The construction of free algebras via a functor on partial algebras

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(joint work with Sam van Gool)

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Free algebra as colimit of a chain

Setting: variety $\mathbf{V}$ with a well-understood locally finite reduct $\mathbf{V}^-$.

Examples:

- Modal algebras = Boolean algebras + $\Diamond$,
- Heyting algebras = Distributive lattices + $\rightarrow$.

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Aim: construct finitely generated $\mathbf{V}$-algebras $F_{\mathbf{V}}(x_1, \ldots, x_m)$.

Idea: Regard $F_{\mathbf{V}}(x_1, \ldots, x_m)$ as colimit of a chain of finite algebras in the reduced signature, and add the additional operation(s) step-by-step.

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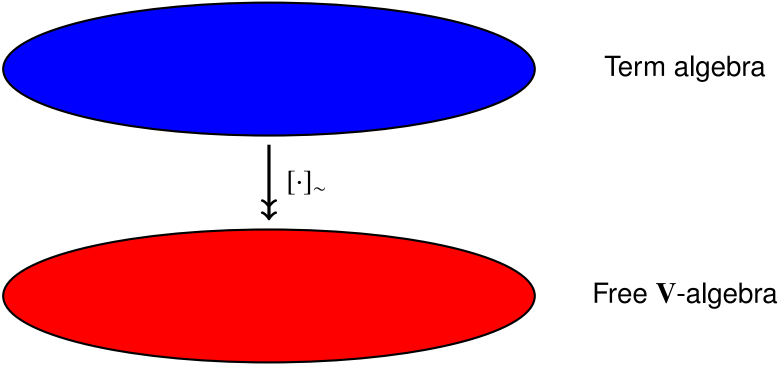
Setting: $\mathbf{V} = \mathbf{V}^- + f$.



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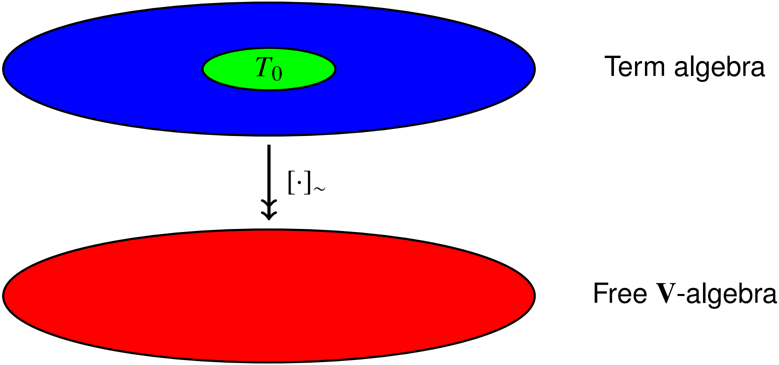

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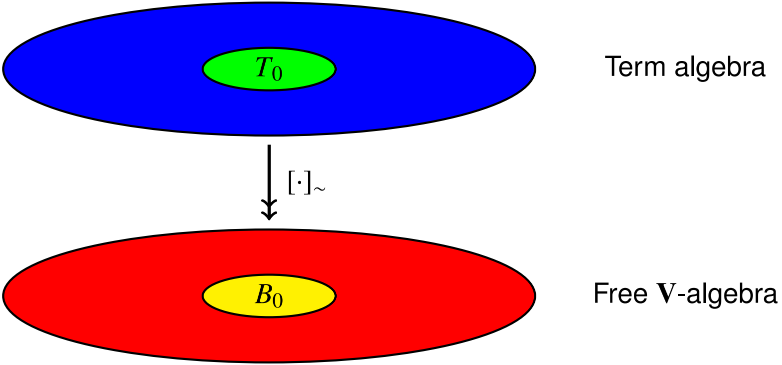

