Modal Compact Hausdorff Spaces I

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Overview

Our aim is to lift aspects of modal logic from the setting of zero-dimensional compact Hausdorff spaces, to that of general compact Hausdorff spaces.

The talk will be in two parts.
Preliminaries — Modal Logic

**Definition**  A modal space \((X, R)\) is a pair where \(X\) is a Stone space and \(R\) is a relation on \(X\) that satisfies

1. \(R[x]\) is closed for each \(x \in X\)
2. \(R^{-1}U\) is cloopen for each clopen \(U \subseteq X\).

**Definition**  A modal algebra \((B, \Diamond)\) is a pair where \(B\) is a Boolean algebra and \(\Diamond\) is an operation that satisfies

1. \(\Diamond 0 = 0\)
2. \(\Diamond (x \lor y) = \Diamond x \lor \Diamond y\).
Proposition For \((X, R)\) a modal space \((\text{Clop } X, \Diamond)\) is a modal algebra where \(\Diamond U = R^{-1} U\).

Proposition For \((B, \Diamond)\) a modal algebra \((B^*, R)\) is a modal space where \(B^*\) is the space of prime filters and \(p R q \iff \Diamond q \subseteq p\).

These extend to a dual equivalence between the category MS of modal spaces and continuous \(p\)-morphisms \((f \circ R = R \circ f)\), and the category MA of modal algebras and their homomorphisms.
**Definition**  For $X$ a Stone space, let $\mathcal{F} X$ be its closed sets and for each open $U \subseteq X$ define

\[
\diamond U = \{ F \in \mathcal{F} X : F \cap U \neq \emptyset \}
\]
\[
\Box U = \{ F \in \mathcal{F} X : F \subseteq U \}
\]

The Vietoris space $\mathcal{V} X$ is the space on the set $\mathcal{F} X$ having the sets $\diamond U$ and $\Box U$ where $U \subseteq X$ is open as a sub-basis.

**Theorem**  $\mathcal{V}$ extends to a functor from the category Stone of Stone spaces to itself.
Relations $R$ on a set $X$ correspond to functions from $X$ to its power set $\mathcal{P}X$ by considering $x \sim R[x]$.

**Proposition** For a relation $R$ on a Stone space $X$, TFAE

1. $(X, R)$ is a modal space.
2. The map $x \sim R[x]$ is a continuous map from $X$ to $\mathcal{V}X$.

Continuous maps from $X$ to $\mathcal{V}X$ are coalgebras for the Vietoris functor on Stone. They form a category with morphisms certain commuting squares, and this category is isomorphic to $\text{MS}$. 
Modal Compact Hausdorff Spaces

The Vietoris construction was introduced as a generalization of the Hausdorff metric to general Hausdorff spaces. It is well known that \( \mathcal{V} \) yields a functor from the category \( \text{KHaus} \) of compact Hausdorff spaces to itself. This leads naturally to the following ...

**Definition**  A modal compact Hausdorff space is pair \((X, R)\) where

1. \( X \) is a compact Hausdorff space.
2. The map \( x \sim R[x] \) is a continuous map from \( X \) to \( \mathcal{V}X \).

In effect, MKH-spaces are defined to be concrete realizations of coalgebras for the Vietoris functor on \( \text{KHaus} \). Indeed ...
Modal Compact Hausdorff Spaces

Proposition  With morphisms being continuous p-morphisms, the category MKHaus of modal compact Hausdorff spaces is equivalent to the category of coalgebras for the Vietoris functor $\mathcal{V}$ on KHaus.

For a more direct description ...

Proposition  For $R$ a relation on a compact Hausdorff space $X$, then $(X, R)$ is an MKH-space iff

1. $R[x]$ is closed for each $x \in X$.
2. $R^{-1} U$ is open for each open $U \subseteq X$.
3. $R^{-1} F$ is closed for each closed $F \subseteq X$. 
Any modal space is a modal compact Hausdorff space, and one does not have to look hard to find new examples.

**Example** The real unit interval $[0, 1]$ with the usual topology and relation $\leq$ is an MKH-space that is not a modal space.

We seek algebraic counterparts to MKH-spaces to play the role modal algebras play for modal spaces. This is Part II.

First we pave the way ...
Definition  A compact regular frame $L$ is a frame whose top is compact and satisfies for each $a \in L$

$$a = \bigvee \{ b : b < a \}$$

Here $b < a$ means $\neg b \lor a = 1$ where $\neg b$ is pseudocomplement.

