Applications of Duality on Unification Type Classification

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In this work we will show that Priestley dualities for Bounded Distributive Lattices [5] and subvarieties of Pseudocomplemented Lattices [6], natural dualities for Kleene [2] and De Morgan algebras [1], and the duality between finitely presented MV-algebras and rational polyhedra [4], are useful tools to determine the unification type of these classes of algebras. Moreover, using those dualities we will present procedures to determine the type of the unification problems of Bounded Distributive Lattices and some subvarieties of Pseudocomplemented Lattices, Kleene algebras and De Morgan algebras and some classes of MV-algebras.

Given an equational theory E over an algebraic language $\mathcal{L}$, an E-*unification problem* is a finite set of pair of $\mathcal{L}$-terms $U = \{(t_1, s_1), \ldots, (t_n, s_n)\}$. An E-*solution* (E-*unifier*) for U is a substitution σ, defined on the variables of the terms of U, such that $\sigma(t_i)$ is equivalent to $\sigma(s_i)$ modulo E, for each $i \in \{1, \ldots, n\}$.

If σ is an E-unifier for U, we can obtain a family of solutions from σ as follows: let γ be a substitution defined in all the variables of the terms $\{\sigma(t_i), \sigma(s_i) \mid i \in \{1, \ldots, n\}\}$, then clearly $\gamma \circ \sigma$ is also an E-unifier for U. In this case we say that σ is *more general* than $\gamma \circ \sigma$, in symbols $\gamma \circ \sigma \preceq \sigma$. The relation $\preceq$ determines a preorder on the set of E-unifiers of U (denoted by $\mathfrak{U}_{\mathsf{E}}(U)$). This preorder allows us to classify the unification problems depending on its properties.

A μ-*set* for $\mathfrak{U}_{\mathsf{E}}(U)$ is a subset $M \subseteq \mathfrak{U}_{\mathsf{E}}(U)$ such that for all $\sigma_1, \sigma_2 \in M$ if $\sigma_1 \preceq \sigma_2$ then $\sigma_1 = \sigma_2$, and for every $\gamma \in \mathfrak{U}_{\mathsf{E}}(U)$ there exists $\sigma \in M$ such that $\gamma \preceq \sigma$. We say that U has E-type ($type_{\mathsf{E}}(U)$):

- 0: if $\mathfrak{U}_{\mathsf{E}}(U)$ has no μ-sets;
- ∞: if $\mathfrak{U}_{\mathsf{E}}(U)$ has a μ-set of infinite cardinality;
- n: if $\mathfrak{U}_{\mathsf{E}}(U)$ has a finite μ-set of cardinality n.

We say that the equational theory E has type:

- 0: if $\{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \cap \{0\} \neq \emptyset$;
- ∞: if $\infty \in \{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{\infty, 1, 2, \ldots\}$;
- ω: if $\{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{1, 2, \ldots\}$ and there is no n such that $type_{\mathsf{E}}(U) \leq n$ for each E-unification problem U;
- n: if $n \in \{type_{\mathsf{E}}(U) \mid U \text{ an } \mathsf{E}\text{-unification problem}\} \subseteq \{1, \ldots, n\}$.

In [3] Ghilardi, translates the traditional equational unification of a theory E to the algebraic unification of the equational class of algebras $\mathcal{V}$ that E determines. In this translations unification problems become finitely presented algebras, and unifiers are homomorphisms from that algebras into projective algebras in the class $\mathcal{V}$. It is here where dualities play a central role, since different kinds of dualities have been used in the literature to describe free, projective and finitely presented algebras of equational classes of algebras.

In this talk we will present a list of cases where the use of dualities have been central to provide algorithms to calculate the unification type of certain E-unification problems.

References

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