Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Sergio A. Celani, Ramón Jansana and María Esteban

Departamento de Lógica, Historia y Filosofía de la Ciencia. Universidad de Barcelona

Duality workshop, 15-17 August, Oxford

Overview

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

Priestley-style spaces

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

$\langle X, \tau, \leq, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is **compact**
- $\langle X, \tau, \leq \rangle$ is **totally order-disconnected**

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

$\langle X, \tau, \leq, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is **compact**
- $\langle X, \tau, \leq \rangle$ is **totally order-disconnected**

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

$\langle X, \tau, \leq, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is **compact**
- $\langle X, \tau, \leq \rangle$ is **totally order-disconnected**

Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces

$\langle X, \tau, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is a **sober**
- $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

$\langle X, \tau, \leq, \ldots \rangle$ such that:

- $\langle X, \tau \rangle$ is **compact**
- $\langle X, \tau, \leq \rangle$ is **totally order-disconnected**

- We present Spectral-like and Priestley-style dualities for **Distributive Hilbert Algebras with Infimum**

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on A

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\rightarrow\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \rightarrow, 1\rangle$ such that

(1) $a \rightarrow (b \rightarrow c) = 1$,

(2) $(a \rightarrow (b \rightarrow b)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1$,

(3) $a \rightarrow b = 1 = b \rightarrow a$ implies $a = b$.

Natural ordering on A

$a \leq b$ iff $a \rightarrow b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that

(1) $a \to (b \to c) = 1$,

(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,

(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms

$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (I)

- $\{\to\}$-fragment of Intuitionistic logic

$\mathbf{A} = \langle A; \to, 1\rangle$ such that
(1) $a \to (b \to c) = 1$,
(2) $(a \to (b \to b)) \to ((a \to b) \to (a \to c)) = 1$,
(3) $a \to b = 1 = b \to a$ implies $a = b$.

Natural ordering on $\mathbf{A}$
$a \leq b$ iff $a \to b = 1$

Homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that
(1) $h(1_1) = 1_2$,
(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

Semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that
(1) $h(1_1) = 1_2$,
(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

$\Rightarrow$ H_S - category of Hilbert algebras and semi-homomorphisms
$\Rightarrow$ H_H - category of Hilbert algebras and homomorphisms

Hilbert algebras (II)

- Spectral-like dualities for H_S and H_H in *Celani, Cabrer, and Montangie [2009]*

Hilbert algebras (II)

- Spectral-like dualities for H_S and H_H in *Celani, Cabrer, and Montangie [2009]*

***H*-space** $\langle X, \tau_\kappa \rangle$:

- $\kappa \subseteq \mathcal{KO}(X)$ basis,
- $\forall U, V \in \kappa$, $\mathrm{sat}(U \cap V^c) \in \kappa$,
- $\langle X, \tau_\kappa \rangle$ is sober.

***H*-relation** $R \subseteq X_1 \times X_2$: (duals of semi-hom.)

- $R^{-1}(U) \in \kappa_1$ for every $U \in \kappa_2$
- $R(x)$ closed of X_2, for all $x \in X_2$

***H*-functional relation** when moreover: (duals of hom.)

- $(x, y) \in R$ implies $\exists z \in X_1 (x \leq z$ and $R(z) = \mathrm{cl}(y))$

Hilbert algebras (II)

- Spectral-like dualities for H_S and H_H in *Celani, Cabrer, and Montangie [2009]*

Hilbert algebras (II)

- Spectral-like dualities for H_S and H_H in *Celani, Cabrer, and Montangie [2009]*
- Priestley-style dualities for H_S and H_H in *Celani and Jansana [2012b, (to appear)]*

Hilbert algebras (II)

- Spectral-like dualities for H_S and H_H in *Celani, Cabrer, and Montangie [2009]*
- Priestley-style dualities for H_S and H_H in *Celani and Jansana [2012b, (to appear)]*

***-augmented Priestley space** $\langle X, \tau, \leq, S\rangle$:

- $\langle X, \tau, \leq\rangle$ Priestley space
- S family of clopen upsets, that satisfies some conditions [...]
- $\forall U, V \in S$, $(\downarrow(U \cap V^c))^c \in S$.

***-augmented Priestley semi-morphism** $R \subseteq X_1 \times X_2$:

- $(x_1, x_2) \notin R$ implies $\exists U \in S_2 \big(x_2 \notin U$ and $R(x_1) \subseteq U\big)$
- $\{x_1 \in X_1 : R(x_1) \subseteq U\} \in S_2$ for all $U \in S_1$

***-augmented Priestley morphism** when moreover:

- $\forall x_1 \in X_1, x_2 \in X_{S_2}$,
  $(x_1, x_2) \in R$ implies $\exists x_1' \in X_{S_1} \big(x_1 \leq x_1'$ and $R(x_1') = {\uparrow} x_2\big)$

Hilbert algebras + supremum

Hilbert algebras with Supremum

$\mathbf{A} = \langle A; \to, \vee, 1 \rangle$ such that

(1) $\langle A; \to, 1 \rangle$ Hilbert algebra

(2) $\langle A; \vee, 1 \rangle$ join semilattice with top element

(3) $a \vee b = b$ iff $a \to b = 1$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a \vee b = b$

- $\{\to, \vee\}$-fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

Hilbert algebras + supremum

Hilbert algebras with Supremum

$\mathbf{A} = \langle A; \to, \vee, 1 \rangle$ such that

(1) $\langle A; \to, 1 \rangle$ Hilbert algebra

(2) $\langle A; \vee, 1 \rangle$ join semilattice with top element

(3) $a \vee b = b$ iff $a \to b = 1$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a \vee b = b$

- $\{\to, \vee\}$-fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

Hilbert algebras + supremum

Hilbert algebras with Supremum

$\mathbf{A} = \langle A; \to, \vee, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \vee, 1\rangle$ join semilattice with top element

(3) $a \vee b = b$ iff $a \to b = 1$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a \vee b = b$

- $\{\to, \vee\}$-fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

Hilbert algebras + supremum

Hilbert algebras with Supremum

$\mathbf{A} = \langle A; \to, \vee, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \vee, 1\rangle$ join semilattice with top element

(3) $a \vee b = b$ iff $a \to b = 1$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff

$a \vee b = b$

- $\{\to, \vee\}$-fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

Hilbert algebras + supremum

Hilbert algebras with Supremum

$\mathbf{A} = \langle A; \to, \vee, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \vee, 1\rangle$ join semilattice with top element

(3) $a \vee b = b$ iff $a \to b = 1$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff

$a \vee b = b$

- $\{\to, \vee\}$-fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

▶ Spectral-like duality in *Celani and Montangie [2011, (to appear)]*

Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1\rangle$ such that

(1) $\langle A; \rightarrow, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \rightarrow b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff $a = a \wedge b$

- $\{\rightarrow, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1 \rangle$ such that

