Generalised Gelfand Spectra For Noncommutative Operator Algebras and Multi-Valued Logic for Quantum Systems

Workshop 2 on Duality Theory
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“Those who do not understand the nature of sin and virtue are attached to duality; they wander around deluded.”

Sri Guru Granth Sahib

Introduction

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- To each commutative algebra $\mathcal{A}$, the set of algebra homomorphisms $\lambda : \mathcal{A} \to \mathbb{C}$ is assigned. Conversely, to each compact Hausdorff space X, the algebra of continuous functions $f : X \to \mathbb{C}$ is assigned.

Introduction (2)

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Big aim: generalise Gelfand-Naimark duality to noncommutative operator algebras, provide spatial counterparts to algebraic constructions.

Introduction (3)

In the first half of the talk, I will sketch how some ideas from

- noncommutative operator algebras,
- topos theory,
- geometric model theory,
- and quantum physics

may help to get closer to a solution. A number of open questions remain.

The topos approach and the spectral presheaf

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More generally, one can think of **UnitC***, the category of unital C^*-algebras and unital $*$-homomorphisms.

The topos approach to quantum theory

In the topos approach to quantum theory (Isham, Butterfield ('97–'02); Isham, D ('06-'12)), we associate with each NC operator algebra (C^*- or von Neumann algebra; here mostly the latter) a topos and a distinguished spectral object in the topos.

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- The base category of the topos is the poset $\mathcal{V}(\mathcal{N})$ of nontrivial commutative von Neumann subalgebras of $\mathcal{N}$ (which share the unit element with $\mathcal{N}$). The topos itself is $\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}$, presheaves over $\mathcal{V}(\mathcal{N})$.

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Physically, the abelian subalgebras $V \in \mathcal{V}(\mathcal{N})$ are interpreted as 'classical perspectives' on the quantum systems, also called **contexts**. Hence, $\mathcal{V}(\mathcal{N})$ is the context category.

The spectral presheaf

Definition

Let $\mathcal{N}$ be a von Neumann algebra, and let $\mathcal{V}(\mathcal{N})$ be its context category. The **spectral presheaf** $\underline{\Sigma}$ of $\mathcal{N}$ over $\mathcal{V}(\mathcal{N})$ is given

(a) on objects: for all $V \in \mathcal{V}(\mathcal{N})$, let $\underline{\Sigma}_V := \Sigma_V$, the Gelfand spectrum of the abelian von Neumann algebra V,

(b) on arrows: for all inclusions $i_{V'V} : V' \hookrightarrow V$, let

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Question: Is $\underline{\Sigma}$ anything like the spectrum of $\mathcal{N}$?

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Here, we consider $\mathcal{N}$ as a Jordan algebra, replacing the noncommutative product with the commutative, but nonassociative symmetrised product

$$\forall \hat{A}, \hat{B} \in \mathcal{N} : \hat{A} \circ \hat{B} := \frac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}).$$

Jordan automorphisms

Let $\phi : \mathcal{N} \to \mathcal{N}$ be an ultraweakly continuous Jordan automorphism. This induces

$$\begin{aligned} \tilde{\phi} : \mathcal{V}(\mathcal{N}) &\longrightarrow \mathcal{V}(\mathcal{N}) \\ V &\longmapsto \phi(V), \end{aligned}$$

which gives a geometric automorphism $\Phi : \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}$. One can use the inverse image part to pull back $\underline{\Sigma}$,

$$\forall V \in \mathcal{V}(\mathcal{N}) : (\Phi^*(\underline{\Sigma}))_V = \underline{\Sigma}_{\tilde{\phi}(V)}.$$

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The $\mathcal{G}_V$ are the components of a natural transformation $\mathcal{G} : \Phi^*(\underline{\Sigma}) \rightarrow \underline{\Sigma}$, so we get an invertible map (automorphism)

$$\mathcal{G} \circ \Phi^* : \underline{\Sigma} \longrightarrow \underline{\Sigma}.$$

Algebra automorphisms

Every algebra automorphism $\phi : \mathcal{N} \to \mathcal{N}$ of the von Neumann algebra is also a Jordan automorphism. Hence, there is a map

$$\begin{aligned} \mathrm{Aut}(\mathcal{N}) &\longrightarrow \mathrm{Aut}(\underline{\Sigma}) \\ \phi &\longmapsto \mathcal{G} \circ \Phi^*, \end{aligned}$$

but this map is not surjective (since there are more Jordan automorphisms than algebra morphisms).

In this view $\mathrm{Aut}(\underline{\Sigma})$ is too large, so $\underline{\Sigma}$ has too little structure.

