The structure of skew distributive lattices

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Motivation

Skew lattices: noncommutative lattices.

Andrej Bauer, KCV, Mai Gehrke, Sam Van Gool and Ganna Kudryavtseva: *A non-commutative Priestley duality*, preprint.

Sam will explain the contents of the paper.

Skew lattices

A *skew lattice* is an algebra $(S; \wedge, \vee)$ of type $(2, 2)$ such that $\wedge$ and $\vee$ are both idempotent and associative, and they dualize each other in that

$$x \wedge y = x \text{ iff } x \vee y = y \text{ and}$$
$$x \wedge y = y \text{ iff } x \vee y = x.$$

A *skew lattice with zero* is an algebra $(S; \wedge, \vee, 0)$ of type $(2, 2, 0)$ such that $(S; \wedge, \vee)$ is a skew lattice and $x \wedge 0 = 0 = 0 \wedge x$ for all $x \in S$.

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Rectangular bands

- A *rectangular band* $(A, \wedge)$:
  - $\wedge$ is idempotent and associative
  - $x \wedge y \wedge z = x \wedge z$
- It becomes a skew lattice if we define $x \vee y = y \wedge x$.
- For sets X and Y define $\wedge$ on $X \times Y$:

$$(x_1, y_1) \wedge (x_2, y_2) = (x_1, y_2)$$

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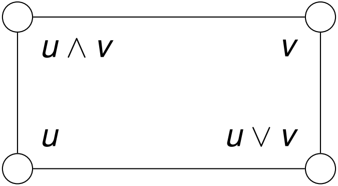

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The order and Green's relation $\mathcal{D}$

S a skew lattice.

Natural preorder: $x \preceq y$ iff $x \wedge y \wedge x = x$ (and dually $y \vee x \vee y = y$).

Natural partial order: $x \leq y$ iff $x \wedge y = x = y \wedge x$ (and dually $y \vee x = y = x \vee y$).

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Theorem (Leech, 1989)

- *$\mathcal{D}$ is a congruence;*
- *$S/\mathcal{D}$ is a lattice: the maximal lattice image of S, and*
- *each $\mathcal{D}$-class is a rectangular band.*

So: a skew lattice is a lattice of rectangular bands.

Fact: $x \preceq y$ in S iff $\mathcal{D}_x \leq \mathcal{D}_y$ in $S/\mathcal{D}$.

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A SL S is:

- *left handed* if it satisfies $x \wedge y \wedge x = x \wedge y$ and $x \vee y \vee x = y \vee x$.
- *right handed* if it satisfies $x \wedge y \wedge x = y \wedge x$ and $x \vee y \vee x = x \vee y$.

Most natural examples of SLs are either left or right handed.

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Skew distributive lattices

A *skew distributive lattice* is a skew lattice S which satisfies the identities

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S a skew distributive lattice $\Rightarrow S/\mathcal{D}$ is a distributive lattice.

Given any $x \in S$: $x \wedge S \wedge x = \{x \wedge y \wedge x \,|\, y \in S\}$ is a distributive lattice.

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Skew Boolean algebras

A *skew Boolean algebra* is an algebra $(S; \wedge, \vee, \setminus, 0)$ of type $(2, 2, 2, 0)$ where

- $(S; \wedge, \vee, 0)$ is a skew distributive lattice with 0,
- $x \wedge S \wedge x = \{x \wedge y \wedge x \mid y \in S\}$ is a Boolean lattice for all x, and
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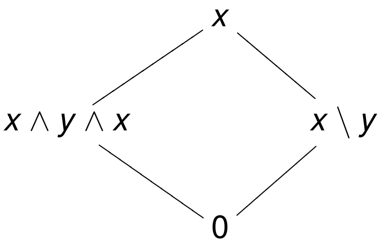

Example of a skew Boolean algebra: $X \rightharpoonup Y$

- For partial maps $f, g : X \rightharpoonup Y$ define:

$$
\begin{aligned}
0 &= \emptyset \\
f \wedge g &= f\restriction_{\mathrm{dom} f \cap \mathrm{dom} g} \\
f \vee g &= f\restriction_{\mathrm{dom} g \setminus \mathrm{dom} f} \cup g \\
f \setminus g &= f\restriction_{\mathrm{dom} f \setminus \mathrm{dom} g}
\end{aligned}
$$

  With these operations $X \rightharpoonup Y$ is a skew Boolean algebra.
- The maximal lattice image: $(X \rightharpoonup Y)/\mathcal{D} = \mathcal{P}(X)$.
- $X \rightharpoonup Y$ is left-handed: $f \wedge g \wedge f = f \wedge g$.