(1) $\langle A; \to, 1 \rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1 \rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$
$a \leq b$ iff $a \to b = 1$ iff
$a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1 \rangle$ such that

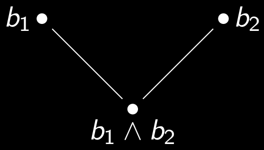
(1) $\langle A; \to, 1 \rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1 \rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$
$a \leq b$ iff $a \to b = 1$ iff
$a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**



Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

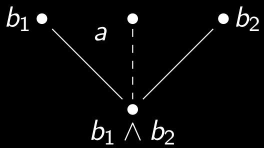
(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff

$a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**



Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

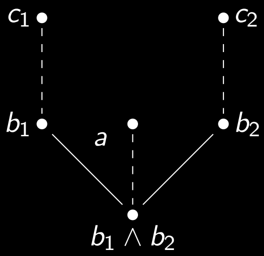
(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff

$a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**



Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1\rangle$ such that

(1) $\langle A; \rightarrow, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

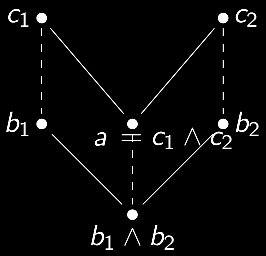
(3) $a \wedge b \leq c$ iff $c \leq a \rightarrow b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff

$a = a \wedge b$

- $\{\rightarrow, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**



Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

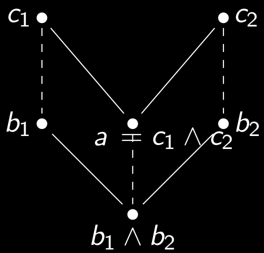
(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

▶ Spectral-like duality in *Celani [2003]*



Hilbert algebras + infimum (I)

Implicative semilattices

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \wedge b \leq c$ iff $c \leq a \to b$ (Residuation law)

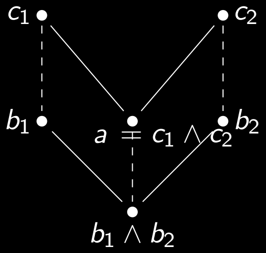
Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \to b = 1$ iff $a = a \wedge b$

- $\{\to, \wedge\}$-fragment of Intuitionistic logic
- The underlying meet semilattice is **distributive**

▶ Spectral-like duality in *Celani [2003]*

▶ Priestley-style duality in *G. Bezhanishvili and Jansana [2011b]*



Hilbert algebras + infimum (II)

Hilbert algebras with Infimum ($H^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ such that

(1) $\langle A; \to, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \to b = 1$ iff $a = a \wedge b$

Natural ordering on A

$a \leq b$ iff $a \to b = 1$ iff $a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad [2006]*

$\wedge$-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) = h(a) \to_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

$\wedge$-semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \to_1 b) \leq h(a) \to_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Hilbert algebras + infimum (II)

Hilbert algebras with Infimum ($H^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1\rangle$ such that

(1) $\langle A; \rightarrow, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \rightarrow b = 1$ iff $a = a \wedge b$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff $a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad [2006]*

$\wedge$-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

$\wedge$-semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Hilbert algebras + infimum (II)

Hilbert algebras with Infimum ($H^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1\rangle$ such that

(1) $\langle A; \rightarrow, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \rightarrow b = 1$ iff $a = a \wedge b$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff $a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad [2006]*

$\wedge$-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

$\wedge$-semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Hilbert algebras + infimum (II)

Hilbert algebras with Infimum ($H^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1\rangle$ such that

(1) $\langle A; \rightarrow, 1\rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1\rangle$ meet semilattice with top element

(3) $a \rightarrow b = 1$ iff $a = a \wedge b$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff

$a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad [2006]*

$\wedge$-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

$\wedge$-semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Hilbert algebras + infimum (II)

Hilbert algebras with Infimum ($H^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1 \rangle$ such that

(1) $\langle A; \rightarrow, 1 \rangle$ Hilbert algebra

(2) $\langle A; \wedge, 1 \rangle$ meet semilattice with top element

(3) $a \rightarrow b = 1$ iff $a = a \wedge b$

Natural ordering on $\mathbf{A}$

$a \leq b$ iff $a \rightarrow b = 1$ iff $a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad [2006]*

$\wedge$-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

(2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

$\wedge$-semi-homomorphism

$h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that

(1) $h(1_1) = 1_2$,

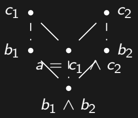
(2) $h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \to, \wedge, 1 \rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive



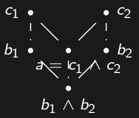
$\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms

$\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \to, \wedge, 1\rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive



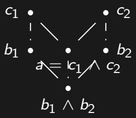
$\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms

$\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1 \rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive

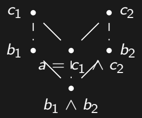


- $\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms
- $\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \to, \wedge, 1 \rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive



$\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms

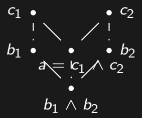
$\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

- We present Spectral-like and Priestley-style dualities for the two categories $\mathsf{DH}^{\wedge}_{S}$ and $\mathsf{DH}^{\wedge}_{H}$:
  - that are based on the ones for Hilbert algebras
  - from which we can recover the ones for implicative semilattices as a particular case

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1 \rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive



$\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms

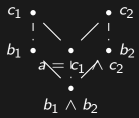
$\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

- We present Spectral-like and Priestley-style dualities for the two categories $\mathsf{DH}^{\wedge}_{S}$ and $\mathsf{DH}^{\wedge}_{H}$:
  - that are based on the ones for Hilbert algebras
  - from which we can recover the ones for implicative semilattices as a particular case

Hilbert algebras + infimum (III)

Distributive Hilbert algebras with Infimum ($DH^{\wedge}$-algebras)

$\mathbf{A} = \langle A; \rightarrow, \wedge, 1 \rangle$ Hilbert algebra with Infimum, s.t. the underlying meet semilattice is distributive



$\Rightarrow$ $\mathsf{DH}^{\wedge}_{S}$ - category of $DH^{\wedge}$-algebras and $\wedge$-semi-homomorphisms

$\Rightarrow$ $\mathsf{DH}^{\wedge}_{H}$ - category of $DH^{\wedge}$-algebras and $\wedge$-homomorphisms

- We present Spectral-like and Priestley-style dualities for the two categories $\mathsf{DH}^{\wedge}_{S}$ and $\mathsf{DH}^{\wedge}_{H}$:
  - that are based on the ones for Hilbert algebras
  - from which we can recover the ones for implicative semilattices as a particular case

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

Definition

$F \subseteq A$ is an **implicative filter** (or $\to$-filter) when:

- $1 \in F$,
- $a, a \to b \in F$ implies $b \in F$

$\mathrm{Fi}_{\to}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\to}(\mathbf{A})$ meet-irreducible elements