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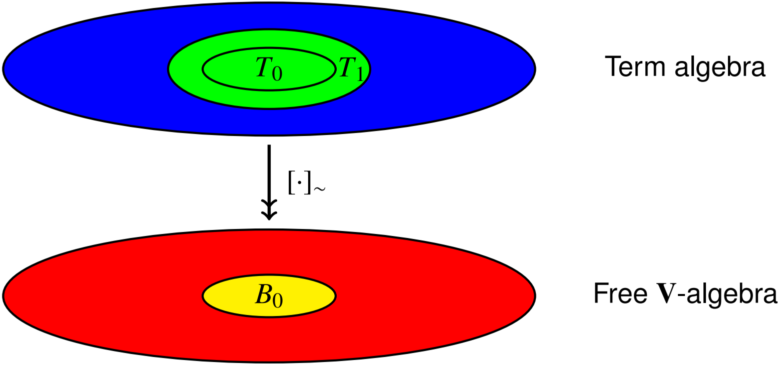

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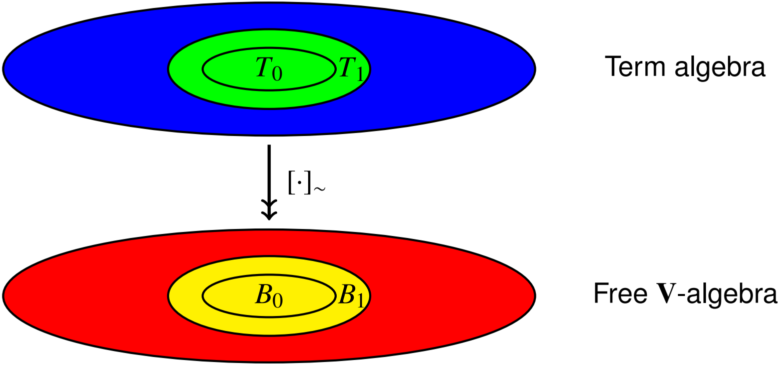

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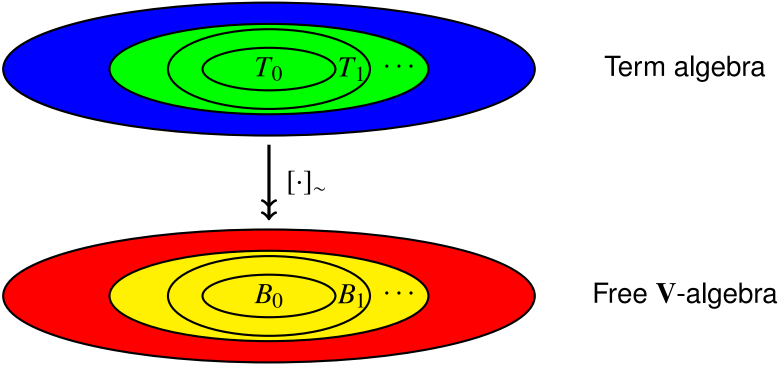

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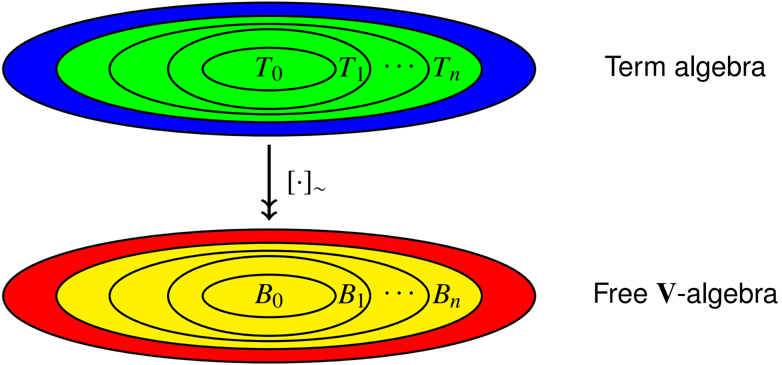

