Canonical and natural extensions in finitely-generated varieties of lattice-based algebras

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Abstract

The paper investigates completions in the context of finitely-generated lattice-based varieties of algebras. In particular the structure of canonical extensions in such varieties is explored, and the role of the natural extension in providing a realisation of the canonical extension is discussed. The completions considered are topological algebras with respect to the interval topology, and the methodology and thrust of the paper are topological in style.

Keywords: topological algebra, interval topology, canonical extension, natural extension, profinite completion, natural duality

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1. Introduction

A lattice-based algebra is an algebraic structure which is a lattice equipped with a (possibly empty) set of additional operations. Such algebras are sometimes called lattice expansions, especially in the literature of canonical extensions. This paper focuses on completions of algebras in finitely generated varieties of lattice-based algebras. Let us fix until further notice such a variety \( \mathcal{A} \). Underpinning our approach are some well-known facts from universal algebra, to be found for example in [3]. The variety \( \mathcal{A} \) is necessarily congruence distributive and so, by Jónsson’s Lemma and Birkhoff’s Subdirect Prod-

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uct Theorem, can be expressed as $A = \text{ISP}(\mathcal{M})$, where $\mathcal{M} = \{M_1, \ldots, M_\ell\}$ is a finite set of finite algebras, each having a lattice reduct; here $\text{ISP}(\mathcal{M})$ denotes the class of isomorphic copies of subalgebras of products of algebras drawn from $\mathcal{M}$. The completions we consider are concretely built using the representation of $A$ as $\text{ISP}(\{M_1, \ldots, M_\ell\})$, and thus the assumption of finite generation is built in from the outset.

We show how each member $A$ of $\mathcal{A}$ has a completion $n_A(A)$—its natural extension—with very amenable properties and structure. By virtue of its construction, $n_A(A)$, when regarded as an algebra, belongs to $\mathcal{A}$, and, as a lattice, it is complete. Furthermore, when equipped with the interval topology derived from the lattice order, $n_A(A)$ is a Boolean topological algebra in which $A$ sits as a dense subalgebra. All the results we shall present for $\mathcal{A}$ (as specified above) are already well known in the special case that $\mathcal{A} = \mathcal{D}$, the variety of bounded distributive lattices (see [7, 14, 25]); here $\ell = 1$ and $M_1$ is the 2-element bounded lattice. The concept of natural extension was introduced and investigated by Davey et al. [6]. The theory presented in [6] does not require the algebras to have lattice reducts. Our interest in the lattice-based case stems from our desire to relate natural extensions to canonical extensions.

A little general background on canonical extensions may be helpful. However we point out that one of our main aims in writing this paper is to show the canonical extensions fraternity how topological techniques can be profitably applied to finitely generated varieties; these techniques are different from, and more direct than, those employed for varieties which are not necessarily finitely generated; see [14, 16, 28]. We draw attention in particular to recent work of Gehrke and Vosmaer [16, 28]. This considers, in part, finitely generated varieties of lattice-based algebras. Below we shall contrast their methods with ours. The starting point for the theory of canonical extensions was the classic work of Jónsson and Tarski in 1951 [23] on Boolean algebras with operators (BAOs), of which the 1993 survey article by B. Jónsson [22] provides a valuable outline. More than forty years after [23] appeared the Jónsson–Tarski construction was extended to distributive lattice-based algebras by Gehrke and Jónsson [13, 14]. In addition Gehrke and Harding [12] were able to define canonical extensions for (bounded) lattices and lattice-based algebras. They showed that the canonical extension of a bounded lattice exists, and is uniquely determined by two order-theoretic properties, referred to as density and compactness, which characterise up to isomorphism the way a lattice sits in its completion; the definitions of these properties
are recalled in Section 2. For a Boolean algebra $B$ the canonical extension can be taken to be the powerset of the set of ultrafilters of $B$, which is a complete Boolean algebra. The terms ‘dense’ and ‘compact’ in the context of canonical extensions were coined by Jónsson and Tarski. To quote from Jónsson’s survey on BAOs [22, p. 240]: ‘This [the pair of properties] is an algebraic way of describing the extension which arises from Stone’s Duality Theorem, which asserts that every Boolean algebra is isomorphic to the field of all clopen subsets of a Boolean space.’ Later, in the context of distributive lattices, Davey, Haviar and Priestley [7] gave a more overtly topological interpretation of the density and compactness conditions. We note in addition that the density condition is reminiscent of density with respect a liminf-topology, as this is defined in the theory of continuous lattices, and dually. In Remark 3.7 below we elaborate on this observation.

One aim of Jónsson and Tarski’s pioneering work was to devise an algebraic means of analysing additional operations on Boolean algebras, by lifting these operations to the canonical extensions. This has remained a central plank of canonical extension theory as the scope of the theory has widened. The methodology is of most value for varieties which are canonical, that is, closed under the passage to canonical extensions. Further background can be found in [12, 14, 16, 28]; in particular [16] indicates how canonical extensions are important in the semantic modelling of a range of logics.

For almost as long as canonical extensions have been studied there has been evidence that they must behave especially well in the finitely generated case. However a complete analysis tailored to this case has not hitherto appeared in the literature. This paper supplies a full explanation, from a topological viewpoint, of how and why finite generation leads to a very satisfying structure theory. The canonical extensions with which we deal are particular complete lattices: our results show inter alia that they are algebraic and dually algebraic, and are Boolean topological lattices with respect to the interval topology. As such, they come within the scope of the theory of (linked bi-)continuous lattices [17, 18]. However in our very special setting the interval topology is simple to work with. Therefore we elect to give a presentation based on a bare minimum of material from [17, 18].

We show in Section 2 that the natural extension is just one of a family of possible manifestations of dense and compact completions (see Theorem 2.4). In Section 4 we focus solely on the canonical extension as concretely realised via the natural extension $n_A(A)$ for $A \in \mathcal{A}$. Since all the algebraic operations lift pointwise from $A$ to $n_A(A)$, this lifting is intrinsic to $\mathcal{A}$; the extended
operations are thus hardwired into the construction and the special properties these operations possess are quite transparent. In addition it is immediate that $\mathcal{A}$ is canonical. (We note too that the natural extension construction, in its full generality, is easily shown to be functorial [6, Proposition 3.2].) This should be contrasted with the approach traditionally used to obtain completions of lattices with additional operations whereby one first forms canonical extensions of the underlying lattices and thereafter superimposes extensions of the non-lattice operations and of homomorphisms. Thanks to the properties characterising canonical extensions, there are two natural ways to extend a map $f$, in the manner of an envelope built as a liminf or a limsup; these extensions are denoted $f^\sigma$ and $f^\pi$. In general they do not coincide, but when they do, then, in the terminology of [14], $f$ is said to be smooth. We show by an argument which is topological in nature that, when $n_\mathcal{A}(A)$ is regarded as the canonical extension of the underlying lattice $L_A$ of $A$, then the pointwise lifting of each basic operation $f$ coincides with both $f^\sigma$ and $f^\pi$ (see Proposition 4.2 and Theorem 4.3); as a by-product, $f$ is smooth.

