

Dualities of Stably Compact Spaces

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Locally Compact Spaces

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The **order of specialization** on X is given by $x \leq y$ iff $x \in \overline{\{y\}}$. The relation $\leq$ is a pre-order and is a partial order iff X is a T_0 -space. We write $\downarrow y = \{x : x \leq y\} = \overline{\{y\}}$. (In general, $\downarrow A := \{y : \exists x \in A, y \leq x\}$.)

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The **saturation** of a subset A is the intersection of all open sets containing it, and a set is **saturated** if it is equal to its saturation. The saturation of A is equal to $\uparrow A := \{y : \exists x \in A, x \leq y\}$. The saturation of a compact set is again compact, so we can replace K in the definition of locally compact by its saturation.

Stably Compact Spaces

A space X is **coherent** if the intersection of any two compact saturated sets is again compact and is **stably compact** if it is a compact, locally compact, coherent sober space. One may replace the sobriety condition by requiring the space to be T_0 and **well-filtered**, that is, if a descending family of compact saturated sets has intersection contained in some open set, then some member of the family must be in the set.

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One may alternatively obtain all stably compact spaces by considering all **compact pospaces**, compact (Hausdorff) spaces equipped with closed partial orders ($\leq := \{(x, y) : x \leq y\}$ is closed in $X \times X$), and passing to the topology consisting of the open upper sets.

The Fundamental Duality

For a topological space (X, τ) , we define the **cocompact** topology τ^c by taking as a base for the topology the complements of compact saturated subsets. The topological space (X, τ^c) is called the **de Groot dual** of X and denoted X^d . The de Groot dual X^d has order of specialization $\geq$, the reverse of the order of specialization of X .

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Proposition. (de Groot Duality) Given a stably compact space X , the de Groot dual X^d is again stably compact. Furthermore, $X^{dd} = X$.

The Patch Topology

(1) The **patch topology** of a topological space is generated by the join of the topology and its de Groot dual topology. If X is stably compact, then the patch topology is a compact Hausdorff topology for which the original order of specialization is a closed order, i.e., $(X, \text{patch}, \leq)$ is a compact pospace.

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(2) If X is a compact Hausdorff space equipped with a closed order, then $X^\uparrow$, X equipped with the topology of open upper sets, is stably compact with order of specialization the given order.

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- (2) If X is a compact Hausdorff space equipped with a closed order, then $X^\uparrow$, X equipped with the topology of open upper sets, is stably compact with order of specialization the given order.
- (3) Constructions (1) and (2) are inverse constructions.

Equivalence of Categories

The preceding equivalence extends to a categorical equivalence between the category Stab^* of stably compact spaces and perfect maps (continuous maps for which inverse images of compact, saturated sets are compact) and the category of CP of compact pospaces and continuous maps, where the inverse equivalence functors are the identity on maps.

The Fundamental Duality II

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Given a stably compact space X and the corresponding compact pospace $(X, \text{patch}, \leq)$, the compact pospace corresponding to X^d is $(X, \text{patch}, \geq)$, the compact pospace of X with the order reversed. Thus de Groot duality corresponds to the duality of order reversal at the pospace level.

Inner and Outer Pavings

A **paving** on a set X is a non-empty collection of subsets. Assume that $X \neq \emptyset$ is equipped with two pavings, an **inner paving** $\mathcal{I}$ and an **outer paving** $\mathcal{O}$ satisfying:

- (i) $\emptyset \in \mathcal{I}; X \in \mathcal{O}$.
- (ii) for each $I \in \mathcal{I}$ and $O_1, O_2 \in \mathcal{O}$ such that $I \subseteq O_1 \cap O_2$, there exists $O_3 \in \mathcal{O}$ such that $I \subseteq O_3 \subseteq O_1 \cap O_2$. We say that $\mathcal{O}$ *filters to* $\mathcal{I}$.
- (iii) for each $O \in \mathcal{O}$ and $I_1, I_2 \in \mathcal{I}$ such that $I_1 \cup I_2 \subseteq O$, there exists $I_3 \in \mathcal{I}$ such that $I_1 \cup I_2 \subseteq I_3 \subseteq O$. We say $\mathcal{I}$ is *directed to* $\mathcal{O}$.

