

The duality between direct and predicate transformer semantics

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Basic intuition

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Direct semantics: A function $t: X \rightarrow Y$ that describes the input-output behaviour of the program.

Predicate transformer semantics: A function $s: 2^Y \rightarrow 2^X$ that for every predicate $p \subseteq Y$ concerning the output yields the weakest condition $s(p) \subseteq X$ on the input that guarantees that p holds for the output. This function s preserves propositional connectors, that is, it is a homomorphism of Boolean algebras.

Basic intuition (ctd.)

The two semantics are equivalent in the following sense:

Given $t: X \rightarrow Y$ define $s: 2^Y \rightarrow 2^X$ by

$$s(p) = t^{-1}(p) \text{ for every } p \subseteq 2^Y$$

Given a Boolean homomorphism $s: 2^Y \rightarrow 2^X$, define $t: X \rightarrow Y$ by

$$t(x) = y \text{ if } x \in s(\{y\})$$

Thus the duality between direct and predicate transformer semantics is the duality between the category of finite sets and the category of finite Boolean algebras. The two element Boolean algebra 2 plays the role of an algebra of observations. We can observe whether, for a program, a certain property is true or false.

Basic intuition (ctd.)

Above we have made an implicit hypothesis:

- For a given input a program produces exactly one output.

But if, **firstly**, we allow – for example – nondeterministic choice, a program may produce different outputs from a single input and, **secondly**, in the presence of recursion, there are nonterminating programs.

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Concerning the **first feature**, the programmer can use a choice operator $P \sqcup Q$ which allows to continue with program P or with program Q without any preference. On the semantic side, mimic this choice operator by a binary operation $\sqcup$. Natural identities:

$$p \sqcup p = p, \quad p \sqcup q = q \sqcup p, \quad (p \sqcup q) \sqcup r = p \sqcup (q \sqcup r)$$

that is, $\sqcup$ should be a semilattice operation.

Basic intuition (ctd.)

Such operators in the language have to be mimicked in the semantics. The operations in the model should not impose any property that is not enforced by these natural requirements, that is, the set Y together with the operation $\sqcup$ should generate the free semilattice FY over Y . Thus, a nondeterministic program will be interpreted by a map (a state transformer)

$$t: X \rightarrow FY$$

The predicate transformers $s: 2^Y \rightarrow 2^X$ corresponding to such state transformers are the $\sqcup$ -semilattice homomorphisms.

From universal algebra we know that the free semilattice over Y can be obtained as the subsemilattice $FY \subseteq 2^{2^Y}$ generated by the projections $\hat{y} = (f \mapsto f(y)): 2^Y \rightarrow 2$.

Directed complete posets (dcpos)

There are properties of programs that cannot be observed as, for example, nontermination. For this and other reasons, D.S. Scott, G.D. Plotkin, M.B. Smyth and others advocated the use the category DCPO instead of SET for semantics.

Objects: Directed complete posets (= dcpos), that is, posets X such that each directed subset D has a least upper bound $\sup D$.

Morphisms: Continuous maps $f: X \rightarrow Y$ between dcpos, that is, maps preserving (1) the order and (2) suprema of directed sets.

Examples: $\mathbf{2} = \{0, 1\}$ with $0 < 1$.

(1 stands for termination, 0 for nontermination.)

$\overline{\mathbb{R}}_+ = \{r \in \mathbb{R} \mid r \geq 0\} \cup \{+\infty\}$, usual order.

$[0, 1]$, usual order.

Observable properties

Observable property:

In $\mathbf{2}$ only 1 (termination) is observable, but not 0 (nontermination).

In $[0, 1]$: One can observe if a program terminates with probability $p > r$, but not, if it terminates with probability $p < r$ or $= r$.

In a dcpo X :

Continuous map $p: X \rightarrow \mathbf{2} \cong U = p^{-1}(1)$

$\cong$ **Scott-open** subset of X , that is, U is an upper set that cannot be reached by suprema of directed subsets disjoint from U .

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Remark: The category SET can be considered as a full subcategory of DCPO; just endow a set with the **discrete** partial order $=$.