We can now contrast our approach with that adopted by Gehrke and Vosmaer. Their recent paper [16] employs entirely different strategies from those used here, so the two papers complement each other with little overlap. Gehrke and Vosmaer make heavy use of the established theory of canonical extensions in general. The restriction to a variety $\mathcal{A}$ generated by a finite algebra $K$ is handled by exploiting the fact that, by Jónsson’s Lemma, $\mathcal{A}$ can be represented as $\text{HSP}_B(K)$, where $\text{P}_B$ denotes Boolean product; the key point is that ultraproducts are absent in this special setting (recall [14, Section 3]). It is then proved, by applying successively the operators $\text{P}_B$, $\text{S}$ and $\text{H}$, that, for each basic operation $f$, the customary extensions $f^\sigma$ and $f^\pi$ coincide on the canonical extension of any algebra in $\mathcal{A}$. We note that Theorems 3.5 and 4.1 in combination give properties of canonical extensions which are also obtained, but by different arguments, in [16]. Canonicity, and continuity for the interval topology of the extended operations, is also shown in [16], but this topology plays a less prominent role there than it does in our approach. A more comprehensive account of canonical extensions in arbitrary lattice-based varieties, exploiting topological ideas, is given in Vosmaer’s thesis [28]. The second author of the present paper acknowledges with gratitude fruitful discussions with Mai Gehrke on canonical extensions in general and on the finitely generated case in particular.

We should also acknowledge here another approach to completions, that whereby profinite completions are used to model canonical extensions of al-
gebras in a finitely generated lattice-based variety \(A\). This approach is in the same spirit as ours: the profinite completion of an algebra in \(A\) is, at the lattice level, a dense and compact completion, and comes equipped, \textit{ab initio}, with an extension for each basic operation. The extension of any operation \(f\) coincides with both \(f^\sigma\) and \(f^\pi\). (See the paper of J. Harding [21] and the recent generalisation of its main theorem by M. J. Gouveia [20].) We stress that the constructions and proofs in the present paper make no reference to profinite completions. However we do conclude, in Section 5, with some brief remarks on the direct relationship between profinite completions and natural extensions which are valid under an assumption of residual finiteness [6], and so hold in a much wider context than that in which canonical extensions of algebras are defined.

A few remarks should be made on how far our natural extension construction relies on duality theory. For the theorem asserting that the natural extension acts as a canonical extension, no duality theory is involved, save for the motivation this provides for the definition of the natural extension as given in [6]: there is no requirement that the variety under consideration should possess a duality. Dualisability is not needed either for the explicit description of the natural extension provided by Theorem 4.5 below (a specialisation of [6, Theorem 4.1]). Nevertheless, having a duality to hand does have merit, since it allows the description of the natural extension to be refined. It is this refined description, applied in the case of Priestley duality, that yields the representation of the canonical extension of a distributive lattice in terms of its dual space which was used by Gehrke and Jónsson [13] to define it concretely and so to extend the theory from BAOs to distributive lattices with operators.

2. Complete sublattices of products of finite lattices

First of all we briefly recall some basic definitions from the theory of canonical extensions that we shall need. Let \(L\) be a sublattice of a complete lattice \(C\). Then \(C\) is called a \textit{completion} of \(L\). (More generally, if \(e : L \rightarrow C\) is an embedding of the lattice \(L\) into the complete lattice \(C\), then the pair \((e, C)\) is also called a \textit{completion} of \(L\).) Write \(T \subseteq S\) to mean that \(T\) is a finite subset of \(S\). The completion \(C\) of \(L\) is said to be \textit{dense} if every element of \(C\) can be expressed both as a join of meets and as a meet of joins of elements of \(L\). This can be seen as a weakening of the join- and meet-density condition characterising the well-known MacNeille completion. In addition,
C is called a compact completion of L if, for all non-empty subsets A and B of L, we have $\bigwedge A \leq \bigvee B$ implies $\bigwedge A_0 \leq \bigvee B_0$, for some $A_0 \subseteq A$ and $B_0 \subseteq B$, or equivalently, if for every filter $F$ of $L$ and every ideal $I$ of $L$, we have $\bigwedge F \leq \bigvee I$ implies $F \cap I \neq \emptyset$. A canonical extension of a lattice $L$ is a completion $C$ of $L$ that is both dense and compact. Gehrke and Harding [12] proved that every bounded lattice $L$ has a canonical extension and that any two canonical extensions of $L$ are isomorphic via an isomorphism that fixes the elements of $L$; an alternative approach can be found in [15].

Now assume $A = \mathbb{ISP}(M)$ be the quasivariety generated by $M$, where $M$ is a finite set of finite lattice-based algebras. (So the algebras in $M$, and therefore those in $A$, are of the form $\langle A; \lor, \land, F \rangle$, for some set $F$ of operations, with the reduct $\langle A; \lor, \land \rangle$ a lattice.) Our aim will be to recognize suitable subalgebras of products of algebras in $M$ as candidates for the canonical extensions of algebras on $A$. So we shall begin with some generalities concerning complete sublattices of products of finite lattices and the way in which topology and lattice structure interact on such objects. Our treatment will highlight the way in which the lattice-theoretic properties of compactness and denseness characterising canonical extensions are genuinely topological conditions. Our first result, Lemma 2.1 below, generalises [7, Lemma 2.2]. It shows that, under rather general conditions, the density condition satisfied by the canonical extension of a lattice equates to a condition of topological density.

In preparation, we note the following well-known description of the closure in topological products. Let $\{M_s\}_{s \in S}$ be a family of topological spaces indexed by a non-empty set $S$. Let $A$ be a subset of $\prod_{s \in S} M_s$. An element $x$ of $\prod_{s \in S} M_s$ is locally in $A$ if, for every $T \subseteq S$, there exists $a \in A$ with $x|_T = a|_T$. The set of all elements of $\prod_{s \in S} M_s$ that are locally in $A$ will be denoted by $\text{loc}(A)$. If each $M_s$ is finite and endowed with the discrete topology, then $\text{loc}(A)$ is the topological closure of $A$ in $\prod_{s \in S} M_s$.

We also recall that a non-empty subset $L$ of a complete lattice $K$ is called a complete sublattice of $K$ if it is closed under joins and meets (taken in $K$) of arbitrary non-empty subsets. While the lattices $M_s$ in (ii) and (iii) of Lemma 2.1 and in Lemma 2.2 are assumed to be finite, it is quite easy to show that the conclusions are valid under the weaker assumption that the $M_s$ have no infinite chains: use the fact that lattices with no infinite chains are complete and have the property that every join equals the join of a finite subset (and dually for meets)—see Theorem 2.41 of Davey and Priestley [8].
Lemma 2.1. Let $S$ be a non-empty set, let $M_s$ be a complete lattice, for all $s \in S$, and let $L$ be a sublattice of $\prod_{s \in S} M_s$.

(i) Let $x \in \prod_{s \in S} M_s$ and assume that $x$ is locally in $L$. Then, with the joins and meets calculated pointwise in the product, the following hold:

(a) $x = \bigvee \{ \bigwedge A_i \mid i \in I \}$, for some non-empty set $I$ and non-empty subsets $A_i$ of $L$, and

(b) $x = \bigwedge \{ \bigvee A_i \mid i \in I \}$, for some non-empty set $I$ and non-empty subsets $A_i$ of $L$.

