Modal Compact Hausdorff Spaces I

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Duality Workshop, Oxford, June 2012

Overview

Our aim is to lift aspects of modal logic from the setting of zero-dimensional compact Hausdorff spaces, to that of general compact Hausdorff spaces.

The talk will be in two parts.

Preliminaries — Modal Logic

Definition A modal space (X, R) is a pair where X is a Stone space and R is a relation on X that satisfies

1. $R[x]$ is closed for each $x \in X$
2. $R^{-1}U$ is clopen for each clopen $U \subseteq X$.

Definition A modal algebra $(B, \Diamond)$ is a pair where B is a Boolean algebra and $\Diamond$ is an operation that satisfies

1. $\Diamond 0 = 0$
2. $\Diamond(x \vee y) = \Diamond x \vee \Diamond y$.

Proposition For (X, R) a modal space $(\text{Clop } X, \Diamond)$ is a modal algebra where $\Diamond U = R^{-1}U$.

Proposition For $(B, \Diamond)$ a modal algebra (B^*, R) is a modal space where B^* is the space of prime filters and $p\,R\,q \Leftrightarrow \Diamond q \subseteq p$.

These extend to a dual equivalence between the category MS of modal spaces and continuous p-morphisms $(f \circ R = R \circ f)$, and the category MA of modal algebras and their homomorphisms.

Preliminaries — Coalgebras and Modal Logic

Definition For X a Stone space, let $\mathcal{F}X$ be its closed sets and for each open $U \subseteq X$ define

$$\begin{aligned} \Diamond U &= \{F \in \mathcal{F}X : F \cap U \neq \varnothing\} \\ \Box U &= \{F \in \mathcal{F}X : F \subseteq U\} \end{aligned}$$

The Vietoris space $\mathcal{V}X$ is the space on the set $\mathcal{F}X$ having the sets $\Diamond U$ and $\Box U$ where $U \subseteq X$ is open as a sub-basis.

Theorem $\mathcal{V}$ extends to a functor from the category Stone of Stone spaces to itself.

Preliminaries — Coalgebras and Modal Logic

Relations R on a set X correspond to functions from X to its power set $\mathcal{P}X$ by considering $x \rightsquigarrow R[x]$.

Proposition For a relation R on a Stone space X, TFAE

1. (X, R) is a modal space.
2. The map $x \rightsquigarrow R[x]$ is a continuous map from X to $\mathcal{V}X$.

Continuous maps from X to $\mathcal{V}X$ are coalgebras for the Vietoris functor on Stone. They form a category with morphisms certain commuting squares, and this category is isomorphic to MS.

Modal Compact Hausdorff Spaces

The Vietoris construction was introduced as a generalization of the Hausdorff metric to general Hausdorff spaces. It is well known that $\mathcal{V}$ yields a functor from the category KHaus of compact Hausdorff spaces to itself. This leads naturally to the following ...

Definition A modal compact Hausdorff space is pair (X, R) where

1. X is a compact Hausdorff space.
2. The map $x \rightsquigarrow R[x]$ is a continuous map from X to $\mathcal{V}X$.

In effect, MKH-spaces are defined to be concrete realizations of coalgebras for the Vietoris functor on KHaus. Indeed ...

Modal Compact Hausdorff Spaces

Proposition With morphisms being continuous p-morphisms, the category MKHaus of modal compact Hausdorff spaces is equivalent to the category of coalgebras for the Vietoris functor $\mathcal{V}$ on KHaus.

For a more direct description ...

Proposition For R a relation on a compact Hausdorff space X, then (X, R) is an MKH-space iff

1. $R[x]$ is closed for each $x \in X$.
2. $R^{-1}U$ is open for each open $U \subseteq X$.
3. $R^{-1}F$ is closed for each closed $F \subseteq X$.

Modal Compact Hausdorff Spaces

Any modal space is a modal compact Hausdorff space, and one does not have to look hard to find new examples.

Example The real unit interval $[0, 1]$ with the usual topology and relation $\leq$ is an MKH-space that is not a modal space.

We seek algebraic counterparts to MKH-spaces to play the role modal algebras play for modal spaces. This is Part II.

First we pave the way ...

Preliminaries — Isbell duality

Definition A compact regular frame L is a frame whose top is compact and satisfies for each $a \in L$

$$a = \bigvee \{b : b \prec a\}$$

Here $b \prec a$ means $\neg b \vee a = 1$ where $\neg b$ is pseudocomplement.

Example If X is a KHaus space the frame $L = \Omega X$ of open sets is compact regular. Here $\neg B = \mathbf{I} - B$ and $B \prec A$ means $\mathbf{C}B \subseteq A$.