Example  If $X$ is a KHaus space the frame $L = \Omega X$ of open sets is compact regular. Here $\neg B = 1 - B$ and $B < A$ means $CB \subseteq A$. 
Preliminaries — Isbell duality

Proposition  In any compact regular frame

1. $0 < 0$
2. $1 < 1$
3. $a < b \Rightarrow a \leq b$
4. $a \leq b < c \leq d \Rightarrow a < d$
5. $a, b < c, d \Rightarrow a \lor b < c \land d$
6. $a < b \Rightarrow \exists c \text{ with } a < c < b$ (interpolation)
7. $a = \lor \{ b : b < a \}$.

These are easy to see if we think of $\Omega X$.

Don’t stare at them too long, the point is we’ll see them again.
**Preliminaries — Isbell duality**

**Definition** A point of $L$ is a frame homomorphism $p : L \to 2$. Let $pL$ be the points of $L$ topologized by the sets $\varphi(a) = \{p : p(a) = 1\}$.

**Theorem (Isbell)** The category KRFrm is dually equivalent to the category KHaus via the functors $\Omega$ and $\text{pt}$.

As a break from preliminaries, we provide an alternate approach ...
Preliminaries — Alternate to Isbell duality

Definition  For $L$ a KRFrm and $A \subseteq L$ define

1. $\uparrow A = \{ b : a < b \text{ for some } a \in A \}$
2. $\downarrow A = \{ b : b < a \text{ for some } a \in A \}$

Call $A$ a round filter if $A = \uparrow A$ and a round ideal if $A = \downarrow A$.

Proposition  The set of prime round filters of $L$ topologized by the sets $\varphi(a) = \{ F : a \in F \}$ is homeomorphic to $pL$.

This gives an alternate path to Isbell duality quite like Stone duality. There is also a version with prime round ideals, and an analog to the prime ideal theorem.
Preliminaries — de Vries duality

Definition  A de Vries algebra \((A, <)\) is a complete Boolean algebra \(A\) with relation \(<\) that satisfies

1. \(0 < 0\)
2. \(1 < 1\)
3. \(a < b \Rightarrow a \leq b\)
4. \(a \leq b < c \leq d \Rightarrow a < d\)
5. \(a, b < c, d \Rightarrow a \lor b < c \land d\)
6. \(a < b \Rightarrow\) exists \(c\) with \(a < c < b\) (interpolation)
7. \(a = \lor\{b : b < a\}\).
8. \(a < b \Rightarrow \neg b < \neg a\).
For $X$ a compact Hausdorff space, the regular open sets $(U = \text{IC} U)$ form a complete Boolean algebra $\mathcal{RO} X$ with finite meets given by intersection and joins by $\text{IC}$ applied to union.

Example $(\mathcal{RO} X, \prec)$ is a de Vries algebra where $U \prec V$ iff $\text{IC} U \subseteq V$.

This is the canonical example, as all occur this way. Roughly, Isbell says we can recover a compact Hausdorff $X$ from its open sets, de Vries that we can recover it from its regular opens and $\prec$.

Still, a few more examples help ...
Preliminaries — de Vries duality

Example  For $B$ a complete Boolean algebra, $(B, \leq)$ is de Vries.

Example  Let $B = \mathcal{P} \mathbb{N}$ be the power set of the natural numbers and define $S < T$ iff $S \subseteq T$ and at least one of $S, T$ is finite or cofinite.

Example  For $B$ any Boolean algebra define $<$ on its MacNeille completion by $x < y$ iff there is $a \in B$ with $x \leq a \leq y$.

Note  The third example includes the second.
Note  We cannot recover $<$ from $B$ as with an MKR frame.
Preliminaries — de Vries duality

**Definition**  A filter $F$ of a de Vries algebra $A$ is round if $F = \uparrow F$. The maximal round filters are called ends. The set $\mathcal{E}A$ of ends of $A$ is topologized by the basis of sets $\varphi(a) = \{E : a \in E\}$.

**Theorem**  $\mathcal{E}A$ is a compact Hausdorff space whose de Vries algebra of regular open sets is isomorphic to $A$.

