Applications of Duality on Unification Type Classification

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University of Oxford

Joint works with S. Bova and V. Marra

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Introduction

Unification

Given an equational theory $\mathbb{E}$ over an algebraic language $\mathcal{L}$, an **$\mathbb{E}$-unification problem** is a finite set of equations (pair of $\mathcal{L}$-terms)

$$U = \{(t_1, s_1), \ldots, (t_n, s_n)\}$$

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An **$\mathbb{E}$-unifier** for U is a substitution σ, such that

$$\sigma(t_i) \cong_{\mathbb{E}} \sigma(s_i)$$

for each $i = 1, \ldots, n$

If σ and σ' are $\mathbb{E}$-unifiers for *U*, we say that σ is **more general** than σ', in symbols $\sigma' \preceq \sigma$, if there exist a subtitution γ such that

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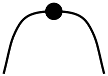
The relation $\preceq$ determines a preorder on the set of $\mathbb{E}$-unifiers of *U* (denoted by $\mathfrak{U}_{\mathbb{E}}(U)$).

Introduction

Unification

An $\mathbb{E}$-unification problem U is said to have type:

$$\mathfrak{U}_{\mathbb{E}}(U)$$

1 

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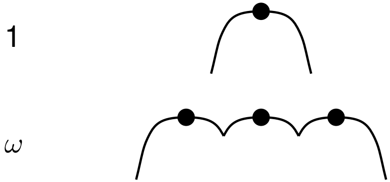

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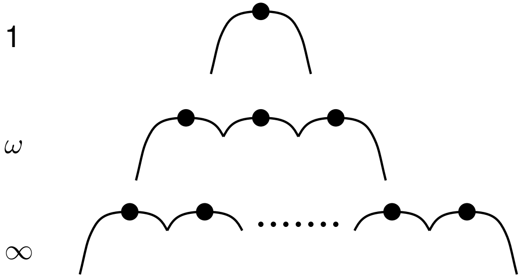

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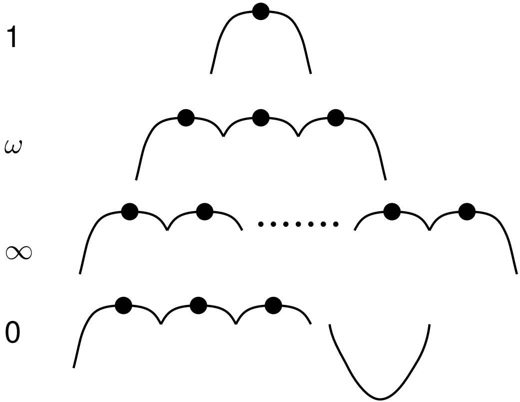

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- ∞ if every $\mathbb{E}$-unification problem U has type 1, ω or ∞ and at least one $\mathbb{E}$-unification problem U_0 has type ∞;
- 0 if at least one $\mathbb{E}$-unification problem U_0 has unification type 0.

[1] S. Ghilardi,
Unification through projectivity,
Journal of Logic and Comp. **7**(6) 733-752, 1997.

Introduction

Algebraic Unification

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$\sigma\colon \mathrm{Var}(U) \to \mathrm{Term}_{\mathcal{L}}$	$h\colon Fp(U) \to P$

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$$\begin{array}{ccc} \mathrm{Var}(U) & \xrightarrow{\ \sigma\ } & \mathrm{Term}_{\mathcal{L}} \\ & {}_{\sigma'}\searrow & \downarrow{\scriptstyle \gamma} \\ & & \mathrm{Term}_{\mathcal{L}} \end{array} \qquad\qquad \begin{array}{ccc} Fp(U) & \xrightarrow{\ h\ } & P \\ & {}_{h'}\searrow & \downarrow{\scriptstyle f} \\ & & P' \end{array}$$

Introduction

The role of Duality

Introduction
The role of Duality

- Description of free algebras

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- Description of free algebras and projective algebras.