There is an obvious forgetful functor from DCPO to SET: for a dcpo X , let $|X|$ denote the underlying set.

The category DCPO is cartesian closed

Products: $\prod_i X_i$ is the set theoretical product of dcpos X_i , ordered pointwise. Suprema of directed families are formed pointwise.

Exponentials: $R^X = [X \rightarrow R]$ denotes the set of all continuous $f: X \rightarrow R$ ordered pointwise. Suprema of directed families of continuous functions are formed pointwise.

Consequence: all functions definable in a natural way are continuous ('Definable in a natural way' means: definable by a λ -expression.)

Basic equivalence between state and predicate transformers

Let R be a fixed dcpo (of observations). To every dcpo X assign the dcpos R^X (predicates, previsions, expectations, ...) and $R^{R^X} = [R^X \rightarrow R]$ (valuations, distributions, ...).

Continuous maps $t: X \rightarrow R^{R^Y} = [R^Y \rightarrow R]$ are **state transformers**.
Continuous maps $s: R^Y \rightarrow R^X$ are **predicate transformers**.

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Equivalence Theorem

$$P: [R^Y \rightarrow R]^X \quad \longleftrightarrow \quad [R^Y \rightarrow R^X]: Q$$

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$$[X \rightarrow [R^Y \rightarrow R]] \cong [X \times R^Y \rightarrow R] \cong [R^Y \times X \rightarrow R] \cong [R^Y \rightarrow [X \rightarrow R]]$$

$$P(t)(g)(x) = t(x)(g)$$

$$Q(s)(x)(g) = s(g)(x)$$

The continuation monad

Let be given a fixed dcpo R .

Assigning to every dcpo X the dcpo $R^{R^X} = [R^X \rightarrow R]$ gives rise to a monad, the **continuation monad**:

Unit: $\delta_X: X \rightarrow [R^X \rightarrow R]$ defined for every $x \in X$ by

$$\delta_X(x)(f) = f(x)$$

the projection on the x -th coordinate. Notation: $\hat{x} = \delta_X(x)$.

Kleisli lifting: For $t: X \rightarrow R^{R^Y}$ define $t^\dagger: R^{R^X} \rightarrow R^{R^Y}$ by

$$t^\dagger(\varphi)(g) = \varphi(x \mapsto t(x)(g))$$

Monads over DCPO

The state transformer semantics of programs with effects is described by a monad $\mathcal{F}$ over DCPO : If X is the input domain and Y the output domain, the (state transformer) semantics of a program is a continuous map $t: X \rightarrow \mathcal{F}Y$.

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Desirables: - An equivalent predicate transformer semantics.

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- An equivalent predicate transformer semantics.
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Desirables: - An equivalent predicate transformer semantics.
- Identify the class of $\mathcal{F}$ -algebras by an equational theory that reflects the natural properties of the effects.

Claim: Continuation monads are the mothers of all monads $\mathcal{F}$ which admit an equivalent predicate transformer semantics, that is, for such monads $\mathcal{F}$ there is a dcpo R of 'observations' such that

$$\mathcal{F}X \subseteq [R^X \rightarrow R],$$

$$\delta_X(x) \in \mathcal{F}X \text{ for all } x \in X,$$

$$t^\dagger \text{ maps } \mathcal{F}X \text{ into } \mathcal{F}Y \text{ for every } t: X \rightarrow \mathcal{F}Y$$

Algebraic background

A **signature** Ω will be a sequence $(\Omega_0, \Omega_1, \Omega_2, \dots)$ of sets of operation symbols of arity $n = 0, 1, 2, \dots$ (e.g. $+$ in Ω_2).

A **d-algebra** will be a dcpo A endowed with continuous operations $\omega^A: A^n \rightarrow A$ for every $\omega \in \Omega_n, n \in \mathbb{N}$.

A **d-homomorphism** (of d-algebras) is a continuous homomorphism $h: A \rightarrow B$, that is, $h(\omega^A(a_1, \dots, a_n)) = \omega^B(h(a_1), \dots, h(a_n))$ (e.g., $h(a_1 + a_2) = h(a_1) + h(a_2)$).