Duality for skew Boolean algebras

A. Bauer, KCV, G. Kudryavtseva:

- A *Boolean space* is a locally compact zero-dimensional Hausdorff space.
- A *Boolean sheaf* is a local homeomorphism $p : E \to B$ where B is a Boolean space.
- We only consider sheaves for which $p : E \to B$ is *surjective.*
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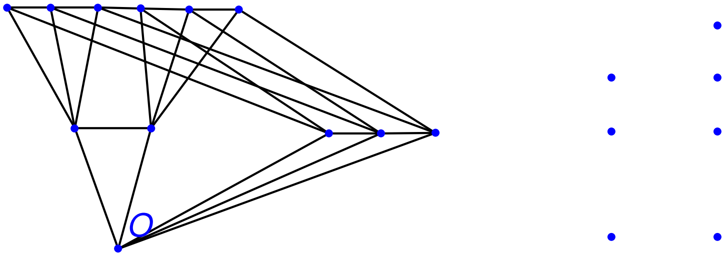
Duality for finite skew Boolean algebras

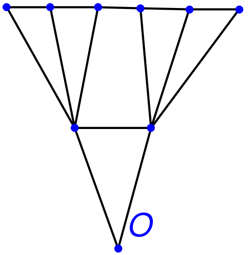
S a finite left-handed SBA, $p : E \to B$ its dual Boolean sheaf.

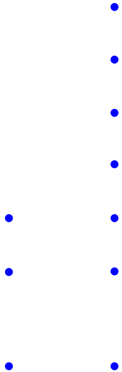
- The points of B are the atoms of $S/\mathcal{D}$.
- Given an atom $\mathcal{D}$-class A in S: the stalk above A consists of the elements of A.
- The elements of S correspond to sections of p (above clopens).



The skew distributive case

Can we do the same with skew distributive lattices?
(Taking join irreducibles instead of atoms.)





We get too many sections over clopen downsets!

The cosets

S SDL with 0; A, B $\mathcal{D}$-classes, $A > B$ in $S/\mathcal{D}$, $a \in A$.

The coset of B in A containing a:

$$B \vee a \vee B = \{b \vee a \vee b' \mid b, b' \in B\}.$$

- The cosets of B in A form a partition of A.
- Given a coset A_i of B in A and $a \in A_i$ there exists a unigue $b \in B$ s. t. $b < a$.
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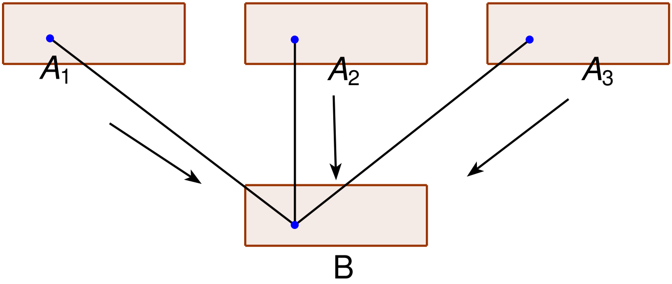
The coset decomposition

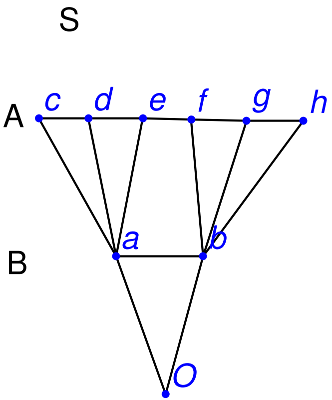
S left-handed DSL, $A > B$ $\mathcal{D}$-classes
Operations on $A \cup B$: $a \in A_i$, $b \in B$, $\varphi_i : A_i \to B$ the coset bijection.
Then:

