Some Remarks on the Metric Vietoris Monad

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The main source of inspiration for this talk is the work of R. Rosebrugh and R.J. Wood on constructive complete distributive lattices where the authors elegantly employ the concepts of adjunction and module [2, 3]. We recall that an order relation on a set X defines a monotone map of type

$$X^{\mathrm{op}} \times X \to \mathsf{2},$$

and from that one obtains the Yoneda embeddings

$$X \to \mathsf{2}^{X^{\mathrm{op}}} =: PX \qquad \text{and} \qquad X \to (\mathsf{2}^{X})^{\mathrm{op}} =: VX.$$

Furthermore, PX and VX are part of monads $\mathbb{P}$ and $\mathbb{V}$ on Ord (the category of ordered sets and monotone maps), hence one obtains full embeddings

$$\mathsf{Ord}_{\mathbb{P}} \to \mathsf{Ord}^{\mathbb{P}} \qquad \text{and} \qquad \mathsf{Ord}_{\mathbb{V}} \to \mathsf{Ord}^{\mathbb{V}}$$

from the Kleisli categories into the Eilenberg–Moore categories $\mathsf{Ord}^{\mathbb{P}} \simeq \mathsf{Sup}$ (the category of complete lattices and sup-preserving maps) and $\mathsf{Ord}^{\mathbb{V}} \simeq \mathsf{Inf}$ (the category of complete lattices and inf-preserving maps) respectively. From that one obtains an equivalence

$$\mathrm{kar}(\mathsf{Ord}_{\mathbb{P}}) \simeq \mathsf{CCD}_{\mathrm{sup}} \qquad \text{and} \qquad \mathrm{kar}(\mathsf{Ord}_{\mathbb{V}}) \simeq \mathsf{CCD}_{\mathrm{inf}}$$

between the idempotent split completion of the Kleisli categories on one side, and the categories of completely distributive complete lattice and sup- respectively inf-preserving maps on the other. These equivalences restrict to

$$\mathsf{Ord}_{\mathbb{P}} \simeq \mathsf{TAL}_{\mathrm{sup}} \qquad \text{and} \qquad \mathsf{Ord}_{\mathbb{V}} \simeq \mathsf{TAL}_{\mathrm{inf}},$$

where "TAL" stands for totally algebraic lattices. Finally, both sides lead to the equivalence

$$\mathsf{Ord}^{\mathrm{op}} \simeq \mathsf{TAL}$$

between the dual category of Ord and the category TAL of totally algebraic lattices and sup- and inf-preserving maps.

Employing a formal analogy between order sets and topological (and other kinds of) spaces, in this talk we will follow the path described above, but now with topological and approach spaces in lieu of ordered sets (the latter representing "metric" topological spaces, see [1]). To illustrate this analogy, note that the ultrafilter convergence of a topological space defines a continuous map

$$(UX)^{\mathrm{op}} \times X \to \mathsf{2}$$

(where UX is the free ordered compact Hausdorff space over X, $(UX)^{\mathrm{op}}$ its Hochster dual, and $\mathsf{2}$ the Sierpiński space), which induces continuous maps

$$X \to \mathsf{2}^{(UX)^{\mathrm{op}}} =: PX \qquad \text{and} \qquad X \to (\mathsf{2}^X)^{\mathrm{op}} =: VX.$$

As it turns out, PX is isomorphic to the filter-of-opens space of X, and VX is the upper Vietoris space. As above, both constructions are parts of monads, but, in contrast to the ordered case, the subsequent development is not symmetric. In this talk we will concentrate on the Vietoris monad, and show how these analogies lead to variations of Isbell, Stone, Priestley and Esakia duality. Finally, by writing $[0,\infty]$ instead of $\mathsf{2}$, we are automatically provided with metric variants of these constructions.

References

[1] R. Lowen, *Approach spaces*, Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1997. The missing link in the topology-uniformity-metric triad, Oxford Science Publications.

[2] R. Rosebrugh and R. J. Wood, *Constructive complete distributivity. IV*, Appl. Categ. Structures, 2 (1994), pp. 119–144.

[3] ——, *Split structures*, Theory Appl. Categ., 13 (2004), pp. No. 12, 172–183.