(ii) Assume that $M_s$ is a finite lattice, for each $s \in S$. Then $\text{loc}(L)$ forms the complete sublattice of $\prod_{s \in S} M_s$ generated by $L$.

(iii) Assume that $M_s$ is a finite lattice, for each $s \in S$. Then following are equivalent for all $x \in \prod_{s \in S} M_s$:

1. $x \in \text{loc}(L)$;
2. $x = \bigvee \{ \bigwedge A_i \mid i \in I \}$, for some non-empty set $I$ and non-empty subsets $A_i$ of $L$;
3. $x = \bigwedge \{ \bigvee A_i \mid i \in I \}$, for some non-empty set $I$ and non-empty subsets $A_i$ of $L$.

(iv) Assume that $M_s$ is a finite lattice endowed with the discrete topology, for each $s \in S$. Then $L$ is a topologically closed sublattice of $\prod_{s \in S} M_s$ if and only if $L$ is a complete sublattice of $\prod_{s \in S} M_s$.

Proof. Assume that $x$ is locally in $L$. For each $T \subseteq S$, define

$$k^T_x := \bigwedge \{ a \in L \mid x \upharpoonright T = a \upharpoonright T \}.$$ 

As $x$ is locally in $L$, we have $k^T_x \upharpoonright T = x \upharpoonright T$. Clearly, $\bigvee \{ k^T_x \mid T \subseteq S \} \supseteq x$, so to prove equality it remains to show that $k^T_x \subseteq x$, for all $T \subseteq S$. Let $T \in S$ and $s \in S$. Then there exists $a \in L$ with $x \upharpoonright \{ T \cup \{ s \} \} = a \upharpoonright \{ T \cup \{ s \} \}$. It follows that $k^T_x \subseteq a$, whence $k^T_x(s) \leq a(s) = x(s)$. Thus, $k^T_x \leq x$, as required. This proves (i)(a), and (i)(b) follows by duality.
Now assume that each $M_s$ is a finite lattice. That $\text{loc}(L)$ forms a complete sublattice of $\prod_{s \in S} M_s$ will follow easily once we prove that (with the joins and meets calculated pointwise in the product):

\[ \emptyset \neq A \subseteq L \implies \bigvee A \in \text{loc}(L) \quad \& \quad \bigwedge A \in \text{loc}(L). \quad (\ast) \]

Let $A$ be a non-empty subset of $L$ and let $x := \bigvee A$. Let $T \subseteq S$ and let $t \in T$. Then, since $M_t$ is finite,

\[ x(t) = \bigvee_{a \in A} a(t) = a_1^t(t) \lor \cdots \lor a_{j_t}^t(t), \]

for some $j_t \in \mathbb{N}$ and $a_1^t, \ldots, a_{j_t}^t \in A$. Define

\[ a := \bigvee \{ a_1^t \lor \cdots \lor a_{j_t}^t \mid t \in T \}. \]

Then $a \in L$ and $a \leq \bigvee A = x$. We have $a(t) \geq a_1^t(t) \lor \cdots \lor a_{j_t}^t(t) = x(t)$, for each $t \in T$. Thus, $x|_T = a|_T$. So $\bigvee A \in \text{loc}(L)$, and $\bigwedge A \in \text{loc}(L)$ by duality.

By replacing $L$ by $\text{loc}(L)$ in $(\ast)$, we conclude at once that $\text{loc}(L)$ is a complete sublattice of $\prod_{s \in S} M_s$. The remainder of (ii) follows from (i). Now another application of $(\ast)$ shows that (iii) follows from (i). Finally, (iv) is an immediate corollary of (ii). \( \square \)

Our next task is to investigate the compactness property demanded of a canonical extension, again working in products of finite lattices. It will now be expedient to use the characterisation of compact completions in terms of filters and ideals.

**Lemma 2.2.** Let $S$ be a non-empty set, let $M_s$ be a finite lattice, for all $s \in S$, and let $L$ be a sublattice of $\prod_{s \in S} M_s$. Let $F$ be a filter of $L$ and let $I$ be an ideal of $L$. Then

\[ \bigwedge F \nleq \bigvee I \iff (\exists z \in S)(\forall a \in F)(\forall b \in I) a(z) \nleq b(z). \]

**Proof.** Since $M_s$ is finite, every filter and every ideal of $\pi_s(L)$ is principal. Thus, for all $s \in S$, there exists $a \in F$ and $b \in I$ such that the filter of $M_s$ generated by $\pi_s(F)$ equals $\uparrow a(s)$ and the ideal of $M_s$ generated by $\pi_s(I)$ equals $\downarrow a(s)$. This is used to justify the last equivalence below.

\[ \bigwedge F \nleq \bigvee I \iff (\exists z \in S) \left( \bigwedge_{a \in F} a(z) \nleq \bigvee_{b \in I} b(z) \right) \iff (\exists z \in S) \prod_{a \in F} a(z) \nleq \prod_{b \in I} b(z) \iff (\exists z \in S)(\forall a \in F)(\forall b \in I) a(z) \nleq b(z). \]

\[ \square \]
The next result, a technical lemma, helps in the multisorted case, that is, when $\mathcal{A} = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ contains more than one algebra. It seems not to have been recorded in the literature.

**Lemma 2.3.** Let $X_1, \ldots, X_\ell$ be complete lattices, let $L$ be a sublattice of the product $X_1 \times \cdots \times X_\ell$ and assume that $X_i$ is a compact completion of $\pi_i(L)$, for all $i \in \{1, \ldots, \ell\}$. Then $X_1 \times \cdots \times X_\ell$ is a compact completion of $L$.

**Proof.** Let $\emptyset \neq A, B \subseteq L$. Then

$$\bigwedge A \leq \bigvee B \implies (\forall i) \bigwedge \pi_i(A) \leq \bigvee \pi_i(B) \text{ in } \pi_i(L)$$

$$\implies (\forall i) (\exists A_i^0 \subseteq A)(\exists B_i^0 \subseteq B) \bigwedge \pi_i(A_i^0) \leq \bigvee \pi_i(B_i^0),$$

as $X_i$ is a compact completion of $\pi_i(L)$. Define the sets

$$A_0 := A_1^0 \cup \cdots \cup A_\ell^0 \text{ and } B_0 := B_1^0 \cup \cdots \cup B_\ell^0.$$ 

Then

$$\bigwedge A_0 \leq (\bigwedge \pi_1(A_1^0), \ldots, \bigwedge \pi_\ell(A_\ell^0))$$

$$\leq (\bigvee \pi_1(B_1^0), \ldots, \bigvee \pi_\ell(B_\ell^0)) \leq \bigvee B_0. \quad \Box$$

The following theorem will be applied in the case that the topological spaces are Boolean, that is, compact and totally disconnected. Given a finite algebra $\mathcal{M}$, algebras of the form $\mathcal{C}(Z, \mathcal{M})$, namely the continuous functions from $Z$ to $\mathcal{M}$ with pointwise-defined operations, where $Z$ is a Boolean space, are called **Boolean powers** of $\mathcal{M}$ and play an important role in various parts of universal algebra.

