An algebraic approach to Gelfand Duality

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Joint work with
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Stone = zero-dimensional compact Hausdorff spaces and continuous maps

BA = Boolean algebras and Boolean homomorphisms

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If we drop zero-dimensionality and work with much larger category $\textbf{KHaus}$ of compact Hausdorff spaces, clopen subsets are no longer representative enough.

Therefore, we have to work with all open subsets. These form a compact regular frame, and Stone duality generalizes to Isbell duality:

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**Isbell Duality (1972):** $\mathbf{KHaus}$ is dually equivalent to the category $\mathbf{KRFrm}$ of compact regular frames.

In the zero-dimensional case, this restricts to the dual equivalence between $\mathbf{Stone}$ and the subcategory $\mathbf{zK Frm}$ of $\mathbf{KRFrm}$ of zero-dimensional compact (regular) frames, which yields Stone duality.
Instead of the frame of all open subsets, we can work with the Boolean frame of regular open subsets. Since our spaces are regular, regular opens form a basis, so they carry all needed information about the topology of the space.

This yields de Vries duality:

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In the zero-dimensional case, this also restricts to Stone duality.

Thus, either \( \text{KRFrm} \) or \( \text{DeV} \) (which are equivalent) give point-free representation of \( \text{KHAus} \).
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If $X$ is not zero-dimensional, then we have to work with a much larger gadget $C(X)$ of all continuous functions on $X$. Obviously $C(X)$ carries all the information about the topology of $X$. 
$C(X) = \text{all continuous functions } f: X \rightarrow \mathbb{R}$
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$C(X)$ is a commutative ring:

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\[ C(X) \text{ is an } \mathbb{R}\text{-algebra:} \]

\[ \mathbb{R}\text{-action: } (\alpha f)(x) = \alpha f(x) \quad \alpha \in \mathbb{R} \]

\[ C(X) \text{ is a Banach algebra:} \]

\[ \text{Norm: } ||f||(x) = \sup\{|f(y)|: y \in X\} \]
\[ C(X) = \text{all continuous functions } f: X \to R \]

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Norm: \[ ||f||(x) = \sup \{|f(y)| : y \in X\} \]

Consequently \( C(X) \text{ is a commutative (real) } C^*\text{-algebra} \)
**Weierstrass Approximation Theorem (1885):** Let $X$ be a compact (closed and bounded) interval of $\mathbb{R}$. The $\mathbb{R}$-subalgebra of $C(X)$ of all polynomial functions on $X$ is uniformly dense in $C(X)$. (That is, it is dense in the uniform topology on $C(X)$).

In other words, each continuous function on $X$ can be uniformly approximated by polynomial functions on $X$. 
Karl Weierstrass (1815 –1897)
In two papers published in 1937 and 1948, Stone generalized the Weierstrass Approximation Theorem in two directions.

One is that we can replace a closed and bounded interval on $\mathbb{R}$ by any compact Hausdorff space.

The second is that we can replace the $\mathbb{R}$-algebra of polynomials by any $\mathbb{R}$-subalgebra $A$ of $C(X)$ that **separates** points of $X$. That is, if $x \neq y$ then there exists $f \in A$ such that $f(x) \neq f(y)$.
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**Stone-Weierstrass Theorem:** Let $X$ be a compact Hausdorff space. If $A$ is an $\mathbb{R}$-subalgebra of $C(X)$ that separates points of $X$, then $A$ is uniformly dense in $C(X)$. 
The Stone-Weierstrass Theorem (SW) is the key in generalizing Stone duality in a different (more ring-theoretic) direction.

The resulting duality is known as **Gelfand duality** (or **Stone-Gelfand duality** or **Stone-Gelfand-Naimark duality**).
$\text{KHaus} = \text{compact Hausdorff spaces and continuous maps}$

$\text{C*Alg} = \text{real (commutative) C*-algebras and *-homomorphisms}.$

**Gelfand Duality Theorem (1939):** $\text{KHaus}$ is dually equivalent to $\text{C*Alg}.$
Israel Gelfand (1913—2009)
The contravariant functor $\text{K Haus} \rightarrow \text{C*Alg}$:

$X \rightarrow C(X)$ and $\phi \rightarrow f \circ \phi$

The contravariant functor $\text{C*Alg} \rightarrow \text{K Haus}$:

$A \rightarrow \text{Max}(A)$ and $M \rightarrow \alpha^{-1}(M)$
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The contravariant functor $\text{C*Alg} \to \text{KHaus}$:

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**Fact 1:** $X$ is homeomorphic to $\text{Max}(C(X))$ (follows from compactness of $X$ because each $M$ in $C(X)$ is of the form $M_x = \{f \in C(X) : f(x) = 0\}$, so the correspondence $x \to M_x$ is a homeomorphism).
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**Fact 2:** $A$ is isomorphic to $C(\text{Max}(A))$ (follows from Stone-Weierstrass because $A$ is isomorphic to an $R$-subalgebra of $C(\text{Max}(A))$ that separates points, so by (SW), $A$ is uniformly dense in $C(\text{Max}(A))$. Now as $A$ is a C*-algebra, $A$ is uniformly complete, hence is isomorphic to $C(\text{Max}(A))$.}
Apart from the $\mathbb{R}$-algebra structure, there is also a natural order on $C(X)$:

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In fact, $C(X)$ with this order forms a **lattice**:

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(f \lor g)(x) = \max\{f(x), g(x)\} \quad \text{and} \quad (f \land g)(x) = \min\{f(x), g(x)\}
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$$(f \lor g)(x) = \max\{f(x),g(x)\}$$
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Moreover, it is easy to verify that

$$f \leq g \text{ implies } f + h \leq g + h \text{ for each } h$$

$$0 \leq f,g \text{ implies } 0 \leq fg$$

So $C(X)$ is an \textbf{ℓ-ring} (lattice ordered ring).
We call $A$ an $\mathcal{E}$-algebra if it is an $\mathcal{E}$-ring and an $R$-algebra and

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Let $bal$ be the category of bounded Archimedean $\ell$-algebras and $\ell$-algebra homomorphisms.

Then $bal$ provides a natural generalization of the category $\textbf{C}^*\text{Alg}$ of real commutative $\text{C}^*$-algebras.
Let $A \in bal$. For each $a \in A$ we define the **absolute value** of $a$ by

$$|a| = a \lor -a$$

And define the **norm** by

$$||a|| = \inf\{\mu : |a| \leq \mu\}$$

As $A$ is bounded and Archimedean, this is a well-defined norm on $A$. 
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For $C(X)$ we have

$$||f||(x) = \sup \{|f(y)| : y \in X\} = \inf \{ \mu : |a| \leq \mu \}$$

Consequently, $C^\ast \text{Alg}$ is the subcategory of $\text{bal}$ consisting of those objects in $\text{bal}$ that are norm-complete.
The dual equivalence between $C^*\text{Alg}$ and $K\text{Haus}$ extends to an adjunction between $bal$ and $K\text{Haus}$. 
The dual equivalence between $\text{C}^*\text{Alg}$ and $\text{KHaus}$ extends to an adjunction between $\text{bal}$ and $\text{KHaus}$.

Recall that with each $\text{C}^*$-algebra $A$ we associated the compact Hausdorff space $\text{Max}(A)$ of maximal ideals of $A$. 
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Recall that with each C*-algebra $A$ we associated the compact Hausdorff space $\text{Max}(A)$ of maximal ideals of $A$.

If $A \in \textit{bal}$, then $\text{Max}(A)$ may not in general be Hausdorff. Instead we work with maximal \textit{convex} ideals, the so called maximal $\ell$-\textit{ideals} (that is, ideals that also preserve the lattice structure; i.e. are kernels of $\ell$-homomorphisms).
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For $A \in \text{bal}$ let $X_A$ be the set of maximal $\ell$-ideals of $A$ endowed with the Zariski topology.
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Then $X_A$ is compact Hausdorff.

Note: If $A$ happens to be a C*-algebra, then maximal ideals coincide with maximal $\ell$-ideals, so this construction generalizes the standard construction for C*-algebras.
This defines a functor $bal \to KHaus$, and the functors

$$bal \to KHaus \text{ and } KHaus \to bal$$

form a contravariant adjunction which restricts to Gelfand duality between $C^*\text{Alg}$ and $KHaus$. 
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form a contravariant adjunction which restricts to Gelfand duality between \( \mathbf{C^*Alg} \) and \( \mathbf{KHaus} \).

Consequently, the inclusion functor \( \mathbf{C^*Alg} \to \mathbf{bal} \) has a left adjoint \( \mathbf{bal} \to \mathbf{C^*Alg} \) which sends \( A \in \mathbf{bal} \) to the C*-algebra \( C(X_A) \) (the norm-completion of \( A \)).
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Thus, \( \text{C*Alg} \) is a reflective subcategory of \( \text{bal} \).
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There are several ways to describe the reflector \( r : \text{bal} \to \text{C*Alg} \)

1. \( r(A) \) is isomorphic to \( C(X_A) \) (topology)
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Consequently we obtain the following characterization of commutative (real) C*-algebras:

The following conditions are equivalent for \( A \in \text{bal} \):

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In fact, \( \text{C*Alg} \) is the smallest reflective subcategory of \( \text{bal} \) and a unique reflective epicomplete subcategory of \( \text{bal} \).
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Richard Dedekind (1831–1916)
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Recall that a poset (partially ordered set) $P$ is **Dedekind complete** if each subset of $P$ which has an upper bound (resp. lower bound) has a least upper bound (resp. a greatest lower bound).