Aim: identify 'good' automorphisms of $\underline{\Sigma}$ that come from algebra automorphisms. Equivalently, equip $\underline{\Sigma}$ with more structure.

Ongoing work with John Harding.

Morphisms between different algebras

Up to now, we considered only (Jordan) automorphisms of von Neumann algebras. More generally, an ultraweakly continuous unital Jordan morphism

$$\phi : \mathcal{N}_1 \longrightarrow \mathcal{N}_2$$

preserves commutativity and hence induces a geometric morphism $\Phi : \mathbf{Set}^{\mathcal{V}(\mathcal{N}_1)^{\mathrm{op}}} \to \mathbf{Set}^{\mathcal{V}(\mathcal{N}_2)^{\mathrm{op}}}$ and a morphism

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Every Jordan morphism $\phi : \mathcal{N}_1 \to \mathcal{N}_2$ acts as a $*$-homomorphism on one direct summand of $\mathcal{N}_1$, and as a $*$-antihomomorphism on the other direct summand. Either direct summand may be empty.

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The open task is to identify which of the morphisms $\mathcal{G} \circ \Phi : \underline{\Sigma}_{\mathcal{N}_2} \to \underline{\Sigma}_{\mathcal{N}_1}$ come from (ultraweakly continuous) $*$-homomorphisms $\phi : \mathcal{N}_1 \to \mathcal{N}_2$, i.e., the arrows in **vNA**.

Zariski geometries

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But Lie algebra aspects are not built in yet.

Zariski geometries (2)

There are further connections: one can associate a presheaf (over commutative subalgebras) of Zariski structures with a suitable noncommutative algebras. For more on Zariski geometries, see Boris Zilber's webpage,

http://people.maths.ox.ac.uk/zilber

A new logic for quantum systems

Standard quantum logic

Birkhoff and von Neumann ('36) suggested to use the (projections onto) closed subspaces of a Hilbert space $\mathcal{H}$ as representatives of propositions about a quantum system. This is motivated by the spectral theorem that gives the link between propositions "$A \varepsilon \Delta$" and projection operators $\hat{P} \in \mathcal{B}(\mathcal{H})$.

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This can be generalised to the projections in a von Neumann algebra $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$. The projections form a complete orthomodular lattice $\mathcal{P}(\mathcal{N})$ that is *non-distributive* iff $\mathcal{N}$ is noncommutative.

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Non-distributivity leads to many well-known interpretational problems in standard quantum logic. Other issues are the lack of a deductive system, lack of a material implication, existence of physically meaningless conjunctions, etc.

Many generalisations beyond orthomodular lattices exist. For a review, including conceptual issues, see Dalla Chiara, Giuntini ('01).

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In this talk, I will relate the logic of quantum systems to *bi-Heyting algebras*.

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These are *distributive* lattices with *two* different kinds of implication and hence two kinds of negation.

Hence, we depart from Boolean logic in a different manner than standard quantum logic. This new form of logic for quantum systems is a part of the topos approach to the formulation of quantum theory.

For details on the bi-Heyting aspects, see D, "Topos-Based Logic for Quantum Systems and Bi-Heyting Algebras", `arXiv:1202.2750`.

Bi-Heyting algebras

History

- Rauszer: bi-Heyting algebras in superintuitionistic logic ('73–'77)
- Lawvere: co-Heyting and bi-Heyting algebras in category and topos theory, in particular in connection with continuum physics ('86, '91)
- Reyes/Makkai ('95) and Reyes/Zolfaghari ('96): bi-Heyting algebras and modal logic
- Bezhanishvili et al. ('10): new duality theorems for bi-Heyting algebras based on bitopological spaces
- Majid ('95, '08): Heyting and co-Heyting algebras within a tentative representation-theoretic approach to the formulation of quantum gravity

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As far as I am are aware, nobody has connected quantum systems and their logic with bi-Heyting algebras before.

Bi-Heyting algebras

A **bi-Heyting algebra** K is a lattice which is a Heyting algebra and a co-Heyting algebra. For each $A \in K$, the functor $A \wedge \_ : K \to K$ has a right adjoint $A \Rightarrow \_ : K \to K$, and the functor $A \vee \_ : K \to K$ has a left adjoint $A \Leftarrow \_ : K \to K$. We write $\neg$ for the Heyting negation and $\sim$ for the co-Heyting negation.

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A bi-Heyting algebra K is called **complete** if it is complete as a Heyting algebra and complete as a co-Heyting algebra.

Example

A canonical example of a bi-Heyting algebra is a Boolean algebra $\mathcal{B}$.