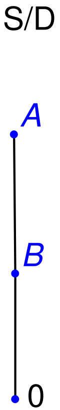
$$b \wedge a = b,\ a \wedge b = \varphi_i(a),\ b \vee a = a,\ \text{and}\ a \vee b = \varphi_i^{-1}(b).$$



Example

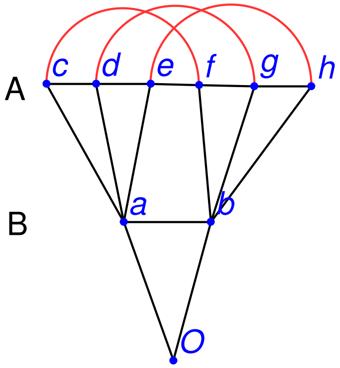
Consider the left-handed skew distributive lattice S:





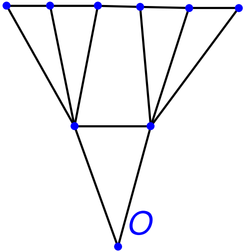
Example

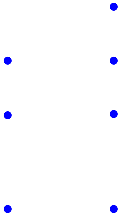
Assume the following coset decomposition:



Take the cosets as the elements of the stalk.

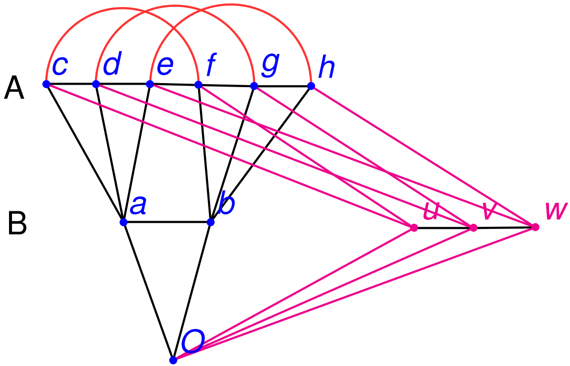
We obtain:





Completing to a SBA

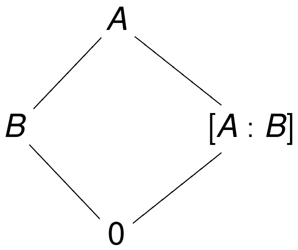
Forgetting the order on the base space, the sections yield a SBA, a "Booleanization" of S:



u, v and w correspond exactly to cosets of B in A.

Denote by $[A : B]$ the set of all cosets of B in A.

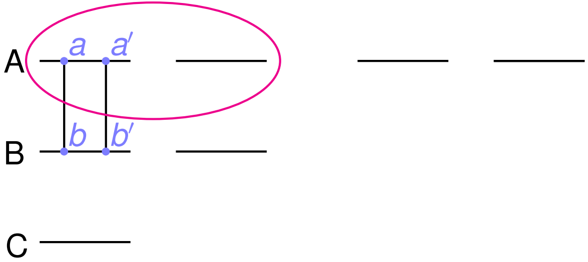
The "Booleanization":



The coset law for a chain

Let $A > B > C$ be a chain of $\mathcal{D}$-classes in a SDL S.
$a, a' \in A$.

Pita Costa, 2011: $C \vee a \vee C = C \vee a' \ C$ iff
$B \vee a \vee B = B \vee a' \vee B$ and $C \vee b \vee C = C \vee b' \vee C$ for
$a > b$, $a' > b'$.