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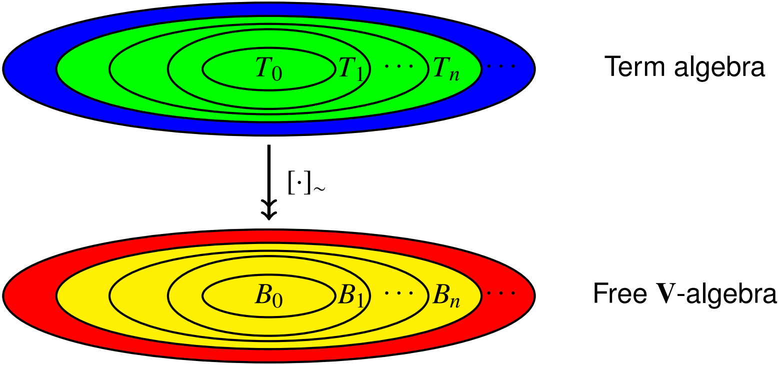

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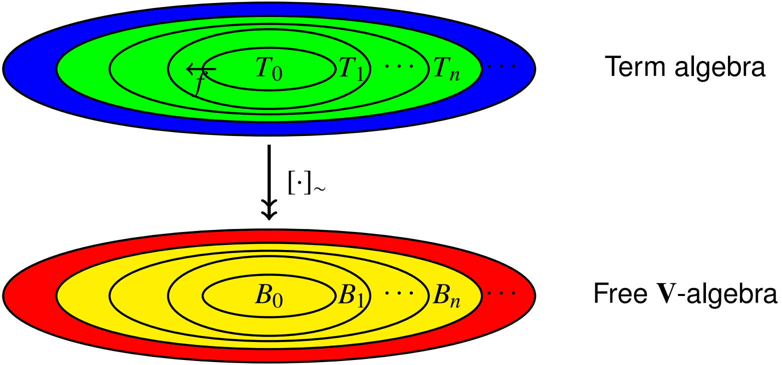

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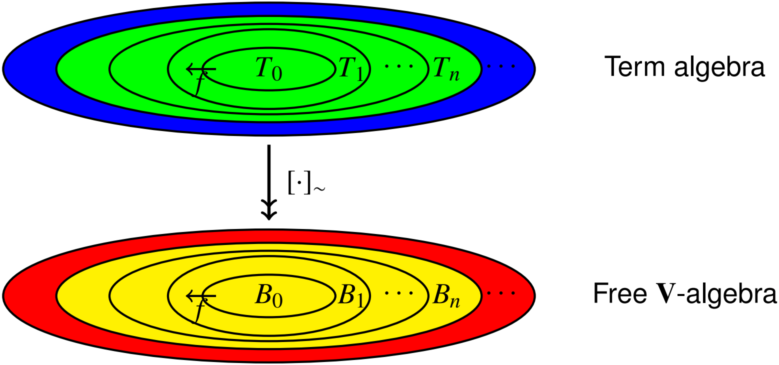

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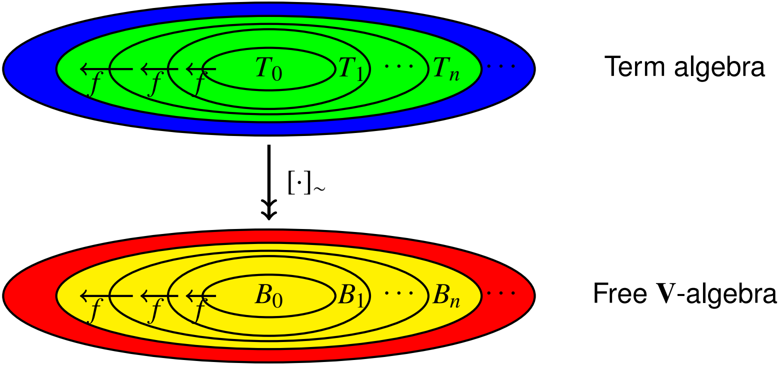