We denote by $[A \multimap B]$ the collection of all d-homomorphisms $h: A \rightarrow B$.

Observe: $[A \multimap B]$ is a sub-dcpo of $[A \rightarrow B]$.

Monad I: the monad of homomorphism

From now on, fix a d-algebra R of 'observations'.

R^X becomes a d-algebra for every dcpo X , the operations being defined pointwise:

$$\text{e.g., } (f_1 + f_2)(x) = f_1(x) + f_2(x).$$

Similarly, $R^{R^X} = [R^X \rightarrow R]$ becomes a d-algebra.

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Properties:

(a) $\hat{x} = \delta_X(x): R^X \rightarrow R$ is a d-homomorphism for every $x \in X$.

(b) For every state transformer $t: X \rightarrow [R^Y \rightarrow R]$, the Kleisli lifting $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$ maps $[R^X \circ \rightarrow R]$ into $[R^Y \circ \rightarrow R]$ and

(c) $t^\dagger: [R^X \rightarrow R] \rightarrow [R^Y \rightarrow R]$ is a d-homomorphism.

The monad of homomorphisms

Result: A

Assigning to each dcpo X the dcpo $[R^X \multimap R]$ of all d-homomorphisms $\varphi: R^X \rightarrow R$ yields a monad over DCPO.

This monad is not of interest in itself, but it behaves well with respect to predicate transformers:

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Result B

The predicate transformers corresponding to the state transformers $t: X \rightarrow [R^Y \multimap R] \subseteq [R^Y \rightarrow R]$ are the d-homomorphisms $s: R^Y \rightarrow R^X$:

$$P: [R^Y \multimap R]^X \quad \longleftrightarrow \quad [R^Y \multimap R^X]: Q$$

Monad II: d-algebra monads

For any dcpo X , let $\mathcal{F}_R X$ be the d-subalgebra of $[R^X \rightarrow R]$ generated by the projections $\widehat{x}, x \in X$ (= the intersection of the d-subalgebras containing all the projections).

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Result C

Assigning to every dcpo X the d-algebra $\mathcal{F}_R X$ yields a monad with unit δ_X and Kleisli lifting $t^\dagger$.

Indeed, consider a continuous map $t: X \rightarrow \mathcal{F}_R Y$. By Property (c) on the previous slide, $t^\dagger$ is a homomorphism, thus it maps $\mathcal{F}_R X$ into the d-subalgebra of $[R^Y \rightarrow R]$ generated by the $t(x), x \in X$, which belong to the d-subalgebra $\mathcal{F}_R Y$ by hypothesis.

Freeness property

In classical universal algebra, a theorem due to G. Birkhoff says:
 $\mathcal{F}_R X$ is the free algebra over X in the 'variety' $\text{HSP}(R)$ of algebras which are homomorphic images of subalgebras of powers of R .

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Result D

$\mathcal{F}_R X$ is the free d-algebra over X in the class $\text{SP}(R)$ of d-algebras isomorphic to a d-subalgebra of some R^Y .

A d-algebra A belongs to $\text{SP}(R)$ if and only if it satisfies the **order separation property**, that is, if for any $a \not\leq a'$ in A , there is a d-homomorphism $h: A \rightarrow R$ such that $h(a) \not\leq h(a')$.

Questions

1. Another theorem of classical universal algebra due to Birkhoff: $\text{HSP}(R)$ consist of those algebras that satisfy all the equational laws that hold in R .

Is there an equational characterization of the class of d-algebras for which $\mathcal{F}_R X$ is free over X for every dcpo X ?

(This class might be bigger than $\text{SP}(R)$.)

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(This class might be bigger than $\text{SP}(R)$.)

2. It looks impossible to characterize the predicate transformers $s: R^Y \rightarrow R^X$ that correspond to the state transformers $t: X \rightarrow \mathcal{F}_R X$ in general. Therefore we ask:

Under what conditions do we have $[R^X \circ \rightarrow R] = \mathcal{F}_R X$ or, at least $\mathcal{F}_R Y \subseteq [R^X \circ \rightarrow R]$?