**Theorem 2.4.** Let $Z_1, \ldots, Z_\ell$ be compact spaces, let $M_1, \ldots, M_\ell$ be finite lattices each equipped with the discrete topology and let $L$ be a sublattice of $\mathcal{C}(Z_1, M_1) \times \cdots \times \mathcal{C}(Z_\ell, M_\ell)$.

(i) The lattice $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a compact completion of $L$.

(ii) The topological closure of $L$ in $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a canonical extension of $L$. 

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Proof. For (i), it suffices, by Lemma 2.3, to consider the case that $\ell = 1$. Let $L$ be a sublattice of the lattice $C(Z, M)$ of continuous functions from $Z$ into $M$, for some compact topological space $Z$ and some finite lattice $M$. Let $F$ be a filter of $L$ and let $I$ be an ideal of $L$ with $F \cap I = \emptyset$. We must prove that $\bigwedge F \niceq \bigvee I$. As $M$ is finite, by Lemma 2.2 it suffices to show that there exists $z \in Z$ such that $a(z) \niceq b(z)$, for all $a \in F$ and all $b \in I$. For all $a, b \in C(Z, M)$, define

$$\lbrack a \niceq b \rbrack := \{ z \in Z \mid a(z) \niceq b(z) \}.$$  

Note that $\lbrack a \niceq b \rbrack$ is the complement in $Z$ of the equaliser of the continuous maps $a \land b$ and $a$. Since $M$ is finite and has the discrete topology, it follows that $\lbrack a \niceq b \rbrack$ is a clopen subset of $Z$, and $F \cap I = \emptyset$ guarantees that $\lbrack a \niceq b \rbrack$ is non-empty, for all $a \in F$ and all $b \in I$. Define $\mathcal{F} := \{ \lbrack a \niceq b \rbrack \mid a \in F \land b \in I \}$. A simple calculation shows that, for all $a_1, a_2 \in F$ and all $b_1, b_2 \in I$, we have

$$\lbrack a_1 \land a_2 \niceq b_1 \lor b_2 \rbrack \subseteq \lbrack a_1 \niceq b_1 \rbrack \cap \lbrack a_2 \niceq b_2 \rbrack.$$  

Hence, $\mathcal{F}$ is a filter base of non-empty closed subsets of $Z$. The compactness of $Z$ guarantees that $\bigcap \mathcal{F}$ is non-empty. Choose $z \in \bigcap \mathcal{F}$, then $a(z) \niceq b(z)$, for all $a \in F$ and all $b \in I$, as required. This completes the proof of (i).

For (ii) we first note that the topological closure $\text{loc}(L)$ of $L$ in the product $M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}$ is a complete sublattice of this product, by Lemma 2.1(ii). It now follows immediately from (i) that $\text{loc}(L)$ is a compact completion of $L$. By Lemma 2.1(i), $\text{loc}(L)$ is also a dense completion of $L$. 

3. The role of the interval topology

In this section we investigate more closely the topological and order-theoretic structure of a topologically closed sublattice of a non-empty product of finite lattices, where these lattices carry the discrete topology. An elementary treatment suffices for the applications we make in Section 4 and we have therefore opted to make our exposition as self-contained as possible. As far as possible we avoid drawing on the theory of continuous lattices as given in [18], or alternatively in its forerunner [17], since this theory is likely to be unfamiliar to many of our readers. However, without proof, we do set our results in context by stating a portmanteau result, Theorem 3.6. The continuous lattices methodology which is required to prove the full theorem

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also provides alternative ways to derive those of our earlier claims which the theorem subsumes.

We begin by recording some elementary order-theoretic facts about lattices of the type considered in Section 2. In the theory of canonical extensions of lattices, completely join- and meet-irreducible elements play a distinguished role, and they will here too. We denote the completely join-irreducible elements of a complete lattice \( C \) by \( J^\infty(C) \) and the completely meet-irreducible elements by \( M^\infty(C) \).

**Lemma 3.1.** Let \( C \) be a complete, equivalently topologically closed, sublattice of a product of finite lattices. Then \( C \) is algebraic and dually algebraic. Consequently, \( C \) is meet-generated by the set \( M^\infty(C) \) and join-generated by the set \( J^\infty(C) \).

**Proof.** We note that any finite lattice is algebraic, a product of algebraic lattices is algebraic (see for example [17, I.4.14]) and a complete sublattice of an algebraic lattice is algebraic. (The last assertion is an elementary exercise; see [8, Exercise 7.7].) For the claims concerning meet- and join-generation, see for example [8, Proposition 10.27].

We recall that the interval topology on an ordered set \( P \) has a sub-basis for its closed sets consisting of all sets of the form \( \uparrow x \) and \( \downarrow x \), for \( x \in P \). We denote this topology by \( \iota_P \). Recall that an ordered topological space \( X \) is a Priestley space if it is compact and totally order-disconnected, that is, for all \( x \leq y \) in \( X \), there exists a clopen up-set containing \( x \) but not \( y \).

**Proposition 3.2.** Assume that \( C \) is a topologically closed sublattice of a product \( \prod_{s \in S} M_s \), where each \( M_s \) is a finite lattice with the discrete topology. Then the induced product topology \( \mathcal{T} \) on \( C \) coincides with the interval topology \( \iota_C \) of \( C \), and with respect to this topology \( C \) is a Priestley space.

**Proof.** Let \( Y := \prod_{s \in S} M_s \). For any \( a \in C \), the set \( \uparrow_C(a) := \{ y \in C \mid y \geq a \} \) is the intersection of \( C \) with the complete sublattice \( \{ y \in Y \mid y \geq a \} \) of \( Y \) and hence itself a complete sublattice of \( Y \). By Lemma 2.1(iv), \( \uparrow_C(a) \) is \( \mathcal{T} \)-closed, and likewise for \( \downarrow_C(a) \). Therefore \( \iota_C \subseteq \mathcal{T} \).

Since the topology on each \( M_s \) is discrete, \( \mathcal{T} \) has a sub-basis for its closed sets consisting of the sets \( \{ a \in C \mid a(s) = m_s \} \), where \( s \) varies over \( S \) and \( m_s \) varies over \( M_s \). It follows immediately that the family of sets of the form

\[
U_{s,m_s} := \{ a \in C \mid a(s) \leq m_s \} \quad \text{and} \quad V_{s,m_s} := \{ a \in C \mid a(s) \geq m_s \}
\]
together form a sub-basis for the $T$-closed sets. Since, by Lemma 2.1(iv), $C$ is closed under joins, calculated pointwise, $b := \bigvee U_{s,m_s}$ exists in $C$ and belongs to $U_{s,m_s}$. Because $U_{s,m_s}$ is a down-set in $C$ we conclude that $U_{s,m_s} = \downarrow C b$, and hence that $U_{s,m_s}$ is $\iota_C$-closed. Likewise, the set $V_{s,m_s}$ is $\iota_C$-closed. Therefore $\mathcal{T} \subseteq \iota_C$. Since each $M_s$ is trivially a Priestley space, the final assertion holds since the class of Priestley spaces is closed under the formation of products and closed subspaces.