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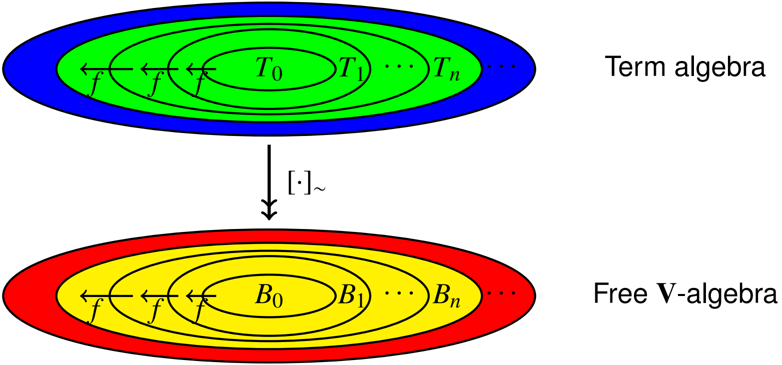

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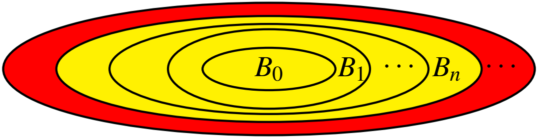
$$F_{\mathbf{V}}(x_1, \ldots, x_m) = \mathrm{colim}_{n \geq 0}\ B_n$$

Research Question



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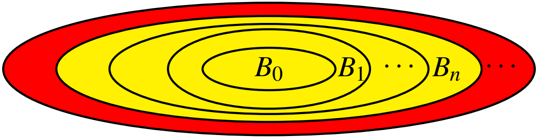

Can B_{n+1} be obtained from B_n by a uniform method?

- Yes, if the variety is defined by pure rank 1 equations [N. Bezhanishvili, Kurz]
- Yes, in some particular cases outside this class: S4 modal algebras [Ghilardi], Heyting algebras [Ghilardi, N. Bezhanishvili & Gehrke].
- Not always, since logics can be undecidable.
- **We give general sufficient conditions** under which this is possible (known cases follow as particular instances).

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- A **homomorphism** $h : A \to B$ of partial algebras is a function which preserves all total operations, and preserves the partial operation f whenever defined.

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Definition:

- A **homomorphism** $h : A \to B$ of partial algebras is a function which preserves all total operations, and preserves the partial operation f whenever defined.
- A homomorphism $h : A \to B$ is **image-total** if the image of h is contained in the domain of f^B.

Free image-total functor

Aim: Obtain the free $\mathbf{V}$-algebra by repeated application of a free image-total functor on partial $\mathbf{V}$- algebras.

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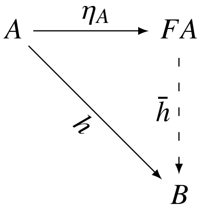
Definition: A functor $F : \mathbf{pV} \to \mathbf{pV}$ is **free image-total** if there is a component-wise image-total natural transformation $\eta : 1_{\mathbf{pV}} \to F$ such that, for all image-total $h : A \to B$, there exists a unique $\bar{h} : FA \to B$ making the following diagram commute:



Free image-total functor

Theorem: Let $\eta : 1 \to F$ be a free image-total functor and $A_0 \in \mathbf{pV}$. Let A_ω be the partial algebra-colimit of the image-total chain

$$\{\eta_{F^n(A_0)} : F^n(A_0) \to F^{n+1}(A_0)\}_{n \geq 0}$$

If A_ω is in $\mathbf{V}$, then A_ω is the free total $\mathbf{V}$-algebra over A_0.

Proof: Category-theoretic arguments.

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To apply this theorem:

- We construct a free image-total functor for any set of equations.
- We give sufficient conditions under which $A_\omega \in \mathbf{V}$.