Entropic algebras

An algebra A is called **entropic** if, for every operation symbol $\omega \in \Omega_n$, the operation $\omega: A^n \rightarrow A$ is an algebra homomorphism, that is, for every $\sigma \in \Omega_m$, ω and σ **commute** in the sense that

$$\omega \circ \sigma^n = \sigma \circ \omega^m$$
$$\begin{array}{ccc} (A^n)^m \cong (A^m)^n & \xrightarrow{\sigma^n} & A^n \\ \omega^m \downarrow & & \downarrow \omega \\ A^m & \xrightarrow{\sigma} & A \end{array}$$

Note that entropicity is defined by a set of equational laws, one for each pair of operation symbols. Thus, entropicity is inherited by products, subalgebras and homomorphic images.

Examples of entropic algebras

In an entropic algebra, all constants agree and form a subalgebra.
A binary operation $*$ commutes with itself if and only if

$$(\text{median law}) \quad (a * b) * (c * d) = (a * c) * (b * d)$$

Thus commutative monoids and semilattices are entropic.

Two binary operations $*$ and $+$ commute iff

$$(a + b) * (c + d) = ((a * c) + (b * d))$$

Thus rings, lattices, ... are not entropic.

A unary operation u commutes with a binary operation $+$ iff
 $u(a + b) = u(a) + u(b)$. Thus vector spaces are entropic.

Entropicity (ctd.)

Result E

A d-algebra R is entropic iff $[A \circ \rightarrow R]$ is a d-subalgebra of R^A for every d-algebra A .

We now have an answer to our second question:

Entropy (ctd.)

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Result F

If R is an entropic d-algebra, then $[R^X \circ \rightarrow R]$ is a subalgebra of $[R^X \rightarrow R]$ and $\mathcal{F}_R X \subseteq [R^X \circ \rightarrow R]$.

I do not see a general property implying equality $\mathcal{F}_R X = [R^X \circ \rightarrow R]$. But one might look at this question more closely.

Example: The angelic powerdomain $\mathcal{H}X$

d-Algebra of observations $\mathbf{2} = \{0 < 1\}$, signature $\Omega = \{\vee, 0\}$, is a d-join-semilattice with bottom, hence entropic.

$\mathcal{H}X = [\mathbf{2}^X \multimap \mathbf{2}]$ is a d-join-semilattice with bottom.

The predicate transformers corresponding to the state transformers $t: X \rightarrow \mathcal{H}Y$ are those $s: \mathbf{2}^Y \rightarrow \mathbf{2}^X$ preserving arbitrary joins.

For $\varphi \in [\mathbf{2}^X \multimap \mathbf{2}]$ we have $\varphi = \sup\{\hat{x} \mid \hat{x} \leq \varphi\}$. Hence $\mathcal{F}_2 X = [\mathbf{2}^X \multimap \mathbf{2}] = \mathcal{H}X$.

For a d-join-semilattice S with bottom the continuous join-semilattice homomorphisms into $\mathbf{2}$ preserving bottom are order separating. Hence $S \in \text{SP}(\mathbf{2})$.

Thus:

$\mathcal{H}X$ is the free d-join-semilattice with bottom over X .

(Extended) probabilistic choice

d-Algebra of observations: $\overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{+\infty\}$ of signature $\Omega = (+, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is entropic.

$\mathcal{V}X = [\overline{\mathbb{R}}_+^X \circ \rightarrow \overline{\mathbb{R}}_+]$ is a d-cone.

The predicate transformers corresponding to the state transformers $t: X \rightarrow \mathcal{V}X$ are the continuous linear maps $s: \overline{\mathbb{R}}_+^Y \rightarrow \overline{\mathbb{R}}_+^X$.

For a **continuous** dcpo X , every $\varphi \in \mathcal{V}X$ is the join of a directed family of finite linear combinations $\sum_i r_i \hat{x}_i$ (C. Jones, R. Tix).

Hence, $\mathcal{F}_{\overline{\mathbb{R}}_+} X = [\overline{\mathbb{R}}_+^X \circ \rightarrow \overline{\mathbb{R}}_+] = \mathcal{V}X$.