Members of $\mathcal{I}$ are called **inner sets**; members of $\mathcal{O}$ are called **outer sets**. The triple $(X, \mathcal{I}, \mathcal{O})$ is called an **$\mathcal{IO}$ -structure**.

Interpolated $\mathcal{IO}$ -Structures

Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -structure. If for each $I \in \mathcal{I}$ and $O \in \mathcal{O}$ with $I \subseteq O$, there exist $I' \in \mathcal{I}$ and $O' \in \mathcal{O}$ such that $I \subseteq O' \subseteq I' \subseteq O$. We say that $\mathcal{I}$ and $\mathcal{O}$ **interpolate** and call the structure an **interpolated** $\mathcal{IO}$ -structure.

Stably Compact $\mathcal{IO}$ -Structures

Let X be a topological space, let the inner sets $\mathcal{I} = \mathcal{Q}$, the compact saturated sets, and the outer sets $\mathcal{O} = \mathcal{G}$, the open sets. Then $(X, \mathcal{Q}, \mathcal{G})$ is an interpolated $\mathcal{IO}$ -structure iff X is locally compact.

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In particular, we obtain an interpolated $\mathcal{IO}$ -structure on a stably compact space X , namely $(X, \mathcal{Q}, \mathcal{G})$, where $\mathcal{Q}$ is the collection of compact saturated sets and $\mathcal{G}$ is the collection of open sets, which we call the **associated** $\mathcal{IO}$ -structure.

Dual $\mathcal{IO}$ -Structures

Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -structure. Let $\mathcal{I}^\Delta$ be the collection of all complements of members of $\mathcal{O}$ and let $\mathcal{O}^\Delta$ consist of all complements of members of $\mathcal{I}$. The triple $(X, \mathcal{I}^\Delta, \mathcal{O}^\Delta)$ is called the **dual structure**.

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Lemma. The dual $X^\Delta := (X, \mathcal{I}^\Delta, \mathcal{O}^\Delta)$ of an $\mathcal{IO}$ -structure $(X, \mathcal{I}, \mathcal{O})$ is again an $\mathcal{IO}$ -structure, which is interpolated iff X is. Furthermore, $X^{\Delta\Delta} = X$.

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The duality $X^{\Delta\Delta} = X$ is called the **$\mathcal{IO}$ -duality**.

$\mathcal{IO}$ -Duality and de Groot Duality

Let X be stably compact. The associated $\mathcal{IO}$ -structure for X^d is the dual $\mathcal{IO}$ -structure X^Δ . Thus the de Groot duality of stably compact spaces corresponds to the $\mathcal{IO}$ -duality of the corresponding $\mathcal{IO}$ -structures, i.e, the $\mathcal{IO}$ -dual of the associated $\mathcal{IO}$ -structure is the the associated $\mathcal{IO}$ -structure of X^d :

$$(X, \mathcal{Q}, \mathcal{G})^\Delta = (X, \mathcal{Q}^\Delta, \mathcal{G}^\Delta) = (X^d, \mathcal{Q}(X^d), \mathcal{G}(X^d)).$$

Topologies for $\mathcal{I}$ and $\mathcal{O}$

Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -structure. The $\mathcal{I}$ -topology on $\mathcal{O}$ has as a basis of open sets all sets of the form $\{O \in \mathcal{O} : I \subseteq O\}$ as I ranges over $\mathcal{I}$. Similarly the $\mathcal{O}$ -topology on $\mathcal{I}$ has as basis all sets $\square(O) := \{I \in \mathcal{I} : I \subseteq O\}$ as O ranges over $\mathcal{O}$. (The defined sets form bases since $\mathcal{I}$ is directed to $\mathcal{O}$ and $\mathcal{O}$ filters to $\mathcal{I}$.)