Quasi-equations

A **quasi-equation** is an expression of the form

$$s_1 = t_1 \,\&\, \ldots \,\&\, s_n = t_n \;\rightarrow\; s = t.$$

We want to work with quasi-equations of rank at most 1.

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Remark: Any variety may be axiomatised by quasi-equations of rank at most 1 using flattening.

Example: the modal axiom

$$\Diamond\Diamond a = \Diamond a$$

is equivalent to the quasi-equation

$$a' = \Diamond a \ \rightarrow\ \Diamond a' = \Diamond a.$$

Free image-total functor

Let $\mathcal{E}$ be a set of quasi-equations (of rank at most 1) axiomatizing the variety $\mathbf{V}$. For $A \in \mathbf{pV}$, define

$$F_{\mathcal{E}}(A) := [A + F_{\mathbf{V}^-}(\mathbf{f}A)]/\theta_A$$

where

- $\mathbf{V}^-$: reduct of $\mathbf{V}$ to the signature of total operations,
- $\mathbf{f}A$: formal elements $\{\mathbf{f}a : a \in A\}$, yielding partial operation $a \mapsto \mathbf{f}a$ for $a \in A$,
- θ_A: smallest $\mathbf{pV}$-congruence on $A + F_{\mathbf{V}^-}(\mathbf{f}A)$ containing $\langle f^A a, \mathbf{f}a \rangle$, for all $a \in \mathrm{dom}(f^A)$.

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The component of $\eta\colon 1 \to F_{\mathcal{E}}$ at A is the composite

$$A \rightarrowtail A + F_{\mathbf{V}^-}(\mathbf{f}A) \twoheadrightarrow F_{\mathcal{E}}(A).$$

Free image-total functor

Lemma: $F_{\mathcal{E}}$ is a free image-total functor with universal arrow η.

Furthermore, if $A_0 \in \mathbf{pV}$ is such that each component $\eta_{F^n_{\mathcal{E}}(A_0)} : F^n_{\mathcal{E}}(A_0) \to F^{n+1}_{\mathcal{E}}(A_0)$ is an embedding, then $A_\omega \in \mathbf{V}$.

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Question: When do you get a chain of embeddings?

Approach in modal algebra setting: apply duality theory.

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$\Diamond\Diamond a \leq \Diamond a$	$xRy \rightarrow \exists y' \sim y.\, R[y'] \subseteq R[x]$

The variety $\mathbf{KB}$

Example: variety of $\mathbf{KB}$ modal algebras.

Signature: $\bot, \top, \vee, \wedge, \neg, \Diamond$.

Axioms: Boolean algebras +

$$\Diamond\bot = \bot$$
$$\Diamond(a \vee b) = \Diamond a \vee \Diamond b$$
$$a \leq \neg\Diamond b \to b \leq \neg\Diamond a.$$

Dually: q-frames $(X, R, \sim)$ where the relation R is quasi-symmetric.

Dual description of $F_{\mathbf{KB}}$

By definition, for a partial **KB** algebra A,

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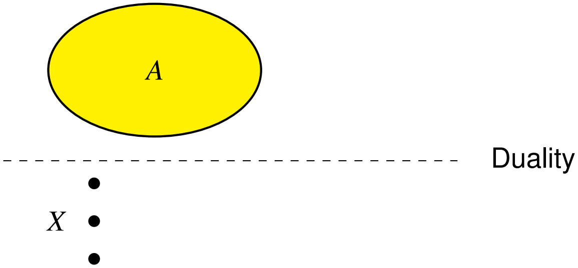

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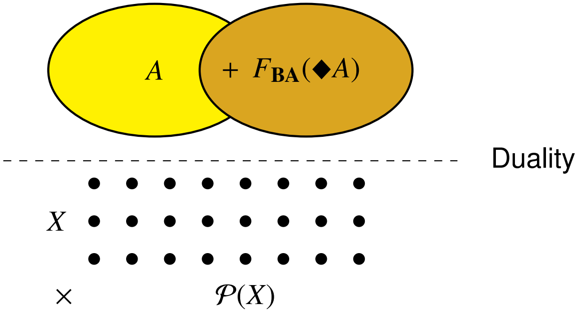