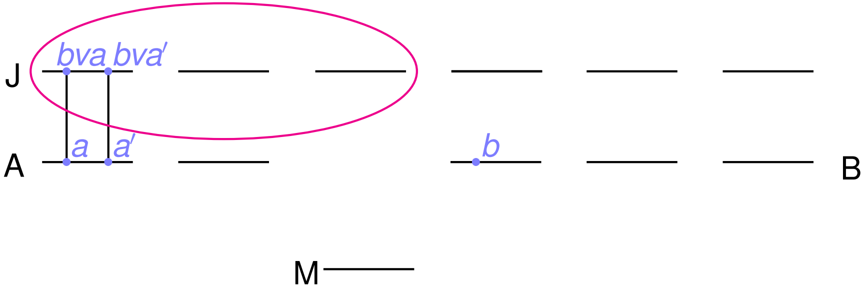


So: a coset of C in A corresponds to a pair of a coset of C in B and a coset of B in A.

The coset law for a diamond

A, B $\mathcal{D}$-classes in a MD LH SL,
$M = A \wedge B$, $J = A \vee B$ in $S/\mathcal{D}$, $a, a' \in A$.

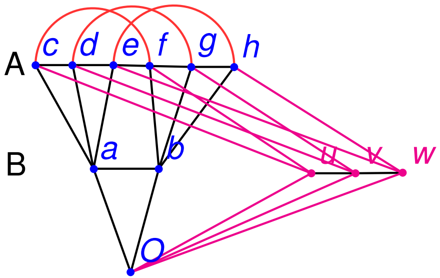
KCV and Pita Costa, 2011: $M \vee a \vee M = M \wedge a' \vee M$ iff
$B \vee (b \vee a) \vee B = B \vee (b \vee a') \vee B$ (for any $b \in B$).

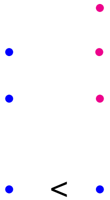


The cosets in the dual space

To obtain the dual of a finite SDL S:

1. Take the join irreducible $\mathcal{D}$-classes.
2. Take the (n-tuples of) cosets along chains in S.



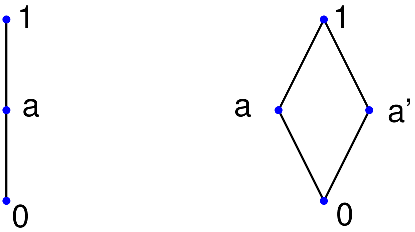


The Mac Neille Theorem

L a DL, B the Booleanization of L.

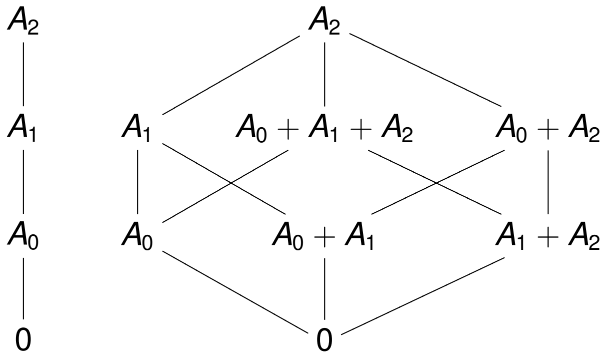
Theorem (Mac Neille, 1945)

Every element $x \in B$ can be expressed as $x = a_1 + \cdots + a_n$ for some $a_1 \leq \cdots \leq a_n$, $a_i \in L$.

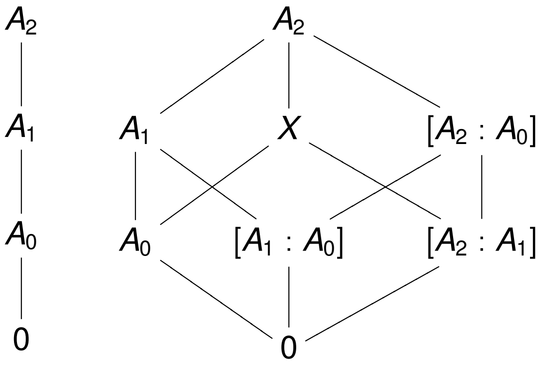


$a' = a + 1$.

Completing a DL to a BA



Completing a SDL to a SBA



Here:
$X = [A_2 : [A_1 : A_0]] = ([A_2 : A_1], [A_1 : [A_1 : A_0]]) =$
$([\mathrm{A}_2 : A_1], [A_0 : 0])$.