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If X is a topological space, $\mathcal{I}$ is the collection of closed sets or compact sets, and $\mathcal{O}$ is the collection of open sets, then the $\mathcal{O}$ -topology is often called the **upper Vietoris topology**. The collection $\diamond(O) := \{C \in \mathcal{I}^\Delta : C \cap O \neq \emptyset\}$ as O varies over $\mathcal{O}$ is called the **lower Vietoris topology** on $\mathcal{I}^\Delta$, the collection of closed sets.

Identifying Topologies on $\mathcal{I}$ and $\mathcal{O}$

Let X be a locally compact sober space with the associated $\mathcal{IO}$ -structure $(X, \mathcal{Q}, \mathcal{G})$. Then $\mathcal{Q}$, equipped with the order of reverse inclusion, is a continuous dcpo, and $\mathcal{G}$ is a continuous lattice. From the Hofmann-Mislove Theorem:

the $\mathcal{G}$ -topology on $\mathcal{Q}$ = the upper Vietoris topology = the Scott topology.

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The lower Vietoris topology on the cocontinuous lattice of closed sets $\mathcal{I}^\Delta$ (ordered by inclusion) agrees with its weak upper topology (defined by taking all principal ideals $\downarrow x$ as a subbasis of closed sets).

Remarks on Complementarity

Since the outer (resp. inner) sets of an $\mathcal{IO}$ -structure X correspond to the inner (resp. outer) sets of X^Δ under the duality of $\mathcal{IO}$ -structures X and X^Δ , general statements about outer sets have dual statements about inner sets, and a statement holds if and only if its dual holds. We call this the **duality of complementarity**.

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Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -structure. The mapping of complementation $O \mapsto O^c$ from $\mathcal{O}$ to $\mathcal{I}^\Delta$ (resp. $I \mapsto I^c$ from $\mathcal{I}$ to $\mathcal{O}^\Delta$) is a homeomorphism.

Stably Compact Domains

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Thus the two topologies, the Scott and the weak lower, are the de Groot duals of each other if (P, Scott) is stably compact, which is the case iff P with the patch=Lawson topology is compact.

An Angelic-Demonic Duality

Let X be a stably compact space. We model demonic nondeterminism with the **Smyth powerdomain** $\mathcal{Q}(X)$ of all compact saturated sets with the reverse inclusion order and the upper Vietoris=Scott topology and angelic determinism by the **Hoare powerdomain** $\mathcal{H}(X)$ consisting of all closed sets ordered by inclusion with the lower Vietoris=weak upper topology.

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From the earlier considerations we have:

Proposition (Goubault-Larrecq). For X stably compact

$$\mathcal{Q}(X^d) = \mathcal{H}(X)^d.$$

Stably Compact Spaces and Capacities

We consider certain powerdomains recently introduced by Goubault-Larrecq that model more elaborate choice procedures based on Choquet's theory of capacities, which are generalizations of measures. One obtains factorization theorems that characterize the choice process as a probabilistic choice followed by a nondeterministic one. In these more sophisticated models, one notes that stable compactness is preserved in their construction, that angelic and demonic remain dual concepts, and that random choice, like erratic choice (the Plotkin powerdomain), is self-dual.

Capacities

Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -paving. A map $c : \mathcal{I} \cup \mathcal{O} \rightarrow [0, 1]$ is called a **capacity** if for each $I \in \mathcal{I}$

$$c(I) = c^*(I) := \inf\{c(O) : I \subseteq O \in \mathcal{O}\} \quad (1)$$

and for each $O \in \mathcal{O}$

$$c(O) = c_*(O) := \sup\{c(I) : O \supseteq I \in \mathcal{I}\}. \quad (2)$$

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$$c(I) = c^*(I) := \inf\{c(O) : I \subseteq O \in \mathcal{O}\} \quad (3)$$

and for each $O \in \mathcal{O}$

$$c(O) = c_*(O) := \sup\{c(I) : O \supseteq I \in \mathcal{I}\}. \quad (4)$$

We could replace $[0, 1]$ in the preceding definition by any closed interval in the extended reals $[-\infty, \infty]$, but restrict to the case of primary interest for us.