Lets turn this into a categorical equivalence ...
Preliminaries — de Vries duality

Definition  A morphism between de Vries algebras $A$ and $B$ is a map $f : A \rightarrow B$ that satisfies

1. $f(0) = 0$
2. $f(a \wedge b) = f(a) \wedge f(b)$
3. $a < b \Rightarrow -f(-a) < f(b)$.
4. $f(a) = \bigvee \{ f(b) : b < a \}$

Definition  The composite $f \star g$ of de Vries morphisms is given by

$$(f \star g)(a) = \bigvee \{(f \circ g)(b) : b < a\}$$
Warning! Composition $\ast$ of de Vries morphisms is not the usual function composition, and de Vries morphisms are not necessarily even Boolean algebra homomorphisms. Odd things can happen.

**Definition**  DeV is the category of de Vries algebras and their morphisms under the composition $\ast$.

**Theorem**  DeV is dually equivalent to KHaus via the end functor $\mathcal{E}$ and regular open functor $\mathcal{RO}$. 
Summary of Dualities

\[ \mathcal{B} = \text{the functor taking all regular elements } (a = \neg\neg a) \text{ of } L \]

\[ \mathcal{K} = \text{the frame of round ideals of a de Vries algebra.} \]
Dualities in the Stone Space Setting

For a Boolean algebra $B$ ...

- Its space $X$ is its Stone space and has a basis of clopens.
- Its frame $L$ is the ideal lattice of $B$. Its complemented elements are join dense.
- Its de Vries algebra $A$ is its MacNeille completion with $<$ as before. Its reflexive elements $(a < a)$ are join dense.
Extended Preliminaries — Vietoris for Frames

As KHaus and KRFrm are dually equivalent, the Vietoris functor \( \mathcal{V} \) on KHaus transfers to a “Vietoris functor” \( \mathcal{W} \) on KRFrm.

MKHaus is equivalent to the category of coalgebras for \( \mathcal{V} \), hence is dually equivalent to the category of algebras for \( \mathcal{W} \) (\( \mathcal{W} L \to L \)).

Johnstone has given a direct construction of \( \mathcal{W} \) ...
Extended Preliminaries — Vietoris for Frames

**Theorem (Johnstone)** For a frame $L$ let $\mathcal{X} = \{\square_a, \Diamond_a : a \in L\}$, $L^*$ be the free frame over $\mathcal{X}$, and $\Theta$ the congruence generated by

1. $\square_a \land b = \square_a \land \square b$
2. $\Diamond_a \lor b = \Diamond_a \lor \Diamond b$
3. $\square_a \lor b \leq \square_a \lor \Diamond b$
4. $\square a \land \Diamond b \leq \Diamond a \land b$
5. $\square \lor S = \lor \{\square s : s \in S\}$ ($S$ directed in $L$)
6. $\Diamond \lor S = \lor \{\Diamond s : s \in S\}$ ($S$ directed in $L$)

Then $\mathcal{W} L = L^*/\Theta$ is the Vietoris frame of $L$.

**Note** Several identities are familiar from positive modal logic.
As DeV and KHaus are dually equivalent, there is a Vietoris functor $\mathcal{Z}$ for de Vries algebras corresponding to $\mathcal{V}$ for spaces.

MKHaus will be dually equivalent to the category of algebras $(\mathcal{Z} A \to A)$ for the Vietoris de Vries functor.

Problem Give a direct construction of $\mathcal{Z}$. 
Before ending, we mention recent work (BH) extending these dualities to stably compact spaces and “regular proximity frames”.

\[
\begin{array}{ccc}
\mbox{RPrFrm} & \xleftarrow{\mathcal{E}} & \mbox{StKSp} \\
\xrightarrow{\mathcal{R}O} & \mathcal{R}I & \mbox{StKFrmp} \\
\xleftarrow{j} & \Omega & \xrightarrow{p}
\end{array}
\]
Extended Preliminaries — Stably Compact Spaces

Recall, $X$ is stably compact if it is compact, locally compact, sober, and the intersection of two compact saturated sets is compact.

A proximity frame $\mathcal{L} = (L, \prec)$ is roughly a frame $L$ with proximity $\prec$ that satisfies the conditions for a de Vries algebra not involving $\neg$. Regularity is subtle and hard to describe quickly.

The functors are clear except $j$ and $\mathcal{RO}$. Here $j$ is “regularization” and $\mathcal{RO}$ is regular open meaning $A = I_\tau C_\pi A$ where $\pi = \text{patch}$.
Conclusions

This concludes the preliminaries. The plan is to lift this situation to the modal setting.

On to Part II ...
Thank you for listening.

Papers at www.doc.ic.ac.uk/~nbezhani/