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

Definition

$F \subseteq A$ is an **implicative filter** (or $\rightarrow$-filter) when:

- $1 \in F$,
- $a, a \rightarrow b \in F$ implies $b \in F$

$\mathrm{Fi}_{\rightarrow}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\rightarrow}(\mathbf{A})$ meet-irreducible elements

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

Definition

$F \subseteq A$ is an **implicative filter** (or $\to$-filter) when:

- $1 \in F$,
- $a, a \to b \in F$ implies $b \in F$

$\mathrm{Fi}_{\to}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\to}(\mathbf{A})$ meet-irreducible elements

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

Definition

$F \subseteq A$ is an **implicative filter** (or $\rightarrow$-filter) when:

- $1 \in F$,
- $a, a \rightarrow b \in F$ implies $b \in F$

$\mathrm{Fi}_{\rightarrow}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\rightarrow}(\mathbf{A})$ meet-irreducible elements

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

Definition

$F \subseteq A$ is an **implicative filter** (or $\rightarrow$-filter) when:

- $1 \in F$,
- $a, a \rightarrow b \in F$ implies $b \in F$

$\mathrm{Fi}_{\rightarrow}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\rightarrow}(\mathbf{A})$ meet-irreducible elements

Spectral-like duality. Basic tools (I)

Definition

$F \subseteq A$ is an **meet filter** (or $\wedge$-filter) when:

- $a \in F$ and $a \leq b$ implies $b \in F$,
- $a, b \in F$ implies $a \wedge b \in F$

$\mathrm{Fi}_{\wedge}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\wedge}(\mathbf{A})$ meet-irreducible elements

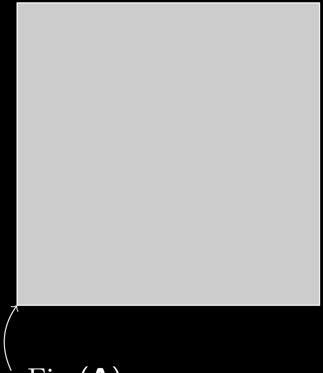
Definition

$F \subseteq A$ is an **implicative filter** (or $\rightarrow$-filter) when:

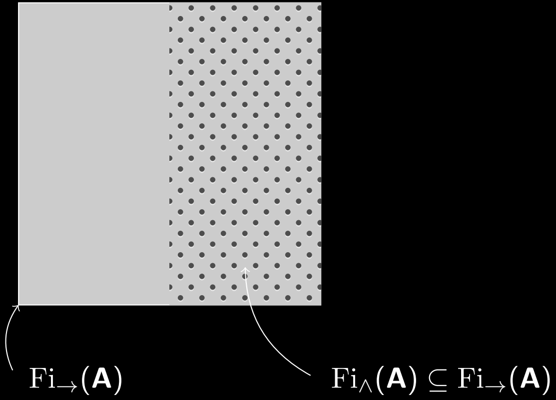
- $1 \in F$,
- $a, a \rightarrow b \in F$ implies $b \in F$

$\mathrm{Fi}_{\rightarrow}(\mathbf{A})$ distributive lattice - $\mathrm{Irr}_{\rightarrow}(\mathbf{A})$ meet-irreducible elements

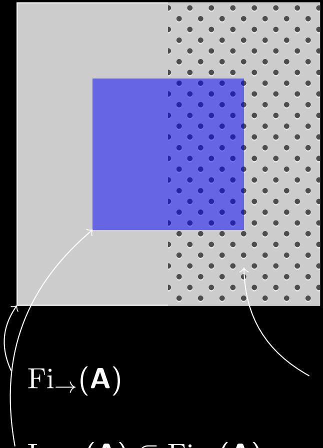
Spectral-like duality. Basic tools (II)



Spectral-like duality. Basic tools (II)



Spectral-like duality. Basic tools (II)

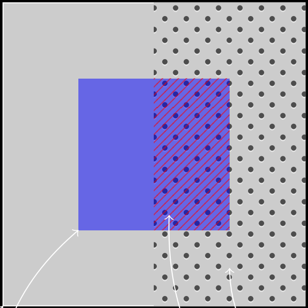


$\mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Fi}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Irr}_{\to}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$

Spectral-like duality. Basic tools (II)



$\mathrm{Fi}_{\rightarrow}(\mathbf{A})$

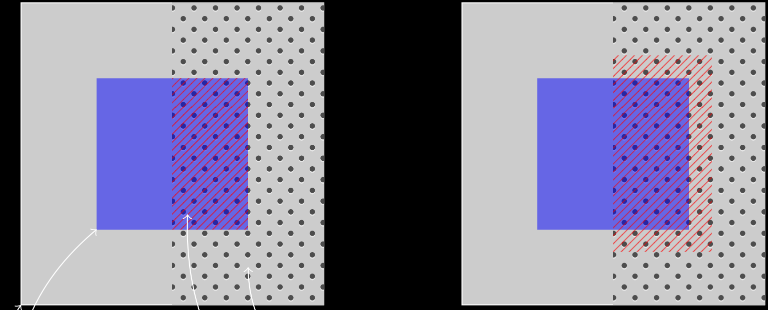
$\mathrm{Fi}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Fi}_{\rightarrow}(\mathbf{A})$

$\mathrm{Irr}_{\rightarrow}(\mathbf{A}) \subseteq \mathrm{Fi}_{\rightarrow}(\mathbf{A})$

$\mathrm{Irr}_{\wedge}(\mathbf{A}) = \mathrm{Irr}_{\rightarrow}(\mathbf{A}) \cap \mathrm{Fi}_{\wedge}(\mathbf{A})$

Spectral-like duality. Basic tools (II)

Non-distributive case



$\mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Fi}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Irr}_{\to}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Irr}_{\wedge}(\mathbf{A}) = \mathrm{Irr}_{\to}(\mathbf{A}) \cap \mathrm{Fi}_{\wedge}(\mathbf{A})$

Spectral-like duality. Basic tools (III)

$$\begin{aligned} \sigma : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Irr}_{\rightarrow}(\mathbf{A})) \\ a &\longmapsto \{F \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}) : a \in F\} \end{aligned}$$

- $\sigma(a) \Rightarrow \sigma(b) := ({\downarrow}(\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \rightarrow b))$
- $\sigma(a) \sqcap \sigma(b) := {\uparrow}(\sigma(a) \cap \sigma(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \rightarrow, \wedge, 1\rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1)\rangle$$

[NOTATION: $\sigma(a)^c$:= complement of $\sigma(a)$, i.e. $\{F \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}) : a \notin F\}$
${\uparrow}$ (${\downarrow}$) := upset (downset) generated, w.r.t. the order $\subseteq$]

Spectral-like duality. Basic tools (III)

$$\begin{aligned}\sigma : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Irr}_{\to}(\mathbf{A})) \\ a &\longmapsto \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\}\end{aligned}$$