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- Description of free algebras and projective algebras.
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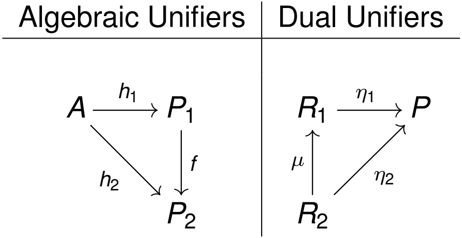

Main Goal

Use dualities to classify unification problems by their unification type

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Use dualities to classify unification problems by their unification type, in the following varieties:

- Bounded Distributive Lattices;
- De Morgan Algebras;
- Kleene Algebras;
- Pseudocomplemented Lattices;
- MV-algebras.

Our results

Bounded Distributive Lattices

[2] H.A. Priestley.
Representation of distributive lattices by means of ordered Stone spaces,
Bull. London Math. Soc. **2** 186-190, 1970.

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Algebra	Spaces
$(L, \vee, \wedge, 0, 1)$	$(X, \leq, \mathcal{T})$

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Homomorphisms	Continuous order preserving maps
Projective	Lattice ordered

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(S. Ghilardi) The unification problem $U = \{(x \wedge y, z \vee t)\}$ has nullary unification type.

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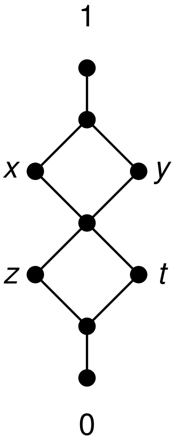

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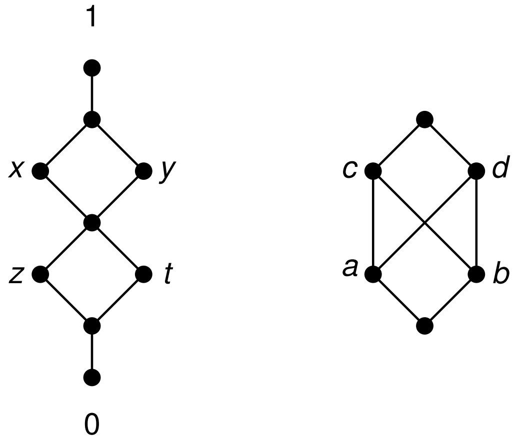

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Bounded Distributive Lattices

Theorem (S. Bova and LMC)

Let U be a unification problem in the language of bounded lattices. Then the unification type of U is:

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Let U be a unification problem in the language of bounded lattices. Then the unification type of U is:

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- ω *iff for every* $x, y \in \mathrm{D}(Fp(U))$ *the interval* $[x, y]$ *is a lattice;*

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- 0 *otherwise.*

Sketch of the proof

Step 1) If D($Fp(U)$) is a lattice, then $Fp(U)$ is projective. Therefore, the identity map is the *Most General Unifier* for S.

Our results

Bounded Distributive Lattices

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Step 2) If D($Fp(U)$) is not a lattice, but for every $x, y \in$ D($Fp(U)$) the interval $[x, y]$ is a lattice. The inclusion maps $\iota_{[x,y]}$ determine a finite set of *Maximal Unifiers*.

Our results

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Step 3) If there exists $x, y \in \mathrm{D}(Fp(U))$ such that the interval $[x, y]$ is not a lattice. Following Ghilardi's proof of nullarity of $\mathcal{BDL}$ we prove that this case is nullary.

Our results
De Morgan algebras

[3] W.H. Cornish and Fowler, P.R.
Coproduct of De Morgan algebras
Bull. Australian Math. Soc. **16** 1-12, 1977.

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Projective	???

