An algebraic approach to Gelfand Duality

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Joint work with
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BA = Boolean algebras and Boolean homomorphisms

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Marshall Stone (1903–1989)

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Therefore, we have to work with all open subsets. These form a compact regular frame, and Stone duality generalizes to Isbell duality:

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Therefore, we have to work with all open subsets. These form a compact regular frame, and Stone duality generalizes to Isbell duality:

Isbell Duality (1972): KHaus is dually equivalent to the category **KRFrm** of compact regular frames.

In the zero-dimensional case, this restricts to the dual equivalence between **Stone** and the subcategory **zKFrm** of **KRFrm** of zero-dimensional compact (regular) frames, which yields Stone duality.

Instead of the frame of all open subsets, we can work with the Boolean frame of regular open subsets. Since our spaces are regular, regular opens form a basis, so they carry all needed information about the topology of the space.

This yields de Vries duality:

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In the zero-dimensional case, this also restricts to Stone duality.

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De Vries Duality (1962): KHaus is dually equivalent to the category **DeV** of de Vries algebras.

In the zero-dimensional case, this also restricts to Stone duality.

Thus, either **KRFrm** or **DeV** (which are equivalent) give point-free representation of **KHaus**.

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If X is not zero-dimensional, then we have to work with a much larger gadget $C(X)$ of all continuous functions on X. Obviously $C(X)$ carries all the information about the topology of X.

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Consequently $C(X)$ is a **commutative (real) C*-algebra**

Weierstrass Approximation Theorem (1885): Let *X* be a compact (closed and bounded) interval of ***R***. The ***R***-subalgebra of *C(X)* of all polynomial functions on *X* is uniformly dense in *C(X)*. (That is, it is dense in the uniform topology on *C(X)*).

In other words, each continuous function on *X* can be uniformly approximated by polynomial functions on *X*.

Karl Weierstrass (1815 –1897)

In two papers published in 1937 and 1948, Stone generalized the Weierstrass Approximation Theorem in two directions.

One is that we can replace a closed and bounded interval on ***R*** by any compact Hausdorff space.

The second is that we can replace the ***R***-algebra of polynomials by any ***R***-subalgebra A of $C(X)$ that ***separates*** points of X. That is, if $x \neq y$ then there exists $f \in A$ such that $f(x) \neq f(y)$.

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Stone-Weierstrass Theorem: Let X be a compact Hausdorff space. If A is an ***R***-subalgebra of $C(X)$ that separates points of X, then A is uniformly dense in $C(X)$.

The Stone-Weierstrass Theorem (SW) is the key in generalizing Stone duality in a different (more ring-theoretic) direction.

The resulting duality is known as **Gelfand duality** (or **Stone-Gelfand duality** or **Stone-Gelfand-Naimark** duality).

KHaus = compact Hausdorff spaces and continuous maps

C*Alg = real (commutative) C*-algebras and *-homomorphisms.

Gelfand Duality Theorem (1939): KHaus is dually equivalent to **C*Alg**.

Israel Gelfand (1913—2009)

The contravariant functor **KHaus** → **C*Alg**:

$X \to C(X)$ and $\phi \to f \circ \phi$

The contravariant functor **C*Alg** → **KHaus**:

$A \to Max(A)$ and $M \to \alpha^{-1}(M)$

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Fact 1: *X* is homeomorphic to *Max(C(X))* (follows from compactness of *X* because each *M* in *C(X)* is of the form $M_x = \{f \in C(X): f(x) = 0\}$, so the correspondence $x \to M_x$ is a homeomorphism).

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Fact 2: *A* is isomorphic to *C(Max(A))* (follows from Stone-Weierstrass because *A* is isomorphic to an ***R***-subalgebra of *C(Max(A))* that separates points, so by (SW), *A* is uniformly dense in *C(Max(A))*. Now as *A* is a C*-algebra, *A* is uniformly complete, hence is isomorphic to *C(Max(A))*.

Apart from the $\boldsymbol{R}$-algebra structure, there is also a natural order on $C(X)$:

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Moreover, it is easy to verify that

$$f \leq g \text{ implies } f+h \leq g+h \text{ for each } h$$
$$0 \leq f,g \text{ implies } 0 \leq fg$$

So *C(X)* is an ***ℓ-ring*** (lattice ordered ring).

We call A an ***ℓ-algebra*** if it is an ℓ-ring and an $\boldsymbol{R}$-algebra and

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Let ***bal*** be the category of bounded Archimedean ℓ-algebras and ℓ-algebra homomorphisms.

Then ***bal*** provides a natural generalization of the category **C*Alg** of real commutative C*-algebras.

Let $A \in$ ***bal***. For each $a \in A$ we define the ***absolute value*** of a by

$$|a| = a \vee -a$$

And define the ***norm*** by

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For $C(X)$ we have

$$||f||(x) = sup\{|f(y)| : y \in X\} = inf\{\mu : |a| \leq \mu\}$$

Consequently, **C*Alg** is the subcategory of ***bal*** consisting of those objects in ***bal*** that are ***norm-complete***.