- $\sigma(a) \Rightarrow \sigma(b) := (\downarrow(\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \to b))$
- $\sigma(a) \sqcap \sigma(b) := \uparrow(\sigma(a) \cap \sigma(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \to, \wedge, 1\rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1)\rangle$$

[NOTATION: $\sigma(a)^c$:= complement of $\sigma(a)$, i.e. $\{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \notin F\}$
$\uparrow (\downarrow)$:= upset (downset) generated, w.r.t. the order $\subseteq$]

Spectral-like duality. Basic tools (III)

$$\sigma : \mathbf{A} \longrightarrow \mathcal{P}(\mathrm{Irr}_{\to}(\mathbf{A}))$$
$$a \longmapsto \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\}$$

- $\sigma(a) \Rightarrow \sigma(b) := (\downarrow(\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \to b))$
- $\sigma(a) \sqcap \sigma(b) := \uparrow(\sigma(a) \cap \sigma(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \to, \wedge, 1\rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1)\rangle$$

[NOTATION: $\sigma(a)^c$:= complement of $\sigma(a)$, i.e. $\{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \notin F\}$
$\uparrow$ ($\downarrow$) := upset (downset) generated, w.r.t. the order $\subseteq$]

Spectral-like duality. Basic tools (III)

$$\begin{aligned}\sigma : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Irr}_{\rightarrow}(\mathbf{A}))\\ a &\longmapsto \{F \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}) : a \in F\}\end{aligned}$$

- $\sigma(a) \Rightarrow \sigma(b) := (\downarrow(\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \rightarrow b))$
- $\sigma(a) \sqcap \sigma(b) := \uparrow(\sigma(a) \cap \sigma(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \rightarrow, \wedge, 1\rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1)\rangle$$

[NOTATION: $\sigma(a)^c$:= complement of $\sigma(a)$, i.e. $\{F \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}) : a \notin F\}$
$\uparrow$ $(\downarrow)$:= upset (downset) generated, w.r.t. the order $\subseteq$]

Spectral-like duality. Basic tools (III)

$$\begin{aligned} \sigma : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Irr}_{\to}(\mathbf{A})) \\ a &\longmapsto \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\} \end{aligned}$$

- $\sigma(a) \Rightarrow \sigma(b) := (\downarrow(\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \to b))$
- $\sigma(a) \sqcap \sigma(b) := \uparrow(\sigma(a) \cap \sigma(b) \cap \mathrm{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \to, \wedge, 1\rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1)\rangle$$

[NOTATION: $\sigma(a)^c :=$ complement of $\sigma(a)$, i.e. $\{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \notin F\}$
$\uparrow (\downarrow) :=$ upset (downset) generated, w.r.t. the order $\subseteq$]

Priestley-style duality. Basic tools (I)

Definition (*Frink [1954]*)

$I \subseteq A$ is a **Frink ideal** when for all $b \in A$ and $I' \subseteq I$ finite subset:

$$\bigcap_{a \in I'} \uparrow a \subseteq \uparrow b \text{ implies } b \in I$$

$$\left(\bigcap\{\downarrow b : I' \subseteq \downarrow b\} \subseteq I\right)$$

Definition (*Celani and Jansana [2012a]*)

$I \in \mathrm{Id}_F(\mathbf{A})$ is **strong Frink ideal** when for all $B \subseteq A$ and $I' \subseteq I$ finite subsets:

$$\bigcap_{a \in I'} \uparrow a \subseteq \langle B \rangle \text{ implies } \langle B \rangle \cap I \neq \emptyset$$

[NOTATION: $\langle B \rangle :=$ implicative filter generated by B]

Priestley-style duality. Basic tools (I)

Definition (*Frink [1954]*)

$I \subseteq A$ is a **Frink ideal** when for all $b \in A$ and $I' \subseteq I$ finite subset:

$$\bigcap_{a \in I'} \uparrow a \subseteq \uparrow b \text{ implies } b \in I$$

$$\left(\bigcap \{\downarrow b : I' \subseteq \downarrow b\} \subseteq I\right)$$

Definition (*Celani and Jansana [2012a]*)

$I \in \mathrm{Id}_F(\mathbf{A})$ is **strong Frink ideal** when for all $B \subseteq A$ and $I' \subseteq I$ finite subsets:

$$\bigcap_{a \in I'} \uparrow a \subseteq \langle B \rangle \text{ implies } \langle B \rangle \cap I \neq \emptyset$$

[NOTATION: $\langle B \rangle :=$ implicative filter generated by B]

Priestley-style duality. Basic tools (I)

Definition (*Frink [1954]*)

$I \subseteq A$ is a **Frink ideal** when for all $b \in A$ and $I' \subseteq I$ finite subset:

$$\bigcap_{a \in I'} \uparrow a \subseteq \uparrow b \text{ implies } b \in I$$

$$\left(\bigcap \{\downarrow b : I' \subseteq \downarrow b\} \subseteq I\right)$$

Definition (*Celani and Jansana [2012a]*)

$I \in \mathrm{Id}_F(\mathbf{A})$ is **strong Frink ideal** when for all $B \subseteq A$ and $I' \subseteq I$ finite subsets:

$$\bigcap_{a \in I'} \uparrow a \subseteq \langle B \rangle \text{ implies } \langle B \rangle \cap I \neq \emptyset$$

[NOTATION: $\langle B \rangle :=$ implicative filter generated by B]

Priestley-style duality. Basic tools (II)

Definition (*G. Bezhanishvili and Jansana [2011a]*)

$F \in \mathrm{Fi}_{\wedge}(\mathbf{A})$ is a **($\wedge$-)optimal** when there is a Frink ideal I such that:

- F maximal element of $\{G \in \mathrm{Fi}_{\wedge}(\mathbf{A}) : G \cap I \neq \emptyset\}$.
- I maximal element of $\{J \in \mathrm{Id}_{F}(\mathbf{A}) : F \cap J \neq \emptyset\}$.

Definition (*Celani and Jansana [2012a]*)

$F \in \mathrm{Fi}_{\rightarrow}(\mathbf{A})$ is **($\rightarrow$-)optimal** when there is a strong Frink ideal I such that:

- F maximal element of $\{G \in \mathrm{Fi}_{\rightarrow}(\mathbf{A}) : G \cap I \neq \emptyset\}$.
- I maximal element of $\{J \in \mathrm{Id}_{sF}(\mathbf{A}) : F \cap J \neq \emptyset\}$.