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De Morgan algebras

Theorem (S. Bova and LMC)

Let A be a finite De Morgan algebra. Then A is projective iff $D(A) = (X, \leq, f, \mathcal{T})$ satisfies the following:

(M1) $(X, \leq)$ *is a nonempty lattice;*

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(M1) $(X, \leq)$ *is a nonempty lattice;*

(M2) *for all* $x \in X$*, if* $x \leq f(x)$*, then there exists* $y \in X$ *such that* $x \leq y = f(y)$*;*

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(M3) *for all* $x, y, z \in X$

$$\left.\begin{array}{l} x \vee y \leq f(x \vee y) \\ y \vee z \leq f(y \vee z) \\ x \vee z \leq f(x \vee z) \end{array}\right\} \Rightarrow x \vee y \vee z \leq f(x \vee y \vee z).$$

Definition (De Morgan Unification Core)

Let A be a finite De Morgan algebra. The **De Morgan unification core** of $D(A) = (X, \leq, f, \mathcal{T})$ is the set

$$X' = \{x \in Q \mid \exists y, z \in Q \text{ such that } y \leq x, f(x), z = f(z)\}.$$

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Given a De Morgan-unification problem U, if P is a finite projective De Morgan Algebra and $h\colon Fp(U) \to P$ is an algebraic unifier, then $D(h)\colon D(P) \to D(Fp(U))$ satisfies

$$\mathrm{img}(D(h)) \subseteq X'.$$

Theorem (S. Bova and LMC)

Let U be a De Morgan-unification problem and X' be the De Morgan unification core of $D(Fp(U))$.

Our results

De Morgan algebras

Theorem (S. Bova and LMC)

Let U be a De Morgan-unification problem and X' be the De Morgan unification core of $D(Fp(U))$. Then the unification type of U is:

1. *iff $(X', \leq, f, \mathcal{T})$ satisfies (M_1), (M_2), and (M_3);*

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- ω *iff* $(X', \leq, f, \mathcal{T})$ *does not satisfy* (M_1)*, but for every* $x \in X'$ *with* $x \leq' i(x)$, $[x, f(x)]_{X'}$ *satisfies* (M_1), (M_2), *and* (M_3)*;*

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- ω *iff $(X', \leq, f, \mathcal{T})$ does not satisfy (M_1), but for every $x \in X'$ with $x \leq' i(x)$, $[x, f(x)]_{X'}$ satisfies (M_1), (M_2), and (M_3);*
- 0 *otherwise.*

Our results

MV-algebras

Definition

An algebra $(A, \oplus, \neg, 0)$ is an **MV-algebra** if

1. $(A, \oplus, 0)$ is a commutative monoid;
2. $\neg\neg a = a$ for each $a \in A$;
3. $\neg 0 \oplus a = \neg 0$ for each $a \in A$;
4. $\neg(\neg a \oplus b) \oplus b = \neg(\neg b \oplus a) \oplus a$ for each $a \in A$.

Our results
MV-algebras

[4] V. Marra and L. Spada.
Duality, projectivity, and unification in Łukasiewicz logic and MV-algebras.
Preprint

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Algebra	Spaces
$(A, \oplus, \neg, 0)$	Rational Polyhedron

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n-generated free	$[0,1]^n$

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$(A, \oplus, \neg, 0)$	Rational Polyhedron
Homomorphisms	Piecewise (affine) linear continuous maps with integer coefficients
n-generated free	$[0, 1]^n$
Projective	Retracts of cubes

Our results

MV-algebras

Theorem (V. Marra and L. Spada)

The unification problem $\{(x \vee y \vee \neg x \vee \neg y, \top)\}$ *in the equational theory of MV-algebras has unification type* 0*.*

Therefore, the equational theory of MV-algebras has unification type 0*.*

Our results
MV-algebras

[5] E. Jeřábek.
Bases of admissible rules of Łukasiewicz logic,
J. Logic and Comp., **20**(6) 1149-1163, 2010 .