We remark that it is easy to see that on each $M_s$ the discrete topology coincides with the interval topology, and that, on a product of complete lattices, the product topology derived from the interval topologies on the factors is the interval topology on the product (see [9, Theorem 2.6]). So on the full product $\prod_{s \in S} M_s$ the product topology is the interval topology. Proposition 3.2 says more than this; in general the interval topology on a closed subspace will not coincide with the subspace topology derived from the interval topology.

We easily obtain the following result covering both lattice structure and any additional operations.

**Proposition 3.3.** Let $\mathcal{M}$ be a set of finite lattice-based algebras (of the same type). Assume that $C$ is a subalgebra of a product of algebras from $\mathcal{M}$ whose lattice reduct is a complete sublattice of the product. Then $C$ is a compact topological algebra with respect to the interval topology.

**Proof.** Equip each algebra in $\mathcal{M}$ with the discrete topology, so each becomes a (compact) topological algebra, and assume that $C$ is a subalgebra of $\prod_{s \in S} M_s$, with $M_s \in \mathcal{M}$, for all $s \in S$. Since (the lattice reduct of) $C$ forms a complete sublattice of $\prod_{s \in S} M_s$, it follows from Lemma 2.1 that $C$ is topologically closed with respect to the product topology. Hence $C$ forms a compact topological algebra with respect to the induced product topology. Since, by Proposition 3.2, the interval topology on $C$ agrees with the induced product topology, we are done.

For our next result we must venture into the foothills of the theory of continuous lattices. The result is not new (cf. [18, 19, 26, 27]), but for completeness we outline a direct proof. It would be easy to formulate a ‘one-sided’ version applicable to algebraic, rather than bi-algebraic, lattices, but we do not need this.
Proposition 3.4. Let $C$ be a bi-algebraic lattice. Then the following conditions are equivalent:

1. $C$ is a Priestley space with respect to $\iota_C$;
2. $\iota_C$ is Hausdorff;
3. the topology $\iota_C$ coincides with the Lawson topology and with the dual Lawson topology on $C$;
4. for each compact element $k$ of $C$,
   \[ C \setminus \uparrow k = \downarrow F \]
   for some finite subset $F$ of $C$, and the order dual assertion holds too.

Proof. We note the following very basic facts from [18]. On any complete lattice $C$ the interval topology $\iota_C$ and the Lawson topology $\lambda(C)$ coincide if $\iota(C)$ is Hausdorff (because $\lambda(C)$ is necessarily compact and $\iota_C \subseteq \lambda(C)$ [18, II.1.9 and III.1.15]), and the dual assertion holds too. Hence (2) implies (3). Assume (3) and consider the Lawson topology on $C$. It is elementary that the Lawson-open up-sets are exactly the Scott-open sets (that is, those whose complements are closed under directed joins) and that any such set is a union of principal up-sets [18, III.1.6 and III.1.9]. Consider a non-empty clopen up-set $U$ in $C$. Every element of $U$ lies above a minimal element of $U$, and each minimal element is compact. Since $U$ is a closed subset of $C$ it is compact. Consequently $U$ has only finitely many minimal elements. Therefore non-empty clopen up-sets are of the form $\uparrow G$, with $G$ a finite set and, order dually, clopen down-sets are of the form $\downarrow F$ with $F$ finite. Hence (4) holds. Certainly either condition in (4) implies (1) because the compact elements in an algebraic lattice are join-dense. \qed

Theorem 3.5 focuses on completely join- and meet-irreducible elements. By Proposition 3.4 and Theorem 2.4 this will apply to the lattice reduct of the canonical extension of an algebra in a finitely generated lattice-based variety. Below we contrast the statements in Theorem 3.5 with weaker properties true of canonical extensions of bounded lattices in general. The proof of Theorem 3.5 takes as its starting point property (4) in Proposition 3.4.

Theorem 3.5. Let $C$ be a bi-algebraic lattice which is a Priestley space in its interval topology $\iota_C$. Let $x \not\leq y$ in $C$. Then there exists $j \in J^\infty(C)$, with
\( j \leq x \) and \( j \not\leq y \), and a finite set \( \mathcal{M}_j \subseteq M^\infty(C) \) such that \( C \uparrow j = \downarrow \mathcal{M}_j \). The order dual statement also holds.

**Proof.** Let \( x \not\leq y \) in \( C \). Because \( C \) is dually algebraic, it is join-generated by \( J^\infty(C) \), so there exists \( j \in J^\infty(C) \) with \( j \leq x \) and \( j \not\leq y \). Since \( C \) is algebraic, \( j \) is the directed join of the compact elements in \( \downarrow j \) and because \( j \in J^\infty(C) \), we deduce that \( j \) is compact. Now we can apply (4) in Proposition 3.4 to write \( C \uparrow j = \downarrow \mathcal{M}_j \), where \( \mathcal{M}_j \) is a finite set and each member of \( \mathcal{M}_j \) can be assumed to be maximal in \( C \uparrow j \) since each member of the clopen down-set \( \downarrow \mathcal{M}_j \) lies below a maximal element belonging to this set. Let \( m \in \mathcal{M}_j \). If \( m \) were not completely meet-irreducible, then we could write \( m \) as the meet of a set \( S \) of elements of \( C \) with \( s > m \), for all \( s \in S \). But then \( s \not\geq j \) for all \( s \in S \), so that \( \bigwedge S \not\geq j \), which is a contradiction. \( \square \)

In the non-distributive case, we cannot hope to strengthen the above result to assert that \( \mathcal{M}_j \) can be taken to be a singleton for every \( j \in J^\infty(C) \). If we had \( \mathcal{M}_j = \{ m \} \), then \( (j, m) \) would be a splitting pair, and \( j \) necessarily completely join-prime. But a complete lattice that is join-generated by its completely join-prime elements is necessarily completely distributive (see for example [7], in particular Theorem 2.5, for this classic result, and full references). When we consider completions of non-distributive lattices, the completion cannot, of course, be completely distributive.

We can assert, however, that the finite set \( \mathcal{M}_j \) above has the property that \( \mathcal{M}_j \gg \mathcal{M}_j \), that is, for any down-directed set \( F \) with \( \bigwedge F \in \downarrow \mathcal{M}_j \), we have \( F \cap \downarrow \mathcal{M}_j \not= \emptyset \) (cf. [27, Proposition 1.7 and Theorem 2.2]). This may be seen as a weakened version of complete meet-primeness—the best we can hope for in the non-distributive setting.