Priestley-style duality. Basic tools (II)

Definition (*G. Bezhanishvili and Jansana [2011a]*)

$F \in \mathrm{Fi}_\wedge(\mathbf{A})$ is a **($\wedge$-)optimal** when there is a Frink ideal I such that:

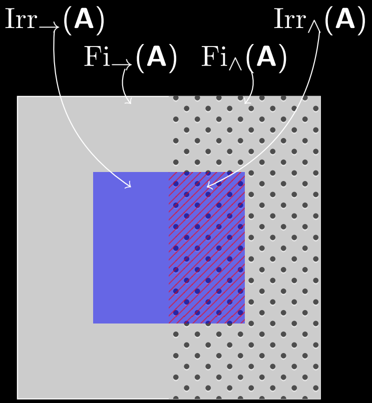
- F maximal element of $\{G \in \mathrm{Fi}_\wedge(\mathbf{A}) : G \cap I \neq \emptyset\}$.
- I maximal element of $\{J \in \mathrm{Id}_F(\mathbf{A}) : F \cap J \neq \emptyset\}$.

Definition (*Celani and Jansana [2012a]*)

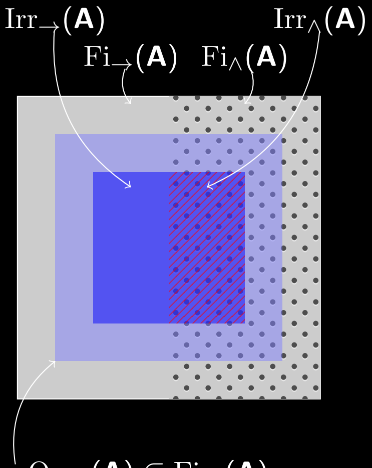
$F \in \mathrm{Fi}_\rightarrow(\mathbf{A})$ is **($\rightarrow$-)optimal** when there is a strong Frink ideal I such that:

- F maximal element of $\{G \in \mathrm{Fi}_\rightarrow(\mathbf{A}) : G \cap I \neq \emptyset\}$.
- I maximal element of $\{J \in \mathrm{Id}_{sF}(\mathbf{A}) : F \cap J \neq \emptyset\}$.

Priestley-style duality. Basic tools (III)



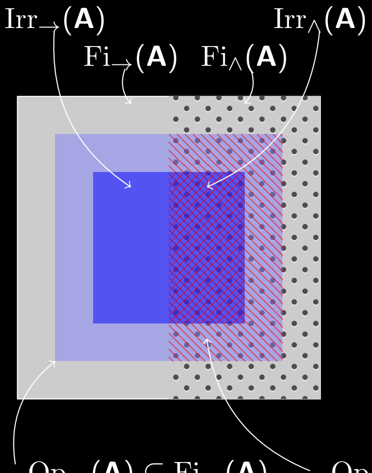
Priestley-style duality. Basic tools (III)



$\mathrm{Op}_{\to}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$

$\mathrm{Irr}_{\to}(\mathbf{A}) \subseteq \mathrm{Op}_{\to}(\mathbf{A})$

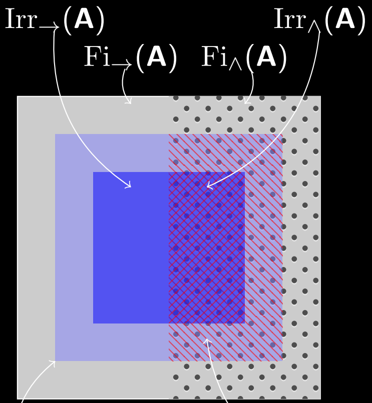
Priestley-style duality. Basic tools (III)



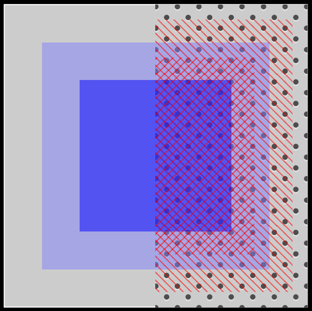
$\mathrm{Op}_{\to}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$ $\qquad$ $\mathrm{Op}_{\wedge}(\mathbf{A}) = \mathrm{Op}_{\to}(\mathbf{A}) \cap \mathrm{Fi}_{\wedge}(\mathbf{A})$

$\mathrm{Irr}_{\to}(\mathbf{A}) \subseteq \mathrm{Op}_{\to}(\mathbf{A})$ $\qquad$ $\mathrm{Irr}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Op}_{\wedge}(\mathbf{A})$

Priestley-style duality. Basic tools (III)



Non-distributive case



$\mathrm{Op}_{\to}(\mathbf{A}) \subseteq \mathrm{Fi}_{\to}(\mathbf{A})$ $\qquad$ $\mathrm{Op}_{\wedge}(\mathbf{A}) = \mathrm{Op}_{\to}(\mathbf{A}) \cap \mathrm{Fi}_{\wedge}(\mathbf{A})$

$\mathrm{Irr}_{\to}(\mathbf{A}) \subseteq \mathrm{Op}_{\to}(\mathbf{A})$ $\qquad$ $\mathrm{Irr}_{\wedge}(\mathbf{A}) \subseteq \mathrm{Op}_{\wedge}(\mathbf{A})$

Priestley-style duality. Basic tools (IV)

$$\varphi : \mathbf{A} \longrightarrow \mathcal{P}(\mathrm{Op}_{\to}(\mathbf{A}))$$
$$a \longmapsto \{F \in \mathrm{Op}_{\to}(\mathbf{A}) : a \in F\}$$

- $\varphi(a) \Rightarrow \varphi(b) := (\downarrow(\varphi(a) \cap \varphi(b)^c))^c \quad (= \varphi(a \to b))$
- $\varphi(a) \sqcap \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \to, \wedge, 1 \rangle \cong \langle \varphi[A], \Rightarrow, \sqcap, \varphi(1) \rangle$$

Priestley-style duality. Basic tools (IV)

$$\begin{aligned}\varphi : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}))\\ a &\longmapsto \{F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : a \in F\}\end{aligned}$$

- $\varphi(a) \Rightarrow \varphi(b) := (\downarrow(\varphi(a) \cap \varphi(b)^c))^c \quad (= \varphi(a \to b))$
- $\varphi(a) \sqcap \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \to, \wedge, 1\rangle \cong \langle \varphi[A], \Rightarrow, \sqcap, \varphi(1)\rangle$$

Priestley-style duality. Basic tools (IV)

$$\begin{aligned} \varphi : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A})) \\ a &\longmapsto \{F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : a \in F\} \end{aligned}$$

- $\varphi(a) \Rightarrow \varphi(b) := (\downarrow(\varphi(a) \cap \varphi(b)^c))^c \quad (= \varphi(a \rightarrow b))$
- $\varphi(a) \sqcap \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \wedge b))$

Theorem

For any DH$^{\wedge}$-algebra,

$$\langle A, \rightarrow, \wedge, 1\rangle \cong \langle \varphi[A], \Rightarrow, \sqcap, \varphi(1)\rangle$$

Priestley-style duality. Basic tools (IV)

$$\begin{aligned} \varphi : \mathbf{A} &\longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A})) \\ a &\longmapsto \{F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : a \in F\} \end{aligned}$$