Note that by Stone's representation theorem, each Boolean algebra is isomorphic to the algebra of clopen, i.e., closed and open, subsets of its Stone space. This gives the connection with the canonical topological examples of Heyting and co-Heyting algebras (open resp. closed sets in a topological space).

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In a Boolean algebra, we have for the Heyting negation that, for all $A \in \mathcal{B}$,

$$A \vee \neg A = 1,$$

which is the characterising property of the co-Heyting negation. In fact, in a Boolean algebra, $\neg = \sim$.

Clopen subobjects of the spectral presheaf

Projections and clopen subsets

If V is a *commutative* von Neumann algebra, then its projection lattice $\mathcal{P}(V)$ is a complete Boolean algebra. Let $\mathcal{C}l(\Sigma_V)$ denote the clopen subsets of the Gelfand spectrum of V. There is an isomorphism of complete Boolean algebras

$$\begin{aligned} \alpha_V : \mathcal{P}(V) &\longrightarrow \mathcal{C}l(\Sigma_V) \\ \hat{P} &\longmapsto \{\lambda \in \Sigma_V \mid \lambda(\hat{P}) = 1\}. \end{aligned}$$

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Hence, within each context, i.e., commutative subalgebra $V \subset \mathcal{N}$, we can freely switch between clopen subsets of Σ_V and projections in V.

We will write $S_{\hat{P}} := \alpha_V(\hat{P})$ and $\hat{P}_S := \alpha^{-1}(S)$.

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Note also that the Gelfand spectrum $\underline{\Sigma}_V$ of an abelian von Neumann algebra V and the Stone spectrum of $\mathcal{P}(V)$ are homeomorphic.

Clopen subobjects of the spectral presheaf

A subobject of the spectral presheaf $\underline{\Sigma}$ is simply a subfunctor $\underline{S}$. It is determined by specifying a sub*set* $\underline{S}_V$ of $\underline{\Sigma}_V$ for each context V such that

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A subobject $\underline{S}$ of $\underline{\Sigma}$ is called **clopen** if $\underline{S}_V \subseteq \underline{\Sigma}_V$ is a clopen subset for all $V \in \mathcal{V}(\mathcal{N})$.

Equivalently, we can consider the family $(\hat{P}_{\underline{S}_V})_{V \in \mathcal{V}(\mathcal{N})}$ of corresponding projections. The subobject condition becomes

$$\forall V, V' \in \mathcal{V}(\mathcal{N}) : V' \subset V \quad \text{implies} \quad \hat{P}_{\underline{S}_V} \leq \hat{P}_{\underline{S}_{V'}}.$$

The set of clopen subobjects of $\underline{\Sigma}$ is denoted as $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$.

Contextuality and coarse-graining

The concept of **contextuality** is implemented by this construction: $\underline{\Sigma}$ is a presheaf over the context category $\mathcal{V}(\mathcal{N})$.

Each context is a 'classical perspective' on the quantum system. Within each context, we have classical Boolean logic; propositions are represented by elements of the complete Boolean algebra $\mathcal{P}(V)$.

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Moreover, the concept of **coarse-graining** is implemented by the fact that we use subobjects of $\underline{\Sigma}$: for each context $V \in \mathcal{V}(\mathcal{N})$, the component $\underline{S}_V \subseteq \underline{\Sigma}_V$ represents a *local proposition* about the value of some physical quantity in V.

If $V' \subset V$, then $\hat{P}_{\underline{S}_{V'}} \geq \hat{P}_{\underline{S}_V}$ (since $\underline{S}$ is a subobject), so $\underline{S}_{V'}$ represents a local proposition at the smaller context $V' \subset V$ that is *coarser* than (i.e., it is weaker than, a consequence of) the local proposition represented by $\underline{S}_V$.

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Clopen subobjects hence are interpreted as *contextualised families of local propositions, compatible w.r.t. coarse-graining*.

$\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ as a lattice

It is obvious that $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ is a partially ordered set if we set

$$\forall \underline{S}, \underline{T} \in \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) : \underline{S} \leq \underline{T} \quad \text{iff} \quad (\forall V \in \mathcal{V}(\mathcal{N}) : \underline{S}_V \subseteq \underline{T}_V).$$

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Meets and joins with respect to this order are defined as follows: for all families $(\underline{S}_i)_{i \in I} \subseteq \mathrm{Sub_{cl}}(\underline{\Sigma})$,

$$\forall V \in \mathcal{V}(\mathcal{N}) : (\bigwedge_{i \in I} \underline{S}_i)_V := \mathrm{cl} \bigcup_{i \in I} \underline{S}_{i;V},$$

$$\forall V \in \mathcal{V}(\mathcal{N}) : (\bigvee_{i \in I} \underline{S}_i)_V := \mathrm{int} \bigcap_{i \in I} \underline{S}_{i;V}.$$

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Since the lattice operations are defined locally, i.e., at each stage $V \in \mathcal{V}(\mathcal{N})$ separately, we obtain a distributive lattice by the fact that for all $V \in \mathcal{V}(\mathcal{N})$,

$$\mathcal{C}l(\underline{\Sigma}_V) \simeq \mathcal{P}(V)$$

is distributive.

