Concrete Coalgebraic Modal Logic

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Introduction

Coalgebra

Logic of Predicate Liftings

Concrete Coalgebraic Modal Logic

What is it about?

- Coalgebras are everywhere in CS,
- e.g. streams, labelled transition systems, Markov processes, image-compact transition systems, automata . . .
- Modal logic for coalgebras.
- An application of duality theory.
- Hennessy-Milner property:
  Behavioural equivalence (Bisimilarity) $\equiv$ Logical equivalence

General frameworks:

- L. Moss' cover modality, 1999.
- D. Pattinsson's predicate liftings, 2002.
- A. Kurz's abstract logic using dualities, 2006.
- B. Klin's abstract logic using dual adjunctions, 2007

Variants and examples:

- C. Kupke's Stone coalgebras, 2003/4.
- M. Bonsangue's π-calculus in logical form, 2007
- Kapulkin's coalgebraic logic over **Poset**, 2012
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Motivation: general study over concrete categories?

- A deeper understanding of abstract frameworks.
- Intuitive definitions.
- Modalities beyond set.
- Still a wide range of applications.

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What is a T-coalgebra?

Given a category $\mathscr{C}$ and a functor $T : \mathscr{C} \to \mathscr{C}$:

$$\begin{array}{ccc} X & \xrightarrow{\ \xi\ } & TX \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle Tf} \\ Y & \xrightarrow[\ \gamma\ ]{} & TY \end{array}$$

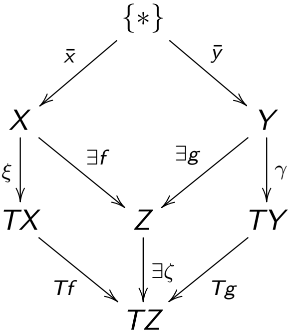
Objects: $\xi, \gamma, \ldots$ morphisms from X to TX.

Morphisms: $f, \ldots$ morphisms making the above diagram commutative.

Behavioural equivalence:

Coalgebraic bisimilarity

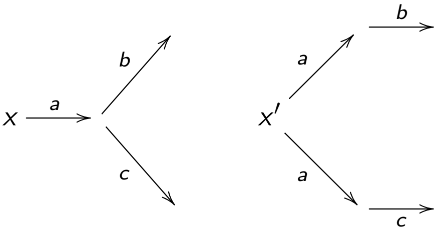
In **Set**, $x \in X$ and $y \in Y$ are behaviourally equivalent if



commutes where $\bar{x}(*) = x$ and $\bar{y}(*) = y$.

Example: labelled transition systems

Coalgebras of $\mathcal{P}(-)^A$: $X \to (\mathcal{P}X)^A$



- Leafs are all behaviourally equivalent.
- x and x' are NOT.

Example: Streams over Σ

Coalgebras of $\Sigma \times -$: $X \to \Sigma \times X$

$$x_0 \xrightarrow{a_0} x_1 \xrightarrow{a_1} x_2 \cdots$$

Each state $x \in X$ gives an infinite sequence $\langle a_i \rangle_{i \in \omega}$ of alphabets over Σ.

- Behaviourally equivalent elements have the same output.

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Modality classically

Given a Kripke frame $\langle X, R \subseteq X \times X \rangle$,

Locally: $x \models \Diamond\varphi$ if and only if $y \models \varphi$ for some y accessible from x, i.e. xRy.

Globally: $[\![-]\!] : \mathrm{Form} \to 2^X$ maps formulae to subsets of X:

$$[\![\Diamond\varphi]\!] = \{x \in X : R[x] \cap [\![\varphi]\!] \neq \emptyset\}$$

and $x \models \varphi$ iff $x \in [\![\varphi]\!]$

where $R[x] = \{y \in X : xRy\}$.

Modalities in general

- The collection 2^X of subsets of X is bijective to $\mathrm{Hom}(X, 2)$.

- Define $\Diamond_X : \mathrm{Hom}(X, 2) \to \mathrm{Hom}(\mathcal{P}X, 2)$ by

$$\varphi \mapsto \lambda S. \bigvee \varphi[S]$$

and $[\![\Diamond\varphi]\!]$ can be decomposed into:

$$\begin{aligned} [\![\Diamond\varphi]\!] &= \bigvee \{\varphi(y) : y \in R[x]\} \\ &= \Diamond_X([\![\varphi]\!]) \circ R[-] \end{aligned}$$

where $\varphi : X \to 2$ and $R[-] : X \to \mathcal{P}X$.