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Bases of admissible rules of Łukasiewicz logic,
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Definition

A formula φ is **admissibly saturated** in $Ł_\infty$ if for every finite set Σ of formulas the following conditions are equivalent:

(i) the rule $(\{\varphi\}, \Sigma)$ is admissible in $Ł_\infty$;

(ii) there exists $\psi \in \Sigma$ such that $(\{\varphi\}, \{\psi\})$ is derivable in $Ł_\infty$.

Our results
MV-algebras

Theorem
A formula α is admissibly saturated iff Fp$(\alpha, \top)$ is isomorphic to a subalgebra of a free MV-algebra

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A formula α is admissibly saturated iff Fp$(\alpha, \top)$ is isomorphic to a subalgebra of a free MV-algebra iff Fp$(\alpha, \top)$ is isomorphic to a subalgebra of a projective MV-algebra.

Our results

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Theorem

A formula α is admissibly saturated iff $Fp(\alpha, \top)$ is isomorphic to a subalgebra of a free MV-algebra iff $Fp(\alpha, \top)$ is isomorphic to a subalgebra of a projective MV-algebra.

Theorem (E. Jeřábek)

A formula α is admissibly saturated iff $P = D(Fp(\alpha, \top))$ satisfies the following conditions:

1. *P is connected;*
2. *$P \cap \mathbb{Z}^n \neq \emptyset$;*
3. *P is strongly regular.*

Our results

MV-algebras

Theorem (E. Jeřábek)

Every formula φ has an **admissible saturated appoximation** *in Ł$_\infty$, that is, a set Δ such that*

1. *Δ is a finite set of admissibly saturated formulas,*
2. *$(\{\varphi\}, \{\psi\})$ is derivable in Ł$_\infty$ for each $\psi \in \Delta$,*
3. *the rule $(\{\varphi\}, \Delta)$ is admissible in Ł$_\infty$.*

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2. *$(\{\varphi\}, \{\psi\})$ is derivable in Ł$_\infty$ for each $\psi \in \Delta$,*
3. *the rule $(\{\varphi\}, \Delta)$ is admissible in Ł$_\infty$.*

Moreover, the elements of Δ can be chosen such that $\alpha, \beta \vdash_{Ł_\infty} \perp$ for each $\alpha \neq \beta$.

Theorem (LMC and V. Marra)

Let α be an admissible saturated formula in the of $Ł_\infty$ the preorder set of unifiers of $(\alpha, \top)$ is upward directed. Therefore type$_{MV}(\alpha, \top) \in \{0, 1\}$.

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Theorem

Let α be a formula in the language of $Ł_\infty$ and $\{\alpha_1, \ldots, \alpha_n\}$ be an admissible saturated approximation of α, such that $\alpha_i, \alpha_j \vdash_{Ł_\infty} \bot$ for each $i \neq j$. Then

$$type_{MV}(\alpha, \top) = \begin{cases} n & \text{if } type_{MV}(\alpha_i, \top) = 1 \text{ for each } i, \\ 0 & \text{otherwise.} \end{cases}$$

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Corollary

There are no infinitary MV-unification problems.

- Classification of unification problems in Kleene algebras.

Our results

Other Results

- Classification of unification problems in Kleene algebras.
- Type of subvarieties of pseudocomplemented lattices.

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- Classification of unification problem in $(\mathcal{B}_1, \mathcal{B}_2)$.

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- Classification of unification problems in Kleene algebras.
- Type of subvarieties of pseudocomplemented lattices.
- Classification of unification problem in $(\mathcal{B}_1, \mathcal{B}_2)$.
- Complete classification of Planar MV-unification problems.

Further work

- Classification of unification problem in MV-algebras.

Further work

- Classification of unification problem in MV-algebras. Classify of admissible saturated formulas.

Further work

- Classification of unification problem in MV-algebras. Classify of admissible saturated formulas.
- Classification of unification problem in $\mathcal{B}_n$ with $n \geq 3$.

Applications of Duality on Unification Type Classification

Thank you for your attention!