Preliminaries — Isbell duality

Proposition In any compact regular frame

1. $0 \prec 0$
2. $1 \prec 1$
3. $a \prec b \Rightarrow a \leq b$
4. $a \leq b \prec c \leq d \Rightarrow a \prec d$
5. $a, b \prec c, d \Rightarrow a \vee b \prec c \wedge d$
6. $a \prec b \Rightarrow$ exists c with $a \prec c \prec b$ (interpolation)
7. $a = \bigvee\{b : b \prec a\}$.

These are easy to see if we think of ΩX.

Don't stare at them too long, the point is we'll see them again.

Preliminaries — Isbell duality

Definition A point of L is a frame homomorphism $p : L \to 2$. Let $\mathfrak{p}L$ be the points of L topologized by the sets $\varphi(a) = \{p : p(a) = 1\}$.

Theorem (Isbell) The category KRFrm is dually equivalent to the category KHaus via the functors Ω and $\mathfrak{pt}$.

As a break from preliminaries, we provide an alternate approach ...

Preliminaries — Alternate to Isbell duality

Definition For L a KRFrm and $A \subseteq L$ define

1. $\twoheaduparrow A = \{b : a \prec b \text{ for some } a \in A\}$
2. $\twoheaddownarrow A = \{b : b \prec a \text{ for some } a \in A\}$

Call A a round filter if $A = \twoheaduparrow A$ and a round ideal if $A = \twoheaddownarrow A$.

Proposition The set of prime round filters of L topologized by the sets $\varphi(a) = \{F : a \in F\}$ is homeomorphic to $\mathfrak{p}L$.

This gives an alternate path to Isbell duality quite like Stone duality. There is also a version with prime round ideals, and an analog to the prime ideal theorem.

Preliminaries — de Vries duality

Definition A de Vries algebra $(A, \prec)$ is a complete Boolean algebra A with relation $\prec$ that satisfies

1. $0 \prec 0$
2. $1 \prec 1$
3. $a \prec b \Rightarrow a \leq b$
4. $a \leq b \prec c \leq d \Rightarrow a \prec d$
5. $a, b \prec c, d \Rightarrow a \vee b \prec c \wedge d$
6. $a \prec b \Rightarrow$ exists c with $a \prec c \prec b$ (interpolation)
7. $a = \bigvee\{b : b \prec a\}$.
8. $a \prec b \Rightarrow \neg b \prec \neg a$.

Preliminaries — de Vries duality

For X a compact Hausdorff space, the regular open sets ($U = \mathbf{IC}U$) form a complete Boolean algebra $\mathcal{RO}\,X$ with finite meets given by intersection and joins by $\mathbf{IC}$ applied to union.

Example $(\mathcal{RO}\,X, \prec)$ is a de Vries algebra where $U \prec V$ iff $\mathbf{C}U \subseteq V$.

This is the canonical example, as all occur this way. Roughly, Isbell says we can recover a compact Hausdorff X from its open sets, de Vries that we can recover it from its regular opens and $\prec$.

Still, a few more examples help ...

Preliminaries — de Vries duality

Example For B a complete Boolean algebra, $(B, \leq)$ is de Vries.

Example Let $B = \mathcal{P}\,\mathbb{N}$ be the power set of the natural numbers and define $S \prec T$ iff $S \subseteq T$ and at least one of S, T is finite or cofinite.

Example For B any Boolean algebra define $\prec$ on its MacNeille completion by $x \prec y$ iff there is $a \in B$ with $x \leq a \leq y$.

Note The third example includes the second.
Note We cannot recover $\prec$ from B as with an MKR frame.

Preliminaries — de Vries duality

Definition A filter F of a de Vries algebra A is round if $F = \Uparrow F$. The maximal round filters are called ends. The set $\mathcal{E} A$ of ends of A is topologized by the basis of sets $\varphi(a) = \{E : a \in E\}$.

Theorem $\mathcal{E} A$ is a compact Hausdorff space whose de Vries algebra of regular open sets is isomorphic to A.

Lets turn this into a categorical equivalence ...