- $\varphi(a) \Rightarrow \varphi(b) := (\downarrow(\varphi(a) \cap \varphi(b)^c))^c \quad (= \varphi(a \rightarrow b))$
- $\varphi(a) \sqcap \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap \mathrm{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \wedge b))$

Theorem

For any $DH^{\wedge}$-algebra,

$$\langle A, \rightarrow, \wedge, 1\rangle \cong \langle \varphi[A], \Rightarrow, \sqcap, \varphi(1)\rangle$$

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

$DH^{\wedge}$-Priestley spaces

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

$DH^{\wedge}$-Priestley spaces

$$\mathfrak{X} = \langle \; X, \tau_{\kappa} \;\; , \;\; \hat{X} \; \rangle$$

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

$DH^{\wedge}$-Priestley spaces

- H-space

$$\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$$

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$$

- Subset of X that generates a Spectral space

$DH^{\wedge}$-Priestley spaces

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$$

- Subset of X that generates a Spectral space
- $\forall U, V \in \kappa$:

(1) $U^c = \mathrm{cl}(U^c \cap \hat{X})$

(2) $\mathrm{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$

(3) $\forall \mathcal{W} \subseteq \kappa$,

$$\mathrm{cl}\Big(\bigcap_{W \in \mathcal{W}} W^c \cap \hat{X}\Big) \subseteq U^c \Rightarrow$$
$$\mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c$$

$DH^{\wedge}$-Priestley spaces

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X , \tau_\kappa , \hat{X} \rangle$$

- Subset of X that generates a Spectral space
- $\forall U, V \in \kappa$:
  (1) $U^c = \mathrm{cl}(U^c \cap \hat{X})$
  (2) $\mathrm{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$
  (3) $\forall \mathcal{W} \subseteq \kappa$,
  $$\mathrm{cl}\Big(\bigcap_{W \in \mathcal{W}} W^c \cap \hat{X}\Big) \subseteq U^c \Rightarrow$$
  $$\mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c$$

$DH^{\wedge}$-Priestley spaces

$$\mathfrak{X} = \langle X , \tau , \leq , S , \hat{X} \rangle$$

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$$

- Subset of X that generates a Spectral space
- $\forall U, V \in \kappa$:
  (1) $U^c = \mathrm{cl}(U^c \cap \hat{X})$
  (2) $\mathrm{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$
  (3) $\forall \mathcal{W} \subseteq \kappa$,
  $\mathrm{cl}\left(\bigcap_{W \in \mathcal{W}} W^c \cap \hat{X}\right) \subseteq U^c \Rightarrow$
  $\mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c$

$DH^{\wedge}$-Priestley spaces

- *-generalized Priestley space

$$\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$$

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X\,,\,\tau_{\kappa}\,,\,\hat{X}\rangle$$

- Subset of X that generates a Spectral space
- $\forall U, V \in \kappa$:

(1) $U^c = \mathrm{cl}(U^c \cap \hat{X})$

(2) $\mathrm{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$

(3) $\forall \mathcal{W} \subseteq \kappa$,

$$\mathrm{cl}\Big(\bigcap_{W\in\mathcal{W}} W^c \cap \hat{X}\Big) \subseteq U^c \;\Rightarrow\; \mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c$$

$DH^{\wedge}$-Priestley spaces

- *-generalized Priestley space

$$\mathfrak{X} = \langle X\,,\tau\,,\leq\,,S\,,\,\hat{X}\rangle$$

- Subset of X that generates a Priestley space

Characterization of the dual spaces of $DH^{\wedge}$-algebras

$DH^{\wedge}$-Spectral spaces

- H-space

$$\mathfrak{X} = \langle X, \tau_\kappa, \hat{X} \rangle$$

- Subset of X that generates a Spectral space
- $\forall U, V \in \kappa$:
  (1) $U^c = \mathrm{cl}(U^c \cap \hat{X})$
  (2) $\mathrm{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$
  (3) $\forall \mathcal{W} \subseteq \kappa$,
  $\mathrm{cl}\left(\bigcap_{W \in \mathcal{W}} W^c \cap \hat{X}\right) \subseteq U^c \Rightarrow$
  $\mathrm{cl}(W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c$

$DH^{\wedge}$-Priestley spaces

- *-generalized Priestley space

$$\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$$

- Subset of X that generates a Priestley space
- $\forall U, V \in S$:
  (1) $U = \uparrow(U \cap \hat{X})$
  (2) $\uparrow(U \cap V \cap \hat{X}) \in S$
  (3) $\forall W$ clopen upset,
  $W^c \cap \hat{X} \subseteq \downarrow(W^c \cap \hat{X} \cap X_S)$ iff
  $W = U \cap \hat{X}$ for some $U \in S$,

From $DH^{\wedge}$-algebras to their dual spaces

For a $DH^{\wedge}$-algebra $\mathbf{A} = \{A; \rightarrow, \wedge, 1\}$

From $DH^{\wedge}$-algebras to their dual spaces

For a $DH^{\wedge}$-algebra $\mathbf{A} = \{A; \to, \wedge, 1\}$

$\langle \mathrm{Irr}_{\to}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}, \mathrm{Irr}_{\wedge}(\mathbf{A}) \rangle$ is a $DH^{\wedge}$-Spectral space

- For each $a \in A$, $\sigma(a) = \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\mathrm{Irr}_{\to}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^c : a \in F\}$

From $DH^{\wedge}$-algebras to their dual spaces

For a $DH^{\wedge}$-algebra $\mathbf{A} = \{A; \to, \wedge, 1\}$

$$\langle \mathrm{Irr}_{\to}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}, \mathrm{Irr}_{\wedge}(\mathbf{A}) \rangle \text{ is a } DH^{\wedge}\text{-Spectral space}$$

- For each $a \in A$, $\sigma(a) = \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\mathrm{Irr}_{\to}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^c : a \in F\}$

From $DH^{\wedge}$-algebras to their dual spaces

For a $DH^{\wedge}$-algebra $\mathbf{A} = \{A; \to, \wedge, 1\}$

$\langle \mathrm{Irr}_{\to}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}, \mathrm{Irr}_{\wedge}(\mathbf{A}) \rangle$ is a $DH^{\wedge}$-Spectral space

- For each $a \in A$, $\sigma(a) = \{F \in \mathrm{Irr}_{\to}(\mathbf{A}) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\mathrm{Irr}_{\to}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^c : a \in F\}$

$\langle \mathrm{Op}_{\to}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \varphi[A], \mathrm{Op}_{\wedge}(\mathbf{A}) \rangle$ is a $DH^{\wedge}$-Priestley space

- For each $a \in A$, $\varphi(a) := \{F \in \mathrm{Op}_{\to}(\mathbf{A}) : a \in F\}$
- $\tau_{\mathbf{A}}$ on $\mathrm{Op}_{\to}(\mathbf{A})$ with **subbasis**: $\{\varphi(a) : a \in A\} \cup \{\varphi(b)^c : b \in A\}$