For a continuous d-cone C the continuous linear maps into $\overline{\mathbb{R}}_+$ are order separating (Tix' Hahn-Banach), whence $C \in \text{SP}(\overline{\mathbb{R}}_+)$.

Thus

For continuous X , $\mathcal{V}X$ is the free continuous d-cone over X .

Continuous domains order theoretically

Given a dcpo P and $u, v \in P$ we say

- $u \ll v$ (u is **way-below** v or u is **relatively compact** in v) if
$$x_i \nearrow v \implies \exists i. x_i \geq u$$

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- $u \ll v$ (u is **way-below** v or u is **relatively compact** in v) if $x_i \nearrow v \implies \exists i. x_i \geq u$
- A **good approximation** of v is a directed family $x_i \ll v$ such that $x_i \nearrow v$
- $B \subseteq P$ is a **basis** if every $v \in P$ has a good approximation by a directed family of elements in B .
- A dcpo P is called **continuous**, if it has a basis.

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Example: $\overline{\mathbb{R}_+} = \mathbb{R}_+ \cup \{+\infty\}$ is a continuous dcpo;
 $r \ll s$ iff $r < s$; the rational numbers form a basis;
the Scott-open sets are the intervals $]r, +\infty]$.

Interlude: Cones

Cones are vector spaces without $-x$. More precisely, the signature consists of a constant 0 a binary operation $+$ and unary operation $r \cdot ?$, $r \in \overline{\mathbb{R}}_+$ (multiplication with scalars). A **cone** is an algebra with a constant 0 and an addition $+$

satisfying the laws of a commutative monoid

and a scalar multiplication $(r, x) \mapsto r \cdot x: \overline{\mathbb{R}}_+ \times C \rightarrow C$ satisfying

$$\begin{array}{ll} r(x + y) &= rx + ry & (r + s)x &= rx + sx \\ & & r(sx) &= (rs)x \\ 0 \cdot x &= 0 & 1 \cdot x &= x \end{array}$$

Cones are entropic. Homomorphisms of cones are **linear maps**, since they are characterised by the equations

$h(x + y) = h(x) + h(y)$ and $h(rx) = rh(x)$ for all $r \in \overline{\mathbb{R}}_+$.

Combining angelic nondeterministic and probabilistic choice

d-Algebra of observations: $\overline{\mathbb{R}}_+$ of signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is **not** entropic.

$[\overline{\mathbb{R}}_+^X \circ \rightarrow \overline{\mathbb{R}}_+]$ is **not** an algebra.

But instead of equality we have still the inequality

$$(a + c) \vee (b + d) \leq (a \vee b) + (c \vee d)$$

This incites us to introduce a relaxed notion of entropicity.

Relaxed entropic algebras

For operations ω and σ on a d-algebra A , say that ω **subcommutes** with σ (and σ supercommutes with ω), if $\omega \circ \sigma^n \leq \sigma \circ \omega^m$. (Note: This is an inequational law.)

We suppose that the signature Ω is the union of two subsets $\Omega_{\leq}$ and $\Omega_{\geq}$ which may overlap.

We say that a d-algebra of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$ is **relaxed entropic** if every $\omega \in \Omega_{\leq}$ subcommutes with every $\sigma \in \Omega$ and every $\omega \in \Omega_{\geq}$ supercommutes with every $\sigma \in \Omega$.

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Example: $\overline{\mathbb{R}}_+$ of signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, is relaxed entropic if we put $+$ in $\Omega_{\leq}$ and $\vee$ in $\Omega_{\geq}$; the constant 0 and the unary operations of multiplication with scalars are in both $\Omega_{\leq}$ and $\Omega_{\geq}$.

Relaxed d-morphisms

A **relaxed d-morphism** between two d-algebras A, B of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$ is a continuous map $h: A \rightarrow B$ such that

$$h(\omega(a_1, \dots, a_n)) \leq \omega(h(a_1), \dots, h(a_n))$$

for every $\omega \in \Omega_{\leq}$ of arity n , and the other way around for $\omega \in \Omega_{\geq}$.

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Example: Consider d-algebras A, B of signature $\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, with $+$ in $\Omega_{\geq}$ and $\vee$ in $\Omega_{\leq}$.

