mathfrak{s}'(\underline{y}_1^+, \dots, \underline{y}_j^+, \overline{y}_{j+1}^-, \dots, \overline{y}_k^-).$$

From the way a left-right inequality interprets in the doubled type we know that one of two cases occurs. Either $\mathfrak{s}'$ is a variable, 0, or 1, in which case this is clearly a first-order statement about the frame, or $\mathfrak{s}' = f(t_1, \ldots, t_m)$, where f is a complete quasioperator of some arity m. We assume here for ease of notation that f is binary of monotonicity type $(1, \partial)$. Then $p \leqslant \mathfrak{s}'$ becomes $p \leqslant f(t_1, t_2)$. By Proposition 5.1 we have

$$p \leqslant f(t_1, t_2) \Longleftrightarrow (\exists x, x' \in X^{\Diamond})((x, x') \neq (\#, \star) \text{ and } pT(x, x') \text{ and } (\underline{x} \leqslant t_1) \text{ and } (\overline{x}' \geqslant t_2)).$$

In this new equivalent statement, the parts that are not clearly first-order statements about the frame are $\underline{x} \leqslant t_1$ and $\overline{x}' \geqslant t_2$. If x is $\star$, then the statement $\underline{x} \leqslant t_1$ is vacuously true. Otherwise $x \in X$ and the statement $\underline{x} \leqslant t_1$ is exactly of the same type as the original statement $p \leqslant f(t_1, t_2)$. This is courtesy of the fact that $\mathfrak{s}' \leqslant \mathfrak{t}'$ is a left-right inequality. This tells us that the operators and dual operators are lined up exactly right. Dually, $\overline{x}' \geqslant t_2$ is vacuous if $x' = \#$ and otherwise it is exactly of the order-dual form of the original $p \leqslant f(t_1, t_2)$. That is, it is of the form $\kappa(p) \geqslant g(t_1, t_2)$, where g is a complete dual quasioperator (except that g may of course in general have an arity different from 2). The statement on the other side of the original inequality, $p \leqslant \mathfrak{t}'$, is equivalent to the negation of the statement $\kappa(p) \geqslant \mathfrak{t}'$. This again is exactly of the order-dual type to $p \leqslant f(t_1, t_2)$ because of the special shape of left-right inequalities. Thus, by induction, we see that the statements resulting from the Algebraic Correspondence Theorem may be turned into first-order statements about the frame dual to C as we have claimed.

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