From $DH^{\wedge}$-algebras to their dual spaces

For a $DH^{\wedge}$-algebra $\mathbf{A} = \{A; \rightarrow, \wedge, 1\}$

$\langle \mathrm{Irr}_{\rightarrow}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}, \mathrm{Irr}_{\wedge}(\mathbf{A}) \rangle$ is a $DH^{\wedge}$-Spectral space

- For each $a \in A$, $\sigma(a) = \{F \in \mathrm{Irr}_{\rightarrow}(\mathbf{A}) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\mathrm{Irr}_{\rightarrow}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^c : a \in F\}$

$\langle \mathrm{Op}_{\rightarrow}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \varphi[A], \mathrm{Op}_{\wedge}(\mathbf{A}) \rangle$ is a $DH^{\wedge}$-Priestley space

- For each $a \in A$, $\varphi(a) := \{F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : a \in F\}$
- $\tau_{\mathbf{A}}$ on $\mathrm{Op}_{\rightarrow}(\mathbf{A})$ with **subbasis**: $\{\varphi(a) : a \in A\} \cup \{\varphi(b)^c : b \in A\}$

From the spaces to the algebras

For a $DH^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X}\rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X\rangle$ is a $DH^{\wedge}$-algebra.

From the spaces to the algebras

For a *DH*$^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_\kappa, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a *DH*$^{\wedge}$-algebra.

From the spaces to the algebras

For a $DH^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a $DH^{\wedge}$-algebra.

From the spaces to the algebras

For a $DH^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_\kappa, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a $DH^{\wedge}$-algebra.

For a $DH^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$

Define on S the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\downarrow(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \uparrow(U \cap V \cap \hat{X})$$

- $\langle S, \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a $DH^{\wedge}$-algebra.

From the spaces to the algebras

For a $DH^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a $DH^{\wedge}$-algebra.

For a $DH^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$

Define on S the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\downarrow(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \uparrow(U \cap V \cap \hat{X})$$

- $\langle S, \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a $DH^{\wedge}$-algebra.

From the spaces to the algebras

For a $DH^{\wedge}$-Spectral space $\mathfrak{X} = \langle X, \tau_\kappa, \hat{X}\rangle$

Define on $D(\mathfrak{X}) := \{U^c : U \in \kappa\}$ the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\mathrm{sat}(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \mathrm{cl}(U \cap V \cap \hat{X})$$

- $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X\rangle$ is a $DH^{\wedge}$-algebra.

For a $DH^{\wedge}$-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X}\rangle$

Define on S the binary operations:

$$U \Rightarrow_{\mathfrak{X}} V := (\downarrow(U \cap V^c))^c$$

$$U \sqcap_{\mathfrak{X}} V := \uparrow(U \cap V \cap \hat{X})$$

- $\langle S, \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X\rangle$ is a $DH^{\wedge}$-algebra.

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations

$DH^{\wedge}$-Priestley relations

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations

$DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$R \subseteq X_1 \times X_2$

- $*$-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = {\uparrow}(R(x) \cap \hat{X}_2)$.

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations

$DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$R \subseteq X_1 \times X_2$

- $*$-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = {\uparrow}(R(x) \cap \hat{X}_2)$.

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations

$DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$R \subseteq X_1 \times X_2$

- *-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = \uparrow(R(x) \cap \hat{X}_2)$.

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations | $DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$DH^{\wedge}$-Spectral relations: $R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$DH^{\wedge}$-Priestley relations: $R \subseteq X_1 \times X_2$

- *-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = {\uparrow}(R(x) \cap \hat{X}_2)$.

Duals of $\wedge$-homomorphisms

$DH^{\wedge}$-Spectral relations: functional - when moreover

- $(x, y) \in R$ implies $\exists z \in X_1 (x \leq z$ and $R(z) = \mathrm{cl}(y))$

$DH^{\wedge}$-Priestley relations: functional - when moreover

- $\forall x_1 \in X_1, x_2 \in X_{S_2}$, $(x_1, x_2) \in R$ implies $\exists x_1' \in X_{S_1} (x_1 \leq x_1'$ and $R(x_1') = {\uparrow}x_2)$

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations	$DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$DH^{\wedge}$-Spectral relations

$R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$DH^{\wedge}$-Priestley relations

$R \subseteq X_1 \times X_2$

- *-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = {\uparrow}(R(x) \cap \hat{X}_2)$.

Duals of $\wedge$-homomorphisms

$DH^{\wedge}$-Spectral relations

functional - when moreover

- $(x, y) \in R$ implies $\exists z \in X_1 (x \leq z$ and $R(z) = \mathrm{cl}(y))$

$DH^{\wedge}$-Priestley relations

functional - when moreover

- $\forall x_1 \in X_1, x_2 \in X_{S_2}$, $(x_1, x_2) \in R$ implies $\exists x_1' \in X_{S_1} (x_1 \leq x_1'$ and $R(x_1') = {\uparrow} x_2)$

Characterization of the dual morphisms of $\wedge$-(semi)-homomorphisms

$DH^{\wedge}$-Spectral relations	$DH^{\wedge}$-Priestley relations

Duals of $\wedge$-semi-homomorphisms

$R \subseteq X_1 \times X_2$

- H-relation
- $\forall x \in \hat{X}_1,\ R(x) = \mathrm{cl}(R(x) \cap \hat{X}_2)$.

$R \subseteq X_1 \times X_2$

- $*$-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1,\ R(x) = \uparrow(R(x) \cap \hat{X}_2)$.

Duals of $\wedge$-homomorphisms

functional - when moreover

- $(x, y) \in R$ implies $\exists z \in X_1 (x \leq z$ and $R(z) = \mathrm{cl}(y))$

functional - when moreover

- $\forall x_1 \in X_1, x_2 \in X_{S_2}$, $(x_1, x_2) \in R$ implies $\exists x_1' \in X_{S_1} (x_1 \leq x_1'$ and $R(x_1') = \uparrow x_2)$

From $\wedge$-(semi)-homomorphisms to their dual relations

For a $\wedge$-semi-homomorphism $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

From $\wedge$-(semi)-homomorphisms to their dual relations

For a $\wedge$-semi-homomorphism $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

$R_h \subseteq \mathrm{Irr}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Irr}_{\rightarrow}(\mathbf{A}_1)$ is given by:

$$(P, Q) \in R_h \text{ iff } h^{-1}[P] \subseteq Q$$

is a *$DH^{\wedge}$-Spectral relation*

From $\wedge$-(semi)-homomorphisms to their dual relations

For a $\wedge$-semi-homomorphism $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

$R_h \subseteq \mathrm{Irr}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Irr}_{\rightarrow}(\mathbf{A}_1)$ is given by:

$$(P, Q) \in R_h \text{ iff } h^{-1}[P] \subseteq Q$$

is a $DH^{\wedge}$-Spectral relation

$R_h \subseteq \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Op}_{\rightarrow}(\mathbf{A}_1)$ is given by:

$$(P, Q) \in R_h \text{ iff } h^{-1}[P] \subseteq Q$$

is a $DH^{\wedge}$-Priestley relation

From $\wedge$-(semi)-homomorphisms to their dual relations

For a $\wedge$-semi-homomorphism $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