Outer Capacities

Recall the $\mathcal{O}$ -topology on $\mathcal{I}$ and $\mathcal{I}$ -topology on $\mathcal{O}$.

Lemma. For $c : \mathcal{O} \rightarrow [0, 1]$, $c^* : \mathcal{I} \rightarrow [0, 1]$ defined by $c^*(I) = \inf\{c(O) : I \subseteq O \in \mathcal{O}\}$ is order preserving and upper semicontinuous on $\mathcal{I}$. Dually $c : \mathcal{I} \rightarrow [0, 1]$ induces an order preserving, lower semicontinuous $c_* : \mathcal{O} \rightarrow [0, 1]$.

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Proposition. For an $\mathcal{IO}$ -structure $(X, \mathcal{I}, \mathcal{O})$ and $c : \mathcal{O} \rightarrow [0, 1]$, the following are equivalent and define **outer capacities**:

1. c is lower semicontinuous;
2. $c = c^*_*$;
3. $c = d_*$ for some $d : \mathcal{I} \rightarrow [0, 1]$.

A dual result holds for $c : \mathcal{I} \rightarrow [0, 1]$ and defines **inner capacities**.

Capacities as Extensions

Corollary. For an $\mathcal{IO}$ -structure $(X, \mathcal{I}, \mathcal{O})$ and an outer capacity $c : \mathcal{O} \rightarrow [0, 1]$, $\tilde{c} : \mathcal{I} \cup \mathcal{O} \rightarrow [0, 1]$ defined by $\tilde{c}(O) = c(O)$ and $\tilde{c}(I) = c^*(I)$ is a capacity, which is unique in the sense that it is the only capacity extending c . Dually for an inner capacity $c : \mathcal{I} \rightarrow [0, 1]$, $\tilde{c}(I) = c(I)$ and $\tilde{c}(O) = c_*(O)$ defines a unique capacity extension.

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The previous extension process defines a one-to-one correspondence between the set of lower semicontinuous functions $LSC(\mathcal{O}, [0, 1])$ and the set of capacities.

Alternatively, we obtain a bijection via extension between $USC(\mathcal{I}, [0, 1])$ and $C_p(X)$. Hence we may regard the capacities in any of these three equivalent ways. This equivalence provides some grounds for understanding why the notion of a capacity has various formulations in the literature. (Also varied terminology: previsions, games, etc.)

Topologizing Capacities

We topologize the set $Cp(X)$ of capacities with respect to an $\mathcal{IO}$ -paving on X with subbasic open sets of the form

$$\begin{aligned}[s < U] &:= \{\mu \in Cp(X) : \mu(U) > s\} \text{ for } s \in \mathbb{R}, U \in \mathcal{O}; \\ [t > I] &:= \{\mu \in Cp(X) : \mu(I) < t\} \text{ for } t \in \mathbb{R}, I \in \mathcal{I}.\end{aligned}$$

The first of these is called the *outer topology* on $Cp(X)$, the second the *inner topology*, and the patch of the two is called the $\mathcal{IO}$ -topology.

C -Spaces

A T_0 -space X is called a C -space (Erné) or an α -space (Ershov) if each of its points has a neighborhood basis of principal filters $\uparrow x = \{y \in X \mid x \leq y\}$ (sometimes called **cores**) with respect to the specialization order. (This means that given $y \in U$, U open, there exists $x \in U$ and V open such that $y \in V \subseteq \uparrow x \subseteq U$.) The definition may be applied to general topological spaces, and in absence of the T_0 -requirement, we call such spaces c -spaces.

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The standard result that sets of the form $\{y : x \ll y\}$ form a basis for the Scott topology in a continuous poset shows that a continuous poset equipped with the Scott topology is a C -space. Conversely, the c -spaces are precisely those with sobrification a continuous dcpo equipped with the Scott topology (or with completely distributive $\mathcal{O}(X)$).