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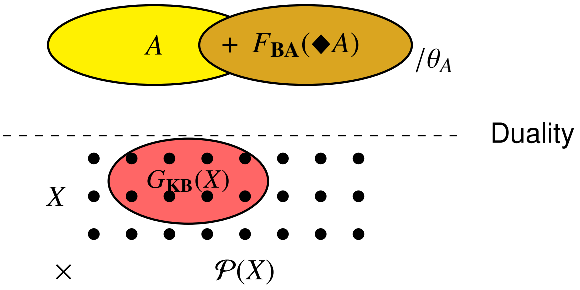

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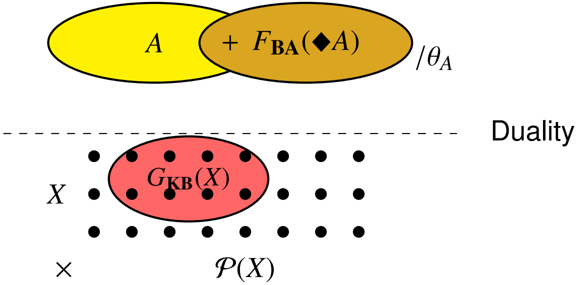
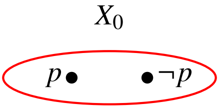


Using **correspondence theory**, one can explicitly calculate a first-order definition of the points in $G_{\mathbf{KB}}(X, R, \sim)$

The chain for **KB**

First steps



The chain for **KB**

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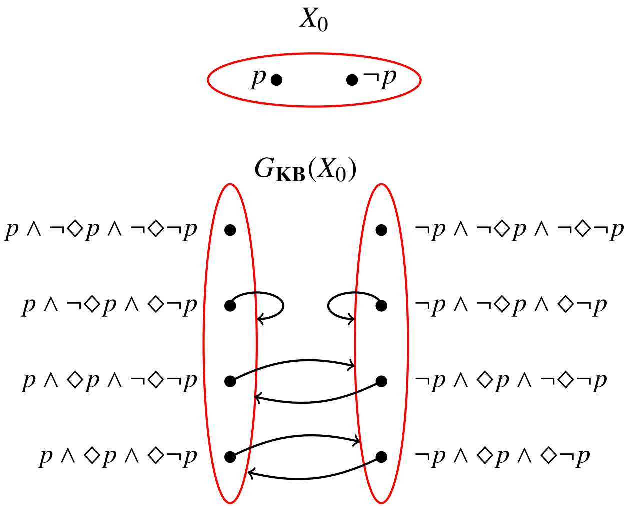
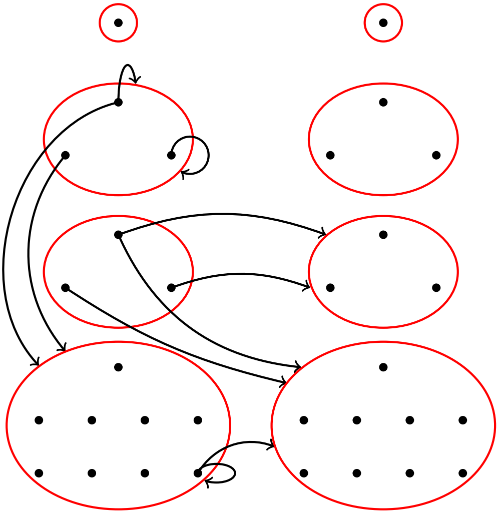


The chain for $\mathbf{KB}$

(part of) $G^2_{\mathbf{KB}}(X_0)$



The chain for **S4**

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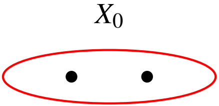

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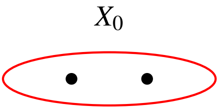