The dual equivalence between **C*Alg** and **KHaus** extends to an adjunction between ***bal*** and **KHaus**.

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If $A \in$ ***bal***, then *Max(A)* may not in general be Hausdorff. Instead we work with maximal ***convex*** ideals, the so called maximal ***ℓ-ideals*** (that is, ideals that also preserve the lattice structure; i.e. are kernels of ℓ-homomorphisms).

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For $A \in$ ***bal*** let X_A be the set of maximal ℓ-ideals of *A* endowed with the Zariski topology.

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Note: If A happens to be a C*-algebra, then maximal ideals coincide with maximal ℓ-ideals, so this construction generalizes the standard construction for C*-algebras.

This defines a functor ***bal*** → **KHaus**, and the functors

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Consequently, the inclusion functor **C*Alg** → ***bal*** has a left adjoint ***bal*** → **C*Alg** which sends $A \in$ ***bal*** to the C*-algebra $C(X_A)$ (the ***norm-completion*** of A).

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Thus, **C*Alg** is a reflective subcategory of ***bal***.

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(1) *r(A)* is isomorphic to *C(*X_A*)* (topology)
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Consequently we obtain the following characterization of commutative (real) C*-algebras:

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In fact, **C*Alg** is the smallest reflective subcategory of ***bal*** and a unique reflective epicomplete subcategory of ***bal***.

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Richard Dedekind (1831–1916)

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Dedekind's Theorem: Every poset P can be embedded into its Dedekind completion $D(P)$, which is a Dedekind complete lattice. The embedding preserves all existing joins and meets in P, and is both ***join-dense*** and ***meet-dense*** (meaning that every element of $D(P)$ is the join of the elements of P beneath it and the meet of the elements of P above it).

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It follows from the work of **Nakano** and **Johnson** that if $A \in$ ***bal*** then $D(A) \in$ ***bal***, and that A is both join-dense and meet-dense in $D(A)$.

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As each Dedekind complete $A \in$ ***bal*** is uniformly complete, we have that **DA** is a (full) subcategory of **C*Alg**.

Note: Although taking the uniform completion is functorial (the reflector ***bal*** $\rightarrow$ **C*Alg**), taking the Dedekind completion is **not** functorial.

Recall:

(1) A ring A is a ***Baer ring*** if each annihilator ideal of A is generated by an idempotent.

(2) A monomorphism $\alpha : A \to B$ in ***bal*** is ***essential*** if $\alpha^{-1}(I) \neq 0$ for each nonzero ℓ-ideal I of B.

(3) B is an ***essential extension*** of A if there is an essential monomorphism $\alpha : A \to B$.

(4) B is ***essentially closed*** if it admits no proper essential extension.

(5) B is the ***essential closure*** of A if B is an essential extension of A and B is essentially closed.

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For $A \in$ ***bal*** the following conditions are equivalent:

(1) A is a Dedekind algebra.

(2) A is a Baer C*-algebra.

(3) A is essentially closed.

(4) A is injective in ***bal***.

For $A, B \in$ ***bal*** the following conditions are equivalent:

(1) B is isomorphic to the Dedekind completion $D(A)$ of A.
(2) B is the injective hull of A in ***bal***.
(3) B is the essential closure of A.
(4) B is isomorphic to $C(X_B)$ and X_B is the Gleason cover of X_A in **KHaus**.

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So for *A* $\in$ ***bal*** we have that the uniform completion of *A* is $C(X_A)$ and the Dedekind completion of *A* is $C(X_B)$ where X_B is the Gleason cover of X_A.

Dedekind completion of A can also be characterized purely in terms of functions on X_A.

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Let f be a bounded function on X_A. Let

$$f^*(x) = inf_{U_x}\{sup_{y\in U_x} f(y)\}$$

$$f_*(x) = sup_{U_x}\{inf_{y\in U_x} f(y)\}$$

Recall that f is ***upper semicontinuous*** if $f^* = f$.

An upper semicontinuous function f is ***normal*** if

$$(f_*)^* = f$$

Let $N(X_A)$ be the set of normal functions on X_A.

Then $N(X_A)$ is isomorphic to the Dedekind completion *D(A)* of *A*.

***A* is a C*-algebra**	***A* is a Dedekind algebra**
A is uniformly complete	*A* is Dedekind complete
A is epicomplete	*A* is essentially closed = *A* is injective
A is isomorphic to $C(X)$ where X is compact Hausdorff	*A* is isomorphic to $C(X)$ where X is compact Hausdorff and extremally disconnected

B* is the uniform completion of *A	***B* is the Dedekind completion of *A***
B is the epicompletion of *A*	*B* is the essential closure of *A* = *B* is the injective hull of *A*
B is isomorphic to $C(X_A)$	*B* is isomorphic to $N(X_A)$ = *B* is isomorphic to $C(X_B)$ where X_B is the Gleason cover of X_A

G. Bezhanishvili, P. J. Morandi, B. Olberding. *Bounded Archimedean ℓ-algebras and Gelfand duality*, under review.

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Thank you!