We remark also that the interval topology on a complete lattice being compact and Hausdorff, as occurs in Proposition 3.4, already signals a weakened form of complete distributivity. In [10, 11] Erné discusses topological equivalents of various distributive laws on complete lattices. In particular, for a complete lattice \( C \), the topology \( \iota_C \) is compact and Hausdorff if and only if \( L \) is ultralimit distributive in the sense that

\[
\bigwedge \left\{ \bigvee Y \mid Y \in \mathcal{Y} \right\} = \bigvee \left\{ \bigwedge Z \mid Z \in \mathcal{Y}^# \right\},
\]

where \( \mathcal{Y} \) is the family of ultrafilters on \( C \) and

\[
\mathcal{Y}^# = \{ Z \subseteq \bigcup \mathcal{Y} \mid \forall Y \in \mathcal{Y} \ (Y \cap Z \not= \emptyset) \};
\]

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We now set our elementary treatment above in a wider context. We recall that a lattice is linked bi-algebraic if it is bi-algebraic and has a Hausdorff interval topology. Combining the conditions of Lemma 3.1 and Proposition 3.4 we see that a complete sublattice of a product of finite lattices (each equipped with the discrete topology) is linked bi-algebraic. We record in Theorem 3.6 equivalent characterisations of linked bi-algebraic lattices. For the proof, we refer to [18, VII.2.6].

**Theorem 3.6.** For a complete lattice $C$, the following conditions are equivalent:

1. $C$ is bi-algebraic and satisfies one, and hence all, of the equivalent conditions given in Proposition 3.4;
2. $C$ is a Boolean topological lattice with respect to $\iota_C$;
3. $C$ is a Boolean topological lattice with respect to some topology.

**Remark 3.7.** We conclude this section with some comments about topological density as regards completions. We have seen that the order-theoretic definition of density correlates with a topological notion of density, as revealed in Lemma 2.1. We have also seen that the completions we consider come within the scope of the rich theory of linked bi-algebraic lattices, on which the Lawson, dual Lawson and interval topologies all coincide. It is also known [18, III.1] that, on a complete lattice, convergence with respect to the Lawson topology may be interpreted in terms of the so-called lim inf-topology. Assume that $L$ is a sublattice (or merely a meet subsemilattice) of a complete lattice $C$, equipped with the Lawson topology. Then the closure of $L$ in $C$ consists of those points $x$ expressible in the form $x = \liminf F := \sup \inf F$, where $F$ is a filter base of subsets of $L$. Equivalently, the Lawson-closure of $L$ consists of liminfs of ultrafilters of subsets of $L$. In the case of a linked bi-algebraic lattice, the closure is also captured via limsups of ultrafilters; indeed, coincidence of liminfs and limsups of arbitrary ultrafilters is a necessary and sufficient condition for the lattice to be linked (see [18, III.3.21] or [19]).

These observations form the basis of an alternative approach to dense completions based on known properties of the Lawson topology: a product
\(C\) of finite lattices is linked bi-algebraic, the closure of a lattice \(L\) embeddable in such a product \(C\) is linked bi-algebraic. In addition the Lawson topology of \(L\) is the restriction of that on \(C\) and coincides with the interval topology, and topological density can be seen as implying order-theoretic density. However, for several reasons, we decided not to base our account of dense and compact completions on the Lawson topology and its relationship to \(\liminf\)-convergence. First of all, the arguments used to establish compactness of completions in Section 2 are different in character from the type of arguments employed in continuous lattice theory and it is not clear that the latter methodology can be applied in a natural way to compact completions. We also contend that \(\liminf\)-convergence is a rather subtle notion in general and that the interval topology is inherently simpler than the \(\liminf\)-topology. It is seductive that, in the setting of algebras in finitely generated lattice-based varieties, order-theoretic density relates rather directly to \(\liminf\) and \(\limsup\)-convergence. This would be significant if it pointed the way to a new topological approach to canonical extensions applicable beyond the finitely generated case. However we cannot expect that this will be so. There exist bounded lattices whose canonical extensions fail to be meet-continuous, and so are not continuous lattices (see [16, Example 3.1]). This suggests that \(\liminf\) convergence is unlikely to work well in canonical extensions in general.

4. The natural extension as a canonical extension

We now focus on a particular candidate, introduced in [6], for a dense and compact completion as supplied by Theorem 2.4. Given the quasivariety \(\mathcal{A} = \mathbb{ISP}(\{M_1, \ldots, M_\ell\})\) and \(A \in \mathcal{A}\), we want to choose compact spaces \(Z_1, \ldots, Z_\ell\) in such a way that \(L_A\), the lattice reduct of \(A\), embeds into \(M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}\) in the manner demanded in Theorem 2.4. Strongly motivated by duality theory, we shall do this by taking \(Z_i := \mathcal{A}(A, M_i)\). Here the underlying set of \(Z_i\) is the set of homomorphisms from \(A\) into \(M_i\) and \(Z_i\) is endowed with the subspace topology derived from the power \(M_i^{A}\), where \(M_i\) is equipped with the discrete topology. Since \(Z_i\) is a closed subspace of the product, it is compact, indeed it is a Boolean space.

We embed \(A\) into \(M_1^{Z_1} \times \cdots \times M_\ell^{Z_\ell}\) by means of the map

\[e_A: A \to \prod_{1 \leq i \leq \ell} M_i^{\mathcal{A}(A, M_i)}\]
given by $e_A(a)(i)(x) = x(a)$, for $i \in \{1, \ldots, \ell\}$ and $x \in A(A, M_i)$; we call the map $e_A(a)$, for $a \in A$, a **multisorted evaluation map**. The map $e_A$ is a homomorphism and, because $A \in \text{ISP}(M)$, it is also an embedding. We then define the **natural extension** $n_A(A)$ of $A$ (relative to $M = \{M_1, \ldots, M_\ell\}$) to be the topological closure of $e_A(A)$ in $\prod_{1 \leq i \leq \ell} M_i \to^{A(A, M_i)}$, where each $M_i$ carries the discrete topology. We need to refine this description a little in order to bring $n_A(A)$ within the scope of Theorem 2.4. We may consider the set $C(Z_i, M_i)$ of continuous maps from $Z_i$ into $M_i$. Since each evaluation map $e_A(a)$ is continuous it follows that we can restrict the codomain of $e_A$ and write

$$e_A: A \to \prod_{1 \leq i \leq \ell} C(Z_i, M_i).$$

The map $e_A$ embeds $A$ as a topologically dense subalgebra of its natural extension.

Before we make any comments on this definition we record in a theorem the key fact that the natural extension does indeed supply a canonical extension. We stress that the theorem is an immediate consequence of Theorem 2.4(ii) and relies solely on the definitions and results in Section 2 and not on any further theory of either form of completion.

**Theorem 4.1.** Let $A = \text{ISP}(M)$ where $M$ is a finite set of finite lattice-based algebras (of the same type). Then, for each $A \in A$, the lattice reduct of the natural extension $n_A(A)$ is a dense and compact completion of the lattice reduct $L_A$, and so a canonical extension.