$\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ as a complete Heyting algebra

In fact, we can say more: each $\mathcal{C}l(\underline{\Sigma}_V)$ is a complete Boolean algebra, so for each $\underline{S} \in \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ the functor

$$\begin{aligned} \_ \wedge \underline{S} : \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) &\longrightarrow \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \\ \underline{R} &\longmapsto \underline{R} \wedge \underline{S} \end{aligned}$$

preserves all joins (note that meets and joins are defined stagewise) and hence has a right adjoint

$$\underline{S} \Rightarrow \_ : \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \longrightarrow \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}).$$

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This map, the **Heyting implication from $\underline{S}$**, makes $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ into a complete Heyting algebra. The Heyting implication is given by the adjunction

$$\underline{R} \wedge \underline{S} \leq \underline{T} \quad \text{if and only if} \quad \underline{R} \leq (\underline{S} \Rightarrow \underline{T}).$$

$\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ as a complete co-Heyting algebra

There is more structure: again since each $\mathcal{C}l(\underline{\Sigma}_V)$ is a complete Boolean algebra, for each $\underline{S} \in \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ the functor

$$\underline{S} \vee \_ : \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \longrightarrow \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$$

preserves all meets, so it has a left adjoint

$$\underline{S} \Leftarrow \_ : \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \longrightarrow \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$$

which we call **co-Heyting implication from $\underline{S}$**. This map makes $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ into a complete co-Heyting algebra. It is characterised by the adjunction

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$$(\underline{S} \Leftarrow \underline{T}) \leq \underline{R} \quad \text{iff} \quad \underline{S} \leq \underline{T} \vee \underline{R},$$

which means that

$$(\underline{S} \Leftarrow \underline{T}) = \bigwedge \{\underline{R} \in \mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}) \mid \underline{S} \leq \underline{T} \vee \underline{R}\}.$$

$\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$ as a complete bi-Heyting algebra

We have shown

Proposition

$(\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma}), \wedge, \vee, \underline{0}, \underline{\Sigma}, \Rightarrow, \neg, \Leftarrow, \sim)$ *is a complete bi-Heyting algebra.*

Daseinisation

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There is a way of 'translating' from standard quantum logic to the new logic based on clopen subobjects of the quantum state space $\underline{\Sigma}$: first consider a single commutative subalgebra $V \subset \mathcal{N}$. There is an embedding

$$\mathcal{P}(V) \hookrightarrow \mathcal{P}(\mathcal{N})$$

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that is a morphism of complete orthomodular lattices, so it preserves all meets in particular.

Hence, it has a left adjoint

$$\begin{aligned} \delta^o_{\mathcal{N},V} : \mathcal{P}(\mathcal{N}) &\longrightarrow \mathcal{P}(V) \\ \hat{P} &\longmapsto \delta^o_{\mathcal{N},V}(\hat{P}) = \bigwedge\{\hat{Q} \in \mathcal{P}(V) \mid \hat{Q} \geq \hat{P}\}. \end{aligned}$$

Note that commutativity of V plays no role here.

Daseinisation (2)

Then consider this map for all contexts $V \in \mathcal{V}(\mathcal{N})$ at once:

Definition

Let $\mathcal{N}$ be a von Neumann algebra, and let $\mathcal{P}(\mathcal{N})$ be its lattice of projections. The map

$$\begin{aligned} \underline{\delta^o} : \mathcal{P}(\mathcal{N}) &\longrightarrow \mathrm{Sub_{cl}}(\underline{\Sigma}) \\ \hat{P} &\longmapsto \underline{\delta^o}(\hat{P}) := (\alpha_V(\delta^o_{\mathcal{N},V}(\hat{P})))_{V \in \mathcal{V}(\mathcal{N})} \end{aligned}$$

is called **outer daseinisation of projections**.

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is called **outer daseinisation of projections**.

