Duality Theory in Algebra, Logic, and Computer Science
Workshop 2 Oxford, August 2012

Adjunctions Induced by the Congruence Lattices of the Free Algebras in a Variety

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Let A be a (finitary) variety of algebras, and let $\mathcal{F}_\kappa^{\mathsf{A}}$ be the free A-algebra generated by a set of cardinality κ. Fix an A-algebra A. For any subset $R \subseteq \mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$, define its *solution set* (in A) to be

$$\mathbb{V}(R) := \{a \in A^\kappa \mid s(a) = t(a) \text{ holds in } A, \text{ for all } (s,t) \in R\}.$$

For any subset $S \subseteq A^\kappa$, define its *associated congruence* (on $\mathcal{F}_\kappa^{\mathsf{A}}$) to be

$$\mathbb{I}(S) := \left\{(s,t) \in \mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}} \mid s(a) = t(a) \text{ holds in } A, \text{ for all } a \in S\right\}.$$

Then the pair $(\mathbb{V}, \mathbb{I})$ yields a Galois connection between the powersets of $\mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$ and A^κ, for each cardinal κ. That is, we have

$$R \subseteq \mathbb{I}(S) \text{ if, and only if, } S \subseteq \mathbb{V}(R).$$

This, in turn, leads to a dual adjunction between A and the following category $\mathsf{Sub}_{\mathrm{def}}^{\mathsf{A}}$ of *subsets of A^κ and definable maps.* An object of $\mathsf{Sub}_{\mathrm{def}}^{\mathsf{A}}$ is an inclusion map $S \subseteq A^\kappa$, for some set S and some cardinal κ. A morphism between $S \subseteq A^\kappa$ and $T \subseteq A^\mu$ is a function $f\colon S \to T$ that agrees over S with some term-definable map $A^\kappa \to A^\mu$.

The construction above is a rather direct abstraction of the dual adjunction in algebraic geometry between affine varieties in k^n, for an algebraically closed field k, and ideals in the polynomial ring $k[X_1, \ldots, X_n]$. It arose in a recent joint paper with L. Spada, for the special case of MV-algebras. In that setting, it is crucial that two further facts hold. Take A to be the standard MV-algebra $[0,1]$. Then the closure operator on $[0,1]^\kappa$ given by the composition $\mathbb{V} \circ \mathbb{I}$ agrees with (and hence *defines*) topological closure in the standard Tychonoff (product) topology on $[0,1]^\kappa$, where $[0,1]$ carries its Euclidean topology. Moreover, the closure operator on $\mathcal{F}_\kappa^{\mathsf{A}} \times \mathcal{F}_\kappa^{\mathsf{A}}$ given by $\mathbb{I} \circ \mathbb{V}$ performs the construction of the *radical congruence* generated by R, i.e. the intersection of all maximal congruences extending R. (Compare Hilbert's Nullstellensatz for k-algebras.) This promptly leads to the main result that semisimple MV-algebras are dually equivalent to compact Hausdorff spaces embedded in Tychonoff cubes, endowed with definable (necessarily continuous) maps as morphisms.

I discuss the two basic questions, which conditions on the variety A and the A-algebra A guarantee that the closure operator $\mathbb{V} \circ \mathbb{I}$ is topological, and that the closure operator $\mathbb{I} \circ \mathbb{V}$ constructs the radical congruence in the sense above.