We have above taken the most direct route possible to the construction of $n_A(A)$. However it is worth drawing attention to the fact that this is just the object part of a covariant functorial construction (valid in the quite general setting of a class $\text{ISP}(M)$, where $M$ is any set of finite algebras of the same type). The codomain $A_T$ of the functor $n_A$ is defined in the following way. We let $M_T$ be the set of members of $M$, each made into a topological algebra by endowing it with the discrete topology. We then define $A_T := \text{ISP}(M_T)$, the **topological prevariety generated by $M_T$**, that is, the class of isomorphic copies of topologically closed substructures of products of members of $M_T$. In the lattice-based setting considered here, $A_T$ is a subclass of the class of topological algebras with algebraic reduct in $A$ and whose underlying ordered topological spaces are Priestley spaces. For details of the functorial construction, in maximum generality and with all its bells and whistles, we refer the reader to [6, Section 2].
Theorem 4.1 reconciles the natural and canonical extensions at the level of the lattice reducts. Now we wish to reconcile the algebraic operations of the natural extension $n_A(A)$ with the $\sigma$- and $\pi$-extensions of the corresponding operations on the copy $e_A$ of $A$. Here we draw heavily on the ideas of Section 2 of [14], the major study by Gehrke and Jónsson of canonical extensions in the distributive case; we have to make only minor adaptations to fit our needs. We shall identify $A$ with $e_A(A)$ and so regard it as a sublattice of $n_A(A)$. As in [12, 14] we can restrict attention to operations which are unary. So, let $f : A \to A$ be any map and define extensions to $C := n_A(A)$ in the customary way:

$$f^\sigma(x) := \bigvee\{\bigwedge\{f(a) \mid a \in A \text{ and } p \leq a \leq q\} \mid p \in K(C), q \in O(C) \text{ and } p \leq x \leq q\},$$

$$f^\pi(x) := \bigwedge\{\bigvee\{f(a) \mid a \in A \text{ and } p \leq a \leq q\} \mid p \in K(C), q \in O(C) \text{ and } p \leq x \leq q\},$$

where $K(C)$ and $O(C)$ are, respectively, the filter elements and the ideal elements of $C$, so that $p \in K(C)$ if and only if $p$ is a meet of elements from $A$ and $q \in O(C)$ if and only if $q$ is a join of elements from $A$. (Filter and ideal elements are known in the older literature as closed and open elements, respectively.) We emphasise that in the proof below we make use of the structural properties of the natural, alias canonical, extension, but require no more than the definitions and the most basic facts about the $\sigma$- and $\pi$-extensions of maps.

**Proposition 4.2.** Let $A$ be an algebra in a finitely generated variety of lattice-based algebras, assume that $C$ is a dense and compact completion of $L_A$ and assume that a basic operation $f$ on $A$ has an $i_C$-continuous extension $g$ on $C$. Then $f^\sigma = f^\pi = g$, and consequently $f$ is smooth.

**Proof.** By Proposition 3.2 and Lemma 3.1, the lattice reduct of the natural extension of $A$ is algebraic, dually algebraic and is a Priestley space with respect to its interval topology. By uniqueness of the canonical extension, the complete lattice $C$ also has these properties and so has the additional properties listed in Theorem 3.5.

Since formation of canonical extensions, their filter and ideal elements, and interval topologies on complete lattices all commute with the formation of finite products (see [12, Section 5] (or [14, Section 2]) and [9, Theo-
rem 2.6), we may follow standard practice and, without loss of generality, assume that \( f \) is unary.

We know that \( f^\sigma \leq f^\pi \) and that each of \( f^\sigma \) and \( f^\pi \) extends \( f \) (see [12]). Therefore, making use of order duality, it will suffice to show that \( g(x) \leq f^\sigma(x) \), for all \( x \in C \). Suppose for a contradiction that this fails for some \( x \).

Then we can find a compact element \( k \in C \) such that \( g(x) \in \uparrow k \) and \( f^\sigma(x) \notin \uparrow k \). By \( \iota_C \)-continuity of \( g \), the set \( W := g^{-1}(\uparrow k) \) is \( \iota_C \)-clopen. The set \( W \) is a finite union of sets of the form \( U \cap V \), where \( U \) is a clopen up-set and \( V \) is a clopen down-set (see for example [8, Lemma 11.22]). Therefore there exist \( p \in J^\infty(C) \) and \( q \in M^\infty(C) \) such that \( x \in [p, q] \subseteq g^{-1}(\uparrow k) \). Note that \( p \) is a filter element of \( C \) and \( q \) is an ideal element, by density (in the canonical extension sense). For every \( a \in [p, q] \cap A \) we have \( f(a) = g(a) \geq k \) and so \( \bigwedge f([p, q] \cap A) \geq k \). The definition of \( f^\sigma(x) \) implies that \( f^\sigma(x) \geq \bigwedge f([p, q] \cap A) \geq k \), and we have the required contradiction.

The following theorem summarises what we have now established.

**Theorem 4.3.** Let \( \mathcal{A} \) be a finitely generated variety of lattice-based algebras. Let \( A \in \mathcal{A} \), and let \( n_A(A) \) be its natural extension.

(i) \( n_A(A) \), viewed as an algebra, belongs to \( \mathcal{A} \) and its lattice reduct is a canonical extension of the lattice reduct of \( A \).

(ii) When equipped with the interval topology, \( n_A(A) \) is a compact and totally order-disconnected topological algebra whose operations are the pointwise liftings of those of \( A \). Moreover, each basic operation \( f \) of \( A \) is smooth, indeed, its extension to \( n_A(A) \) coincides with each of \( f^\sigma \) and \( f^\pi \).

**Corollary 4.4.** Every finitely generated variety of lattice-based algebras is canonical.

Finally in this section we address the question of what the natural extension of an algebra actually looks like. Thus far we have taken advantage of the theory of natural dualities only insofar as we have employed its formalism to arrive at the definition of the natural extension and hence to derive Theorem 4.1. To describe the natural extension \( n_A(A) \) explicitly, we first note that, even without knowing that \( \mathcal{A} \) possesses a natural duality, we may identify the members of the topological closure which defines it as certain multisorted relation-preserving maps. To make this precise we need a few definitions.
As before we fix \( \mathcal{M} = \{M_1, \ldots, M_\ell\} \). By a (finitary) mult\-isor\-ted algebraic relation on \( \mathcal{M} \) we mean a subalgebra of some finite product of algebras from \( \mathcal{M} \). We use \( \bigcup \) to denote disjoint(ified) union. A map \( b \) from the set \( \bigcup_{1 \leq i \leq \ell} A(A, M_i) \) to the set \( \bigcup_{1 \leq i \leq \ell} M_i \) is called an \( \mathcal{M} \)-sorted map if \( b \) maps \( A(A, M_i) \) into \( M_i \), for each \( i \). A multisorted algebraic relation on \( \mathcal{M} \) is lifted pointwise to \( \bigcup_{1 \leq i \leq \ell} A(A, M_i) \) in the obvious way.

**Theorem 4.5** (from [6, Theorem 4.1]). Let \( \mathcal{M} = \{M_1, \ldots, M_\ell\} \) be a finite set of finite algebras, let \( \mathcal{A} := \mathbb{ISP}(\mathcal{M}) \) and let \( A \in \mathcal{A} \). Assume that

\[
b: \bigcup_{1 \leq i \leq \ell} A(A, M_i) \to \bigcup_{1 \leq i \leq \ell} M_i
\]

is an \( \mathcal{M} \)-sorted map. Then the following are equivalent:

(i) \( b \) belongs to \( n_A(A) \);

(ii) \( b \) is locally an evaluation;

(iii) \( b \) preserves every multisorted algebraic relation on \( \mathcal{M} \).