This map is injective (but not surjective), monotone and preserves all joins. For meets, we have

$$\forall \hat{P}, \hat{Q} \in \mathcal{P}(\mathcal{N}) : \underline{\delta^o}(\hat{P} \wedge \hat{Q}) \leq \underline{\delta^o}(\hat{P}) \wedge \underline{\delta^o}(\hat{Q}).$$

Daseinisation (3)

More abstractly, outer daseinisation is a map from the category of complete join-semilattices to the category of complete bi-Heyting algebras,

$$\underline{\delta^o} : \mathbf{CJSLat} \longrightarrow \mathbf{CBiHeyt}.$$

The image is bigger than the Bruns-Lakser construction in general.

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The image is bigger than the Bruns-Lakser construction in general.

Outer daseinisation is based on the left adjoint of the inclusion $i : \mathcal{P}(V) \hookrightarrow \mathcal{P}(\mathcal{N})$. Using the right adjoint of i, we can define **inner daseinisation** analogously, so we get a second map

$$\underline{\delta^i} : \mathbf{CMSLat} \longrightarrow \mathbf{CBiHeyt}$$

that preserves all meets.

Question: Are these maps known? Universal properties?

The daseinisation topology

Instead of considering *all* clopen subobjects of $\underline{\Sigma}$, one can take suitable subframes, subcoframes or sub-bi-Heyting algebras.

E.g. consider the frame D generated by the $\underline{\delta^o}(\hat{P})$, $\hat{P} \in \mathcal{P}(\mathcal{N})$, called the *daseinisation topology on* $\underline{\Sigma}$. Dan Marsden showed that this frame is strictly smaller than $\mathrm{Sub}_{\mathrm{cl}}(\underline{\Sigma})$.

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The choice of a (co)frame of subobjects of $\underline{\Sigma}$ determines

- which propositions we have available in our new form of quantum logic,
- whether propositions are closed under infinite conjunctions and disjunctions,
- which topology we give to the spectral presheaf,
- ...which in turn plays a key role in the question of continuity of maps between spectral presheaves.

Ongoing work with Pedro Resende, Jonathon Funk and Rui Soares Barbosa.

States and truth values

Pure states

In classical physics, a **(pure) state** is given by an element s of the state space $\mathcal{S}$. A **proposition** is represented by a Borel subset $B \subset \mathcal{S}$.

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$$v(B; s) = (s \in B),$$

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For quantum theory, we need an analogue of the pure state $s \in \mathcal{S}$. But:

Theorem

(Isham, Butterfield '00): *The spectral presheaf* $\underline{\Sigma}$ *of a von Neumann algebra* $\mathcal{N}$ *has no global elements if* $\mathcal{N}$ *has no summand of type* I_2*. This is equivalent to the Kochen-Specker theorem.*

Pseudo-states

In standard quantum theory (for $\mathcal{N} = \mathcal{B}(\mathcal{H})$), one uses **vector states**: let $\psi \in \mathcal{H}$ be a unit vector, then

$$\begin{aligned} w^{\psi} : \mathcal{B}(\mathcal{H}) &\longrightarrow \mathbb{C} \\ \hat{A} &\longmapsto \langle \psi, \hat{A}(\psi) \rangle . \end{aligned}$$

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We simply 'daseinise' such a vector state: let $\hat{P}_{\psi}$ the projection onto the line $\mathbb{C}\psi$, then the **pseudo-state** $\underline{\mathfrak{w}}^{\psi}$ is given as

$$\underline{\mathfrak{w}}^{\psi} := \underline{\delta^{o}}(\hat{P}_{\psi}).$$

Truth values

Let $\underline{\mathfrak{w}}^{\psi}$ be a pseudo-state, and let $\underline{S} \in \mathrm{Sub_{cl}}(\underline{\Sigma})$ be a proposition (e.g., $\underline{S} = \underline{\delta^{o}}(\hat{E}[A \,\varepsilon\, \Delta])$). We can interpret the expression

$$v(\underline{S}; \underline{\mathfrak{w}}^{\psi}) := (\underline{\mathfrak{w}}^{\psi} \in \underline{S})$$

in the Mitchell-Benabou language of the topos $\mathbf{Set}^{\mathcal{V}(\mathcal{N})^{\mathrm{op}}}$, which gives a truth value in the multi-valued, intuitionistic logic of the topos.

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Since the base category $\mathcal{V}(\mathcal{N})$ of our topos is a poset, this becomes particularly simple: the global elements of $\underline{\Omega}$ correspond bijectively to the lower sets in $\mathcal{V}(\mathcal{N})$,

$$\Gamma(\underline{\Omega}) = \mathcal{L}(\mathcal{V}(\mathcal{N})).$$

Thanks for listening!