Modalities in general (contd.)

$\Diamond_X$ is natural in X:

$$
\begin{array}{ccccc}
Y & \quad & \mathrm{Hom}(X,2) & \xrightarrow{\Diamond_X} & \mathrm{Hom}(\mathcal{P}X,2) \\
\forall f \big\downarrow & & -\circ f \big\downarrow & & \big\downarrow -\circ\mathcal{P}f \\
X & & \mathrm{Hom}(Y,2) & \xrightarrow[\Diamond_Y]{} & \mathrm{Hom}(\mathcal{P}Y,2)
\end{array}
$$

where $\mathcal{P}f(S) = f[S]$.

Modalities in general (contd.)

- A unary predicate lifting of $T : \mathbf{Set} \to \mathbf{Set}$ is a function

$$\lambda_X : \mathrm{Hom}(X, 2) \to \mathrm{Hom}(TX, 2)$$

  natural in X.

- A T-coalgebra $\xi : X \to TX$ maps propositions on TX to X:

$$\mathrm{Hom}(X, 2) \xrightarrow{\lambda_X} \mathrm{Hom}(TX, 2) \xrightarrow{-\circ\xi} \mathrm{Hom}(X, 2)$$

Syntax

Let Λ a set of predicate liftings of T.
The language $\mathcal{L}(\Lambda)$ with finitary conjunction is defined by

$$\begin{aligned} \varphi := & \bot \\ & |\varphi \wedge \varphi \\ & | \neg\varphi \\ & | \lambda(\varphi) \\ & \text{where } \lambda \in \Lambda. \end{aligned}$$

Semantics

For any T-coalgebra $\langle X, \xi \rangle$, the interpretation is defined by:

$$\begin{aligned}
[\![-]\!]_\xi &: \mathcal{L}(\Lambda) \to \mathrm{Hom}(X, 2) \\
[\![\bot]\!]_\xi &= \lambda x.\bot \\
[\![\neg\varphi]\!]_\xi &= \neg[\![\varphi]\!]_\xi \\
[\![\varphi \wedge \psi]\!]_\xi &= [\![\varphi]\!]_\xi \wedge [\![\psi]\!]_\xi \\
[\![\lambda(\varphi)]\!]_\xi &= \lambda_X([\![\varphi]\!]) \circ \xi
\end{aligned}$$

and write

$$(\xi, x) \models \varphi$$

if $[\![\varphi]\!]_\xi(x) = \top$.

Logical equivalence, adequacy and expressivity

Definition

Given $x \in X$ and $y \in Y$, x and y are *logically equivalent* if

$$(\xi, x) \models \varphi \Leftrightarrow (\gamma, y) \models \varphi$$

or,

$$[\![\varphi]\!]_\xi(x) = [\![\varphi]\!]_\gamma(y)$$

for every $\varphi \in \mathcal{L}(\Lambda)$.

Definition

A logic is adequate if every behaviourally equivalent elements are logically equivalent; a logic is expressive if the converse holds.

Adequacy of logic of predicate liftings

Given a $f : \langle X, \xi \rangle \to \langle Y, \gamma \rangle$, x and $f(x)$ are logically equivalent for any x, i.e.

$$[\![\varphi]\!]_\xi = [\![\varphi]\!]_\gamma \circ f$$

Proof.

Induction on $\mathcal{L}(\Lambda)$.

$[\![\lambda\varphi]\!]_\gamma \circ f = \lambda_Y([\![\varphi]\!]_\gamma) \circ \gamma \circ f = \lambda_X([\![\varphi]\!]_\gamma \circ f) \circ \xi = [\![\lambda\varphi]\!]_\xi(x)$ by

$$\begin{array}{ccccc}
\mathrm{Hom}(Y,2) & \xrightarrow{\lambda_Y} & \mathrm{Hom}(TY,2) & \xrightarrow{-\circ\gamma} & \mathrm{Hom}(Y,2) \\
\downarrow{\scriptstyle -\circ f} & & \downarrow{\scriptstyle -\circ Tf} & & \downarrow{\scriptstyle -\circ f} \\
\mathrm{Hom}(X,2) & \xrightarrow[\lambda_X]{} & \mathrm{Hom}(TX,2) & \xrightarrow[-\circ\xi]{} & \mathrm{Hom}(X,2)
\end{array}$$

□

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\end{array}$$

$\square$

Λ induces a natural transformation

Define L : **BA** $\to$ **BA** by

$$A \to \coprod_{\Lambda} FUA$$

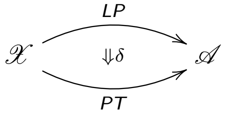
where $F \vdash U$: **BA** $\to$ **Set** and $\delta : LPX \to PTX$ by

$$\langle \lambda, S \rangle \mapsto \lambda_X(S)$$

where $S \in \mathrm{Hom}(X, 2)$ and P maps every set to its Boolean algebra of subsets of X.