$R_h \subseteq \mathrm{Irr}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Irr}_{\rightarrow}(\mathbf{A}_1)$ is given by:

$$(P, Q) \in R_h \text{ iff } h^{-1}[P] \subseteq Q$$

is a $DH^{\wedge}$-Spectral relation

$R_h \subseteq \mathrm{Op}_{\rightarrow}(\mathbf{A}_2) \times \mathrm{Op}_{\rightarrow}(\mathbf{A}_1)$ is given by:

$$(P, Q) \in R_h \text{ iff } h^{-1}[P] \subseteq Q$$

is a $DH^{\wedge}$-Priestley relation

Similarly for $\wedge$-homomorphisms and functional relations

From the relations to the algebraic morphisms

For a $DH^{\wedge}$-Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : D(\mathfrak{X}_2) &\longrightarrow D(\mathfrak{X}_1) \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

From the relations to the algebraic morphisms

For a $DH^{\wedge}$-Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : D(\mathfrak{X}_2) &\longrightarrow D(\mathfrak{X}_1) \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

From the relations to the algebraic morphisms

For a $DH^{\wedge}$-Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : D(\mathfrak{X}_2) &\longrightarrow D(\mathfrak{X}_1) \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

From the relations to the algebraic morphisms

For a $DH^{\wedge}$-Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : D(\mathfrak{X}_2) &\longrightarrow D(\mathfrak{X}_1) \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

For a $DH^{\wedge}$-Priestley (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : S_2 &\longrightarrow S_1 \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

From the relations to the algebraic morphisms

For a $DH^{\wedge}$-Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : D(\mathfrak{X}_2) &\longrightarrow D(\mathfrak{X}_1) \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

For a $DH^{\wedge}$-Priestley (functional) relation $R \subseteq X_1 \times X_2$

$$\begin{aligned} \Box_R : S_2 &\longrightarrow S_1 \\ U &\longmapsto \{x \in X_1 : R(x) \subseteq U\} \end{aligned}$$

- $\Box_R$ is a $\wedge$-semi-homomosphism ($\wedge$-homomorphism).

Dualities for Implicative semilattices as a particular case

$$a \wedge b \leq c \text{ iff } c \leq a \rightarrow b$$

Spectral-like duality

IS spaces (defined in *Celani [2003]*)	$\Leftrightarrow$	$DH^{\wedge}$-Spectral spaces $\langle X, \tau_{\kappa}, \hat{X}\rangle$ such that • $\hat{X} = X$

Dualities for Implicative semilattices as a particular case

$$a \wedge b \leq c \text{ iff } c \leq a \to b$$

Spectral-like duality

IS spaces (defined in *Celani [2003]*)	$\Leftrightarrow$	$DH^{\wedge}$-Spectral spaces $\langle X, \tau_{\mathcal{K}}, \hat{X} \rangle$ such that • $\hat{X} = X$

Dualities for Implicative semilattices as a particular case

$$a \wedge b \leq c \text{ iff } c \leq a \to b$$

Spectral-like duality

IS spaces (defined in *Celani [2003]*) $\Leftrightarrow$ $DH^{\wedge}$-Spectral spaces $\langle X, \tau_{\kappa}, \hat{X} \rangle$ such that

- $\hat{X} = X$

Priestley-style duality

$*$-generalized Esakia spaces (defined in *G. Bezhanishvili and Jansana [2011b]*) $\Leftrightarrow$ $DH^{\wedge}$-Priestley spaces $\langle X, \tau, \leq, S, \hat{X} \rangle$ such that

- $\hat{X} = X$

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \to, \wedge, 1 \rangle$

$a \wedge b = a$ iff $a \to b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction

- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \rightarrow, \wedge, 1 \rangle$

$a \wedge b = a$ iff $a \rightarrow b = 1$

$\rightarrow$ $\wedge$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction

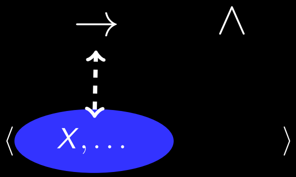
- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \to, \wedge, 1 \rangle$

$a \wedge b = a$ iff $a \to b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction



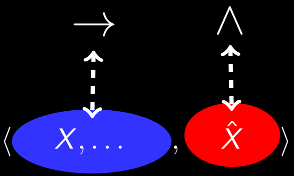
- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \rightarrow, \wedge, 1 \rangle$

$a \wedge b = a$ iff $a \rightarrow b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction



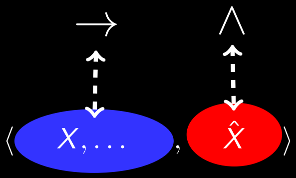
- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \rightarrow, \wedge, 1 \rangle$

$a \wedge b = a$ iff $a \rightarrow b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction



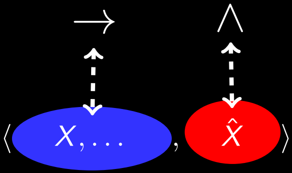
- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Conclusion and further work

- We have considered as objects of our categories, $DH^{\wedge}$-algebras, $\langle \mathbf{A}, \to, \wedge, 1\rangle$

$a \wedge b = a$ iff $a \to b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction



- What about the dualities for $H^{\wedge}$-algebras?
- May this strategy be followed for other classes of algebras?

Thanks!

Bibliography I

S. Celani. Representation of Hilbert algebras and implicative semilattices. *Central European Journal of Mathematics*, 4: 561–572, 2003.

S. Celani and D. Montangie. Hilbert algebras with supremum. submitted to Algebra Universalis, 2011.

S. A. Celani and R. Jansana. On the free implicative semilattice extension of a Hilbert algebra. *Math. Log. Quart.*, pages 1–20, 2012a.

S. A. Celani and R. Jansana. Priestley style duality for Hilbert algebras. (preprint), 2012b.

S. A. Celani, L. M. Cabrer, and D. Montangie. Representation and duality for Hilbert algebras. *Cent. Eur. J. Math.*, 7(3):463–478, 2009. ISSN 1895-1074. doi: 10.2478/s11533-009-0032-5.

Bibliography II

A. Figallo, Jr., G. Ramón, and S. Saad. *iH*-propositional calculus. *Bulletin of the Section of Logic*, 35/4:157–162, 2006.

O. Frink. Ideals in partially ordered sets. *Amer. Math. Monthly*, 61:223–234, 1954.

G. Bezhanishvili and R. Jansana. Priestley style duality for distributive meet-semilattices. *Studia Logica*, 98:83–123, 2011a.

G. Bezhanishvili and R. Jansana. Esakia style duality for implicative semilattices. *Applied Categorical Structures*, September 2011b.