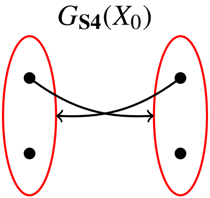

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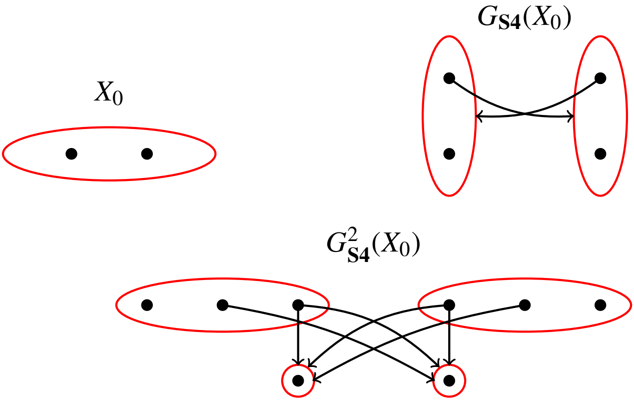

The chain for $\mathbf{S4}$
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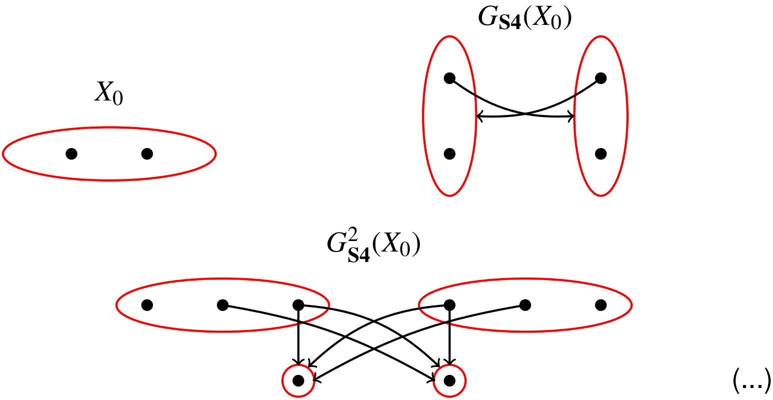

Results

The method works for the modal logics:

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- **K4** (transitive frames)
- **T** (reflexive frames)
- **S4** (transitive and reflexive frames)
- **K5** (Euclidean frames: if xRy and xRz then yRz)

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Question: when does the method apply in general?

- Logical equivalence is not always decidable.
- Do you get embeddings for all decidable logics?
- Do you get embeddings for all varieties axiomatised by rank 0-1 equations?

Free Heyting algebras

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Universal model: U = points of finite height in X.

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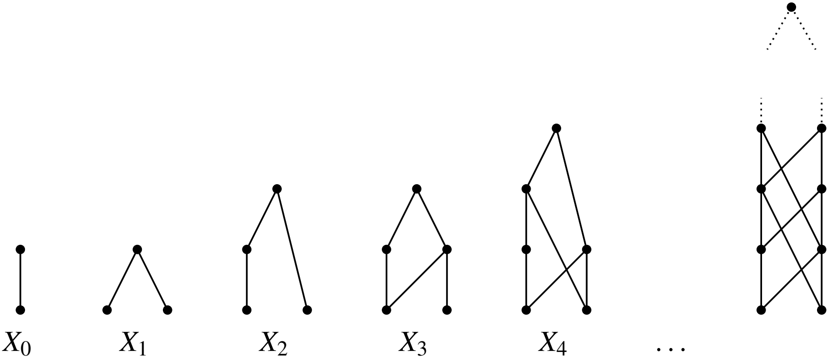
Questions:

- How to characterise the definable subsets of U?
- How does U relate to the posets in the step-by-step construction?

Free Heyting algebra on 1 generator



The construction of free algebras via a functor on partial algebras

Dion Coumans
(joint work with Sam van Gool)

Radboud University Nijmegen

Oxford, August 2012