Abstract logic over logical connection

- An abstract (algebraic) logic of a functor $T : \mathscr{X} \to \mathscr{X}$ over $P \dashv S : \mathscr{A}^{op} \to \mathscr{X}$ consists of
  - a functor $L : \mathscr{A} \to \mathscr{A}$ encoding modal operators and
  - a natural transformation δ



interpreting modalties.

- A T-coalgebra is mapped to a L-algebra by δ:

$$\tilde{P} : (X \xrightarrow{\xi} TX) \mapsto (LPX \xrightarrow{\delta_X} PTX \xrightarrow{P\xi} PX)$$

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Concrete categories (Porst & Tholen, 1991)

A (single-sorted) concrete category $(\mathscr{C}, U)$ has

- a faithful functor $U : \mathscr{C} \to \mathbf{Set}$
- which is representable by some C_0, i.e.

$$UX \cong \mathscr{C}(C_0, X)$$

natural in X.

Hence, an element $x \in UX$ corresponds uniquely to a morphism $\bar{x} : C_0 \to X$.

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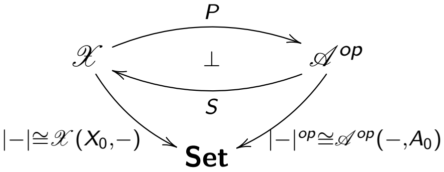
Dual adjunction, or logical connection

A dual adjunction between concrete categories $\mathscr{X}$ and $\mathscr{A}$ (Porst & Tholen, 1991):



and $(-)^\sharp : \mathscr{A}(A, PX) \cong \mathscr{X}(X, SA)$ natural in A and X. Here,

- $\mathscr{A}$ is the category of "propositional logic" of $\mathscr{X}$.

Th object Ω of truth values

- The underlying sets of PX_0 and SA_0 are the same:

$$
\begin{aligned}
|PX_0| &= \mathscr{A}(A_0, PX_0) \\
&\cong \mathscr{X}(X_0, SA_0) = |SA_0|
\end{aligned}
$$

  Ω denotes PX_0 and SA_0, called the objects of truth values.

- The underlying set of PX is given by "homming" into Ω:

$$|PX| \cong \mathscr{X}(X, \Omega),$$

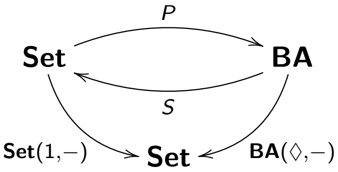
  and $|SA| \cong \mathscr{A}(A, \Omega)$.

Propositions, theories, and models

To know what L is doing, we observe . . .

- A proposition (observable property, or testing) of an object $X \in \mathscr{X}$ is
$$\varphi : X \to \Omega.$$
- The theory of an element $X_0 \xrightarrow{x} X$ of X is
$$Px : PX \to \Omega$$
- The evaluation of $\varphi : A_0 \to PX$ by $x : X_0 \to X$ is composition:
$$X_0 \xrightarrow{x} X \xrightarrow{\varphi^\sharp} \Omega$$
or $A_0 \xrightarrow{\varphi} PX \xrightarrow{Px} \Omega$.

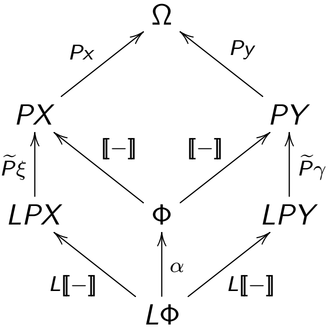
An example



- An element $x \in X : 1 \xrightarrow{x} X$
- The truth-value object $\Omega : 2$
- A proposition of a set $X : X \xrightarrow{\chi_S} 2$
- The theory of $x \in X$: $PX \xrightarrow{x^{-1}} 2$

Logical equivalence

- Suppose that the initial L-algebra $\langle \Phi, \alpha \rangle$ exists.
  $|\Phi|$ is a language and the unique map is the semantics of formulae.
- Given $x \in X \xrightarrow{\xi} TX$ and $y \in Y \xrightarrow{\gamma} TY$, x and y are logically equivalent if



  commutes.