Relaxed d-morphism: Continuous map $h: A \rightarrow B$ satisfying:

$$\begin{aligned} h(0) &= 0 \\ h(ra) &= rh(a) \text{ for all } r \in \overline{\mathbb{R}}_+ \\ h(a + b) &\leq h(a) + h(b) \\ h(a \vee b) &\geq h(a) \vee h(b) \end{aligned}$$

Since the last requirement is satisfied for order preserving maps anyway, it can be omitted. Thus:

relaxed d-morphism = continuous **sublinear** map.

Relax

Fix a d-algebra R of signature $\Omega = \Omega_{\leq} \cup \Omega_{\geq}$.

For every dcpo X , let $[R^X \multimap_r R]$ denote the set of relaxed d-homomorphisms $R^X \rightarrow R$. They form a sub-dcpo of $[R^X \rightarrow R]$.

For every state transformer $t: X \rightarrow [R^Y \rightarrow R]$, the Kleisli lifting $t^\dagger$ maps $[R^X \multimap_r R]$ to $[R^Y \multimap_r R]$. Thus, $X \mapsto [R^X \multimap_r R]$ yields a monad with unit δ and Kleisli lifting $t^\dagger$.

The predicate transformers corresponding to state transformers $t: X \rightarrow [R^Y \multimap_r R]$ are the relaxed homomorphisms $s: R^Y \rightarrow R^X$.

$[R^X \multimap_r R]$ is a subalgebra of $[R^X \rightarrow R]$ if and only if R is relaxed entropic. And in this case $\mathcal{F}_R X$ is a subalgebra of $[R^X \multimap_r R]$.

Example: Combining angelic and probabilistic choice

d-Algebra of observations: $\overline{\mathbb{R}}_+$, signature

$\Omega = (+, \vee, 0, (x \mapsto rx)_{r \in \overline{\mathbb{R}}_+})$, with $+$ in $\Omega_{\leq}$ and $\vee$ in $\Omega_{\geq}$,
is relaxed entropic.

Equational Theory (d-cone join-semilattice): Cone and
join-semilattice axioms connected by the distributivity laws

$$\begin{aligned}x + (y \vee z) &= (x + y) \vee (x + z) \\r \cdot (y \vee z) &= (r \cdot y) \vee (r \cdot z)\end{aligned}$$

$\mathcal{HV}X = [\overline{\mathbb{R}}_+^X \circ \rightarrow_r X]$, the set of sublinear functionals on $\overline{\mathbb{R}}_+^X$ is a
d-subalgebra of $[\overline{\mathbb{R}}_+^X \rightarrow \overline{\mathbb{R}}_+]$.

The state transformers corresponding to the state transformers
 $t: X \rightarrow \mathcal{HV}X$ are the sublinear maps $s: \overline{\mathbb{R}}_+^Y \rightarrow \overline{\mathbb{R}}_+^X$.

Example: Combining angelic and probabilistic choice (ctd.)

If X is a continuous dcpo, then $\mathcal{HV}X = \mathcal{F}_{\overline{\mathbb{R}_+}} X$. (For the proof one needs a Hahn-Banach type argument, that every continuous sublinear functional is the join of a family of continuous linear functionals.)

For every continuous dcpo X , $\mathcal{HV}X$ is the free continuous d-cone join-semilattice over X .

Conclusion

I hope I have extracted the general pattern of those situations in which one can hope for an equivalence (better duality) between state and predicate transformer semantics using the category of dcpos .

The general pattern does not relieve us from substantial work in each particular situation. But the shape of the properties properties to prove is clarified.

Further work

1. Relaxed entropicity and relaxed morphisms make sense for ordered algebras. It makes sense to investigate these notions for monads over the category POSET as in intermediary step between SET and DCPO.

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2. In the (relaxed) entropic setting the monad of (relaxed) homomorphisms contains the algebra monad $\mathcal{F}_R X$. Find reasonable sufficient conditions under which the two monads agree.
3. Are there more good examples fitting into this setting?
4. Is there a relaxed setting for other categories as, for example, qcb-spaces?

The End

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You want to try again?