**Dedekind’s Theorem:** Every poset $P$ can be embedded into its Dedekind completion $D(P)$, which is a Dedekind complete lattice. The embedding preserves all existing joins and meets in $P$, and is both **join-dense** and **meet-dense** (meaning that every element of $D(P)$ is the join of the elements of $P$ beneath it and the meet of the elements of $P$ above it).
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It follows from the work of Nakano and Johnson that if $A \in bal$ then $D(A) \in bal$, and that $A$ is both join-dense and meet-dense in $D(A)$. 
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Let $DA$ be the (full) subcategory of $bal$ whose objects are Dedekind complete.
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Let $DA$ be the (full) subcategory of $\text{bal}$ whose objects are Dedekind complete.

As each Dedekind complete $A \in \text{bal}$ is uniformly complete, we have that $DA$ is a (full) subcategory of $\text{C*Alg}$. 
Recall that a poset (partially ordered set) $P$ is **Dedekind complete** if each subset of $P$ which has an upper bound (resp. lower bound) has a least upper bound (resp. a greatest lower bound).

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Let $DA$ be the (full) subcategory of $bal$ whose objects are Dedekind complete.

As each Dedekind complete $A \in bal$ is uniformly complete, we have that $DA$ is a (full) subcategory of $C^\ast Alg$.

**Note**: Although taking the uniform completion is functorial (the reflector $bal \to C^\ast Alg$), taking the Dedekind completion is not functorial.
Recall:

(1) A ring $A$ is a **Baer ring** if each annihilator ideal of $A$ is generated by an idempotent.
(2) A monomorphism $\alpha : A \rightarrow B$ in $bal$ is **essential** if $\alpha^{-1}(I) \neq 0$ for each nonzero $\ell$-ideal $I$ of $B$.
(3) $B$ is an **essential extension** of $A$ if there is an essential monomorphism $\alpha : A \rightarrow B$.
(4) $B$ is **essentially closed** if it admits no proper essential extension.
(5) $B$ is the **essential closure** of $A$ if $B$ is an essential extension of $A$ and $B$ is essentially closed.
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For $A \in \text{bal}$ the following conditions are equivalent:

(1) $A$ is a Dedekind algebra.
(2) $A$ is a Baer C*-algebra.
(3) $A$ is essentially closed.
(4) $A$ is injective in $\text{bal}$. 
For $A, B \in \text{bal}$ the following conditions are equivalent:

(1) $B$ is isomorphic to the Dedekind completion $D(A)$ of $A$.
(2) $B$ is the injective hull of $A$ in $\text{bal}$.
(3) $B$ is the essential closure of $A$.
(4) $B$ is isomorphic to $C(X_B)$ and $X_B$ is the Gleason cover of $X_A$ in $\text{KHaus}$. 
For $A,B \in \mathbf{bal}$ the following conditions are equivalent:

(1) $B$ is isomorphic to the Dedekind completion $D(A)$ of $A$.
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So for $A \in \mathbf{bal}$ we have that the uniform completion of $A$ is $C(X_A)$ and the Dedekind completion of $A$ is $C(X_B)$ where $X_B$ is the Gleason cover of $X_A$. 
Dedekind completion of $A$ can also be characterized purely in terms of functions on $X_A$. 
Dedekind completion of $A$ can also be characterized purely in terms of functions on $X_A$.

Let $f$ be a bounded function on $X_A$. Let

$$f^*(x) = \inf_{U \ni x} \{ \sup_{y \in U} f(y) \}$$

$$f_*(x) = \sup_{U \ni x} \{ \inf_{y \in U} f(y) \}$$

Recall that $f$ is **upper semicontinuous** if $f^* = f$.

An upper semicontinuous function $f$ is **normal** if

$$(f_*)^* = f$$
Let \( N(X_A) \) be the set of normal functions on \( X_A \).

Then \( N(X_A) \) is isomorphic to the Dedekind completion \( D(A) \) of \( A \).
<table>
<thead>
<tr>
<th>A is a C*-algebra</th>
<th>A is a Dedekind algebra</th>
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<tbody>
<tr>
<td>A is uniformly complete</td>
<td>A is Dedekind complete</td>
</tr>
<tr>
<td>A is epicomplete</td>
<td>A is essentially closed = A is injective</td>
</tr>
<tr>
<td>A is isomorphic to $C(X)$ where $X$ is compact Hausdorff</td>
<td>A is isomorphic to $C(X)$ where $X$ is compact Hausdorff and extremally disconnected</td>
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<tr>
<td>( B ) is the uniform completion of ( A )</td>
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<td>where ( X_B ) is the Gleason cover of ( X_A )</td>
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</tbody>
</table>
G. Bezhanishvili, P. J. Morandi, B. Olberding. *Bounded Archimedean \( \ell \)-algebras and Gelfand duality*, under review.

Thank you!