In fact we can do better still. Because our algebras are lattice-based, \( \mathcal{A} = \mathbb{ISP}(\mathcal{M}) \) has a multisorted duality (see for example [4, Chapter 7]). This allows us to refine the preceding theorem.

**Theorem 4.6** (from [6, Theorem 4.3]). Let \( \mathcal{A} \) be as in Theorem 4.5 and let \( R \) be a set of multisorted algebraic relations which dualises \( \mathcal{A} \) (here the set of all such relations of arity 2 suffices). Then

(i) \( n_A(A) \) is the set of \( \mathcal{M} \)-sorted maps from the set \( \bigcup_{1 \leq i \leq \ell} A(A, M_i) \) to the set \( \bigcup_{1 \leq i \leq \ell} M_i \) which preserve \( R \);

(ii) the isomorphic copy \( e_A(A) \) of \( A \) within \( n_A(A) \) consists of the continuous \( R \)-preserving \( \mathcal{M} \)-sorted maps from \( \bigcup_{1 \leq i \leq \ell} A(A, M_i) \) to \( \bigcup_{1 \leq i \leq \ell} M_i \), where each \( M_i \) carries the discrete topology and the unions carry the usual disjoint union topology.

This theorem specialises to the variety \( \mathcal{A} = \mathcal{D} \) in the following way. We may take \( \mathcal{M} \) to be the one-element set containing the two-element lattice \( 2 \) and take \( R \) to contain the single binary algebraic relation \( \leq \) on \( 2 \). Then \( R \) is a dualising set for \( \mathcal{D} \), with the duality for \( \mathcal{D} \) being Priestley duality in its
hom-functor formulation. Thus Theorem 4.6(i) reduces in this case to the
description of the canonical extension of \( A \in D \) in terms of the Priestley dual
space of \( A \); this is exactly the way the canonical extension was first defined
by Gehrke and Jónsson in [13]. Thus we may view our realisation via the
natural extension of the canonical extension in finitely generated varieties
of lattice-based algebras as supplying the analogue of that initially adopted
for \( D \).

Dualities for well-known finitely generated varieties of lattice-based alge-
bras, especially those for which the lattice reducts are distributive, have been
intensively studied and in many cases optimally economical dualities, with
minimal dualising sets, have been identified. These varieties include many
of interest in algebraic logic and elsewhere. We refer the reader to [4] for
further details and many examples.

5. Concluding remarks

The main part of this paper very deliberately identified the natural ex-
tension as a canonical extension without the intervention of the profinite
completion. We now elaborate a little on how profiniteness fits into the
overall picture.

The general setting for the theory of profinite completions of algebras is
that of classes of algebras of the form \( K := \text{ISP}(M) \), where \( M \) is any set
of finite algebras of the same type. Such a class \( K \) is called an \textit{internally
residually finite prevariety} (IRF-prevariety). Given an IRF-prevariety \( K \),
any algebra in \( K \) embeds in its \( K \)-profinite completion (see [6, Section 2] for
details). In general an IRF-prevariety need not be a variety; when \( K \) is a
variety, it is residually finite in the sense that this term is traditionally used.

It is very well known that a profinite completion of an algebra in any IRF-
prevariety is in fact a Boolean topological algebra (see for example [1, 5] and
also [7, 25] for the case of distributive lattices). Theorem 3.7 of [6] shows that,
for any IRF-prevariety \( K \), the \( K \)-profinite completion and natural extension
of any member of \( K \) are isomorphic, both algebraically and topologically.
IRF-prevarieties encompass many classes of algebras which are not lattice-
based, and so the scope of [6], even for varieties, is much wider than that of
the present paper; see [6, Section 5]. A variety \( A \) of lattice-based algebras of
finite type is known to be residually finite if and only if it is finitely generated
(Kearnes and Willard [24]), so that for such varieties the results of [6] apply
only when \( A \) is finitely generated.
The work of Harding [21] and Gouveia [20] directly relates, as algebras, canonical extensions and profinite completions of algebras in a finitely generated lattice-based variety $\mathcal{A}$. In [21] Harding proved that the profinite completion $\hat{A}$ of $A \in \mathcal{A}$ serves, at the lattice level, as the canonical extension of $L_A$ and that, for each additional operation $f$ which is monotone (in each coordinate), the corresponding operation of $\hat{A}$ agrees with each of the $\sigma$-extension and the $\pi$-extension of $f$. By refining Harding’s arguments, Gouveia [20] removed the monotonicity restriction. It was Harding’s work, and also that of Bezhanishvili et al. in [2], which led the authors of [6] to develop the theory of natural extensions for IRF-prevarieties.

Leaving topological assertions aside, we can see that, for a finitely generated lattice-based variety $\mathcal{A}$ and $A \in \mathcal{A}$, the identifications of

- $A^\delta$, the canonical extension, as constructed and characterised in [12, 15];
- $\hat{A}$, the profinite completion of $A$;
- $n_A(A)$, the natural extension of $A$, as presented in [6],

and hence the full panoply of properties of these algebras, can be arrived at by a variety of routes. Knowing that any two pairs from $A^\delta$, $\hat{A}$ and $n_A(A)$ can be identified (for all $A \in \mathcal{A}$), then the third pair can be identified too. Which of the equivalent representations is most illuminating or most convenient will no doubt depend on the context, and on the preference of the user.

The development of canonical extension methods has been spurred on by the way in which such extensions, for canonical varieties, give rise to relational (Kripke-style) semantics for a wide range of important logics. We note that our natural extension model of the canonical extension carries with it a relational semantics which is different from traditional ones except in the classic cases of Boolean algebras or distributive lattices without additional operations. As we observed in Section 4, any finitely generated variety $\mathcal{A}$ of lattice-based algebras possesses a natural (multisorted) duality. In fact, a dual category equivalence can be set up by natural hom-functors $D$ and $E$ between $\mathcal{A}$ and a category $\mathcal{X}$ of multisorted topological structures (see for example the Multisorted NU Strong Duality Theorem [4, Theorem 7.1.2]). A topological relational semantics is obtained by associating with $A \in \mathcal{A}$ its dual $D(A)$; suppressing the topology of $D(A)$, we have a discrete duality linking, in a functorial way, $n_A(A)$ and $D(A)^\flat$, where $^\flat$ is the obvious forgetful
functor. In the case that $A = \mathcal{D}$ this topological duality is just Priestley duality, and the associated discrete duality is that given by the maps taking a canonical extension to its ordered set of completely join-irreducible elements and by that taking an ordered set to its lattice of down-sets. We shall not pursue these ideas here, and so do not include the details needed to make them precise. We mention them simply to indicate that, for finitely generated lattice-based varieties, relational semantics (with or without topology) are available, and that the theory of natural dualities allows these semantics to be constructed in a uniform way over all such varieties. It is interesting that this is feasible, and confirms that finitely generated varieties behave almost as well as those for which the underlying lattices are distributive; see the remarks following Theorem 3.5.

References