Satisfaction relation $\models$

- For every $\varphi : A_0 \to \Phi$, and $x \in X \xrightarrow{\xi} TX$, the satisfaction relation is defined as:

  $$x \models_v \varphi$$

  if $Px \circ [\![\varphi]\!]_{\tilde{P}\xi} = v$.

- x and y are logically equivalent iff

  $$Px \circ [\![\varphi]\!]_{\tilde{P}\xi} = Py \circ [\![\varphi]\!]_{\tilde{P}\gamma}$$

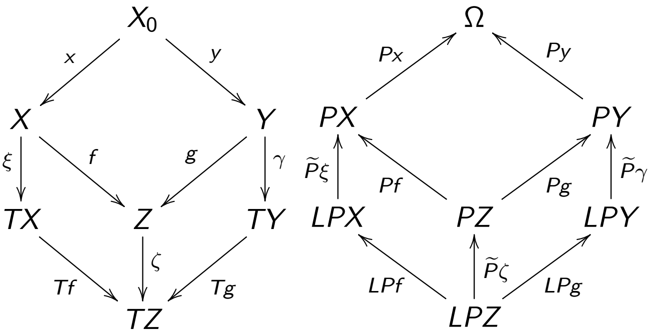
  for every $\varphi \in \Phi$

- Or, by its dual form:

  $$[\![x]\!]^\sharp_{\tilde{P}\xi} = [\![y]\!]^\sharp_{\tilde{P}\gamma}$$

Adequacy

If x and y are behaviourally equivalent,



then they are logically equivalent.

Modalities beyond **Set**

A T-modality of arity $A \in \mathscr{A}$ is a mapping

$$\mathscr{A}(A, PX) \to \mathscr{A}(A_0, PTX)$$

natural in X.

The object of T-modalities of arity A is $PTSA$:

$$\begin{aligned}
& [\mathscr{A}(A, P-), \mathscr{A}(A_0, PT-)] & \\
\cong & [\mathscr{X}(-, SA), \mathscr{X}(T-, SA_0)] & \text{by dual adjunctions} \\
\cong & \mathscr{X}(TSA, SA_0) & \text{by Yoneda lemma} \\
\cong & \mathscr{A}(A_0, PTSA) & \text{by the dual adjunction} \\
\cong & |PTSA| & \text{by definition}
\end{aligned}$$

What is a predicate lifting?

A T-modality of arity A_0 corresponds to

$$[\mathscr{A}(A_0, P-), \mathscr{A}(A_0, PT-)] \cong [\mathscr{X}(-, SA_0), \mathscr{X}(T-, SA_0)] \\ \cong [\mathscr{X}(-, \Omega), \mathscr{X}(T-, \Omega)]$$

a unary predicate liftings.

Corollary

The collection of unary predicate liftings is bijective to the underlying set of $PTSA_0$, or $\mathscr{X}(T\Omega, \Omega)$.

For any two-valued logical connection of **Set**, it is

$$\mathbf{Set}(T2, 2)$$

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Logic of all finitary predicate liftings

- If $U : \mathscr{A} \to \mathbf{Set}$ has a left adjoint F, then $A_0 \cong F1$ and $\coprod_n A_0 \cong Fn$.
- $PTSFn$ consists of n-ary predicate liftings.
- If $\mathscr{A}$ is finitary algebraic, then the logic of all finitary liftings consists of
$$L = \int^{n \in \omega} \mathscr{A}(Fn, -) \bullet PTSFn$$
with a natural transformation $LP \xrightarrow{\rho_P} PTSP \xrightarrow{PT\eta} PT$ where ρ is given by the mediating morphism.

Logic of all finitary predicate liftings (contd.)

- L is finitary by construction.
- Syntax: The initial L-algebra exists, i.e. the colimit of the following sequence

$$F1 \xrightarrow{!} LF1 \xrightarrow{L!} \cdots \longrightarrow L^i F1 \longrightarrow \cdots$$

which is a construction of language.
- Semantics: the interpretation $[\![-]\!]$ is the mediating morphism: applying $\delta : LP \to PT$ step-by-step.

Further issues

- The expressivity of the logic of all finitary prediate liftings.
- Completeness
- Examples: general Stone duality, modal bilattices, and ...
- A more general setting: a multi-sorted and enriched framework.

Thanks for your attention.