Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

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Duality workshop, 15-17 August, Oxford

Spectral-like duality. Basic tools



- Priestley-style duality. Basic tools
- 4 Duality for objects
- 5 Duality for morphisms





Preliminaries Spectral-like duality Priestley-style duality Objects Morphisms Implicative semilattices Conclu • OOOO Spectral-like and Priestley-style dualities for Distributive Hilbert algebras with Infimum

Spectral-like spaces Priestley-style spaces

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Spectral-like spaces

 $\langle X, au, \ldots
angle$ such that:

• $\langle X, \tau \rangle$ is a **sober**

• $\langle X, \tau \rangle$ is **compactly based**, i.e. $\mathcal{KO}(X)$ (compact open subsets) forms a **basis**

Priestley-