The Space of Capacities

Theorem. Let $(X, \mathcal{I}, \mathcal{O})$ be an interpolated $\mathcal{IO}$ -paving.

1. $\mathcal{I}, \mathcal{O}$ are c -spaces.
2. The set $Cp(X) \cong LSC(\mathcal{O}, [0, 1])$ endowed with the pointwise order of the latter is a completely distributive lattice, in particular a continuous lattice.
3. The outer, Scott, and weak upper topologies on $Cp(X)$ all agree with the topology of pointwise convergence of $Cp(\mathcal{O}, [0, 1])$ where $[0, 1]$ is endowed with the Scott topology. Furthermore, the pointwise order is the order of specialization for these topologies.
4. Analogous statements hold for the inner topology via the identification $Cp(X) \cong USC(\mathcal{I}, [0, 1])$. The topology in this case is the de Groot dual of the outer topology.

The Space of Capacities (cont.)

5. The $\mathcal{IO}$ -topology of $C_p(X)$ is the patch of the outer and inner topologies, and is equal to the interval, bi-Scott, and Lawson topologies, all of which collapse together for the case of completely distributive lattices.

Totally Convex Capacities

A capacity μ is **totally convex** (also called **totally monotone**) if $\mu(\emptyset) = 0$ and it satisfies on $\mathcal{O}$ the inclusion-exclusion inequality

$$\mu\left(\bigcup_{i=1}^n U_i\right) \geq \sum_{i=1}^n \mu(U_i) - \sum_{i \neq j} \mu(U_i \cap U_j) + \cdots + (-1)^{n+1} \mu(U_1 \cap \dots \cap U_n)$$

(TCX_n)

for each n . A special case is the case of the *convex capacities*, which satisfy the inequality for $n = 2$. Totally convex capacities are called *credibilities* or *belief functions*.

The *totally concave* capacities, or *plausibilities*, are those for which the reverse inequality (TCC_n) holds.

Stable Compactness and Convexity

Theorem. Assume the the operation of binary union on $\mathcal{O}$ is exists and is continuous.

1. If a monotone function $\mu \in [0, 1]_{\leq}^{\mathcal{O}}$ satisfies (TCX_n) or (TCC_n) , then so does its lsc hull $\rho(\mu) = \mu^\dagger$.
2. For each n , the set of capacities satisfying (TCX_n) or (TCC_n) is compact in the $I\mathcal{O}$ -topology of $Cp(X)$.
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3. The subset of credibilities resp. plausibilities is a stably compact space.

The previous assertions are true if X is a coherent locally compact sober space, $\mathcal{I}$ is the collection of compact saturated sets, and $\mathcal{O}$ is the collection of open sets. In this case binary union is continuous on $\mathcal{O}$.

Duality

We outline a general duality theorem for capacities and in a sequence of steps sharpen it to a result on stably compact spaces.

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Step 1. Let $(X, \mathcal{I}, \mathcal{O})$ be an $\mathcal{IO}$ -paving for which $\emptyset \in \mathcal{I}$ and $X \in \mathcal{O}$. We define for each $\mu \in Cp^1(X)$, the set of normalized capacities ($\mu(X) = 1, \mu(\emptyset) = 0$) on X , the *dual capacity* $\mu^\perp \in Cp^1(X^\Delta)$:

$$\mu^\perp(X \setminus A) = 1 - \mu(A) \text{ for } A \in \mathcal{I} \cup \mathcal{O}.$$

Duality (2)

Step 2.

1. $\mu \in Cp^1(X)$ implies $\mu^\perp \in Cp^1(X^\Delta)$.
2. The map $\mu \mapsto \mu^\perp$ from $Cp^1(X)$ to $Cp^1(X^\Delta)$ has inverse $\mu^\perp \mapsto \mu^{\perp\perp} = \mu$ from $Cp^1(X^\Delta)$ to $Cp^1(X)$.
3. The map $\mu \mapsto \mu^\perp$ from $Cp^1(X)$ to $Cp^1(X^\Delta)$ is an order-preserving bijection (the order being the pointwise order) and a homeomorphism between the outer resp. inner resp. $\mathcal{IO}$ -topology of $Cp^1(X)$ and the inner resp. outer resp. $\mathcal{IO}$ -topology of $Cp^1(X^\Delta)$.

Duality (3)

Step 3. Let us assume now in addition that $(X, \mathcal{I}, \mathcal{O})$ is an interpolated $\mathcal{IO}$ -paving. Then for the space of normalized capacities we have that $Cp^1(X)$ with the outer topology is a stably compact space with dual space the inner topology and patch topology the $\mathcal{IO}$ -topology. We obtain from this result and Step 2 that

$$(Cp^1(X, \text{outer}))^d = Cp^1(X, \text{inner}) \xrightarrow{\perp} Cp^1(X^\Delta, \text{outer}),$$

where the right arrow is $\mu \mapsto \mu^\perp$.

Duality (4)

Step 4. Let X be a stably compact space. Then X^Δ is the standard $\mathcal{IO}$ -paving for X^d . This leads to the following modification of Step 3:

$$\begin{aligned}(Cp^1(X, \text{outer}))^d = Cp^1(X, \text{inner}) &\xrightarrow{\perp} Cp^1(X^\Delta, \text{outer}) \\ &= Cp^1(X^d, \text{outer}).\end{aligned}$$

If we make the convention of equipping the capacity space with the stably compact outer topology, then the preceding equation can be stated: *the de Groot dual of the normalized capacity space is homeomorphic to the normalized capacity space of the de Groot dual via the involution $\mu \mapsto \mu^\perp$.*

Duality (5)

Step 5. The negative sign introduced in $\mu^\perp$ turns a capacity satisfying (TCX_n) to one satisfying (TCC_n) . Using the previous steps and the fact that these inequalities extend to $\mathcal{I}$ and dually, we obtain the following result of Goubault-Larrecq.

Theorem (Convex-Concave Duality). (i) Let X be stably compact. For every normalized capacity μ on X , $\mu^\perp$ is a normalized capacity on X^d . If μ is (totally) convex, then $\mu^\perp$ is (totally) concave and vice-versa. If μ is a continuous valuation, then so is $\mu^\perp$. Finally, $\mu^{\perp\perp} = \mu$.

(ii) The duality via $\perp$ induces a homeomorphism between $J^\cup(X)^d$, the totally convex capacities on X equipped with its co-compact inner topology, and $J^\cap(X^d)$, the totally concave capacities on X^d with its stably compact outer topology.

Duality (6)

Step 6. The preceding theorem shows that random choice as captured by the probabilistic power domain of a stably compact space is a self-dual concept. With a little extra work, one can obtain another angelic-demonic duality by a reformulation of the totally convex-concave duality.

Duality (6)

Step 6. The preceding theorem shows that random choice as captured by the probabilistic power domain of a stably compact space is a self-dual concept. With a little extra work, one can obtain another angelic-demonic duality by a reformulation of the totally convex-concave duality.

We recall what Goubault-Larrecq calls the fundamental theorem of credibilities, which states that credibilities are nothing else than specifications of random choices among sets of possible demonic choices, i.e., they are continuous valuations (we assume everything is normalized) on $\mathcal{Q}(X)$. More specifically, for X stably compact and for any continuous valuation μ on $\mathcal{Q}(X)$, the capacity defined by $\tilde{\mu}(O) = \mu(\Box(O))$ is a credibility on X , and every credibility arises uniquely in this way.

Duality (7)

Step 7. Via this identification (or factorization) the earlier duality theorem can be restated as a duality between the de Groot dual of the probabilistic powerdomain on $\mathcal{Q}(X)$ and the probabilistic powerdomain on the Hoare powerdomain of X^d . This is again a type of demonic-angelic duality, involving as it does the Smyth and Hoare powerdomains.

References

Principal ideas of the preceding material come from items 1. and 2. below. Our aim has been to show that at numerous points the two can be harmoniously blended together and nicely complement each other.

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