Preliminaries — de Vries duality

Definition A morphism between de Vries algebras A and B is a map $f : A \to B$ that satisfies

1. $f(0) = 0$
2. $f(a \wedge b) = f(a) \wedge f(b)$
3. $a \prec b \Rightarrow \neg f(\neg a) \prec f(b)$.
4. $f(a) = \bigvee\{f(b) : b \prec a\}$

Definition The composite $f * g$ of de Vries morphisms is given by

$$(f * g)(a) = \bigvee\{(f \circ g)(b) : b \prec a\}$$

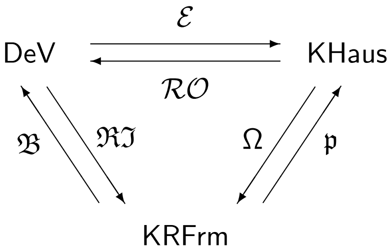
Preliminaries — de Vries duality

Warning! Composition $*$ of de Vries morphisms is not the usual function composition, and de Vries morphisms are not necessarily even Boolean algebra homomorphisms. Odd things can happen.

Definition DeV is the category of de Vries algebras and their morphisms under the composition $*$.

Theorem DeV is dually equivalent to KHaus via the end functor $\mathcal{E}$ and regular open functor $\mathcal{RO}$.

Summary of Dualities



$\mathfrak{B}$ = the functor taking all regular elements ($a = \neg\neg a$) of L

$\mathfrak{RI}$ = the frame of round ideals of a de Vries algebra.

Dualities in the Stone Space Setting

For a Boolean algebra B ...

- Its space X is its Stone space and has a basis of clopens.
- Its frame L is the ideal lattice of B. Its complemented elements are join dense.
- Its de Vries algebra A is its MacNeille completion with $\prec$ as before. Its reflexive elements ($a \prec a$) are join dense.

Extended Preliminaries — Vietoris for Frames

As KHaus and KRFrm are dually equivalent, the Vietoris functor $\mathcal{V}$ on KHaus transfers to a "Vietoris functor" $\mathcal{W}$ on KRFrm.

MKHaus is equivalent to the category of coalgebras for $\mathcal{V}$, hence is dually equivalent to the category of algebras for $\mathcal{W}$ ($\mathcal{W} L \to L$).

Johnstone has given a direct construction of $\mathcal{W}$ ...

Extended Preliminaries — Vietoris for Frames

Theorem (Johnstone) For a frame L let $\mathfrak{X} = \{\Box_a, \Diamond_a : a \in L\}$, L^* be the free frame over $\mathfrak{X}$, and Θ the congruence generated by

1. $\Box_{a \wedge b} \;=\; \Box_a \wedge \Box_b$
2. $\Diamond_{a \vee b} \;=\; \Diamond_a \vee \Diamond_b$
3. $\Box_{a \vee b} \;\leq\; \Box_a \vee \Diamond_b$
4. $\Box_a \wedge \Diamond_b \;\leq\; \Diamond_{a \wedge b}$
5. $\Box_{\bigvee S} \;=\; \bigvee\{\Box_s : s \in S\}$ (S directed in L)
6. $\Diamond_{\bigvee S} \;=\; \bigvee\{\Diamond_s : s \in S\}$ (S directed in L)

Then $\mathcal{W} L = L^*/\Theta$ is the Vietoris frame of L.

Note Several identities are familiar from positive modal logic.

Extended Preliminaries — Vietoris for de Vries?

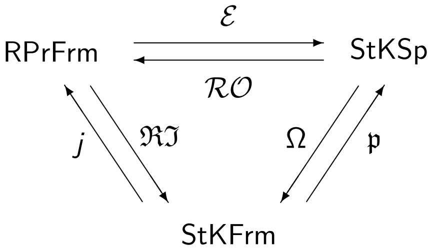
As DeV and KHaus are dually equivalent, there is a Vietoris functor $\mathcal{Z}$ for de Vries algebras corresponding to $\mathcal{V}$ for spaces.

MKHaus will be dually equivalent to the category of algebras $(\mathcal{Z}\, A \to A)$ for the Vietoris de Vries functor.

Problem Give a direct construction of $\mathcal{Z}$.

Extended Preliminaries — Stably Compact Spaces

Before ending, we mention recent work (BH) extending these dualities to stably compact spaces and "regular proximity frames".



Extended Preliminaries — Stably Compact Spaces

Recall, X is stably compact if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact.

A proximity frame $\mathcal{L} = (L, \prec)$ is roughly a frame L with proximity $\prec$ that satisfies the conditions for a de Vries algebra not involving $\neg$. Regularity is subtle and hard to describe quickly.

The functors are clear except j and $\mathcal{RO}$. Here j is "regularization" and $\mathcal{RO}$ is regular open meaning $A = \mathbf{I}_\tau \mathbf{C}_\pi A$ where $\pi =$ patch.

Conclusions

This conclude s the preliminaries. The plan is to lift this situation to the modal setting.

On to Part II ...

Thank you for listening.

Papers at www.doc.ic.ac.uk/~nbezhani/