The Duality Between Direct and Predicate Transformer Semantics

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Together with G. D. Plotkin, R. Tix, Th. Streicher and others I have been involved in establishing a domain theoretical semantics for languages combining probability and nondeterminism. There is still one piece lacking in this enterprise. Everything takes place in the category of directed complete posets and Scott-continuous maps. In all of these cases one has an equivalence between a state transformer semantics and a predicate transformer semantics. In beginning to fill in the lacking piece I began to think about the general framework in which one can hope for an equivalence between the two types of semantics. As far as I can see this framework is rather narrow. The mothers for such equivalences are the continuation monads $R^{R^?}$ over a domain R of observations: There is a natural bijection between continuous maps $t\colon X \to R^{R^Y}$ (state transformers) and continuous maps $s\colon R^Y \to R^X$ (predicate transformers). Specifying an algebraic structure on R leads to two monads subordinate to the continuation monad, the monad $\mathcal{M}$, where $\mathcal{M}X$ is the dcpo of all algebra homomorphisms $h\colon R^X \to R$, and the free algebra monad $\mathcal{F}$, where $\mathcal{F}X$ is the the directed complete subalgebra of R^{R^X} generated by the projections $\widehat{x} = (f \mapsto f(x))\colon R^X \to R$. The equivalence between state and predicate transformer semantics works well for the monad $\mathcal{M}$, but the monad appropriate for semantics is the monad $\mathcal{F}$. We will discuss the question under which conditions the two monads agree. The notion of entropic algebras from universal algebra is crucial. The monads used for semantics of nondeterministic choice and for probabilistic choice fit into the above framework. In these examples one chooses R to be $(\mathbf{2}, \max, 0)$, $(\mathbf{2}, \min, 1)$, $(\overline{\mathbb{R}}_+, +, 0)$. When combining both kinds of choice one has to relax the above framework and use relaxed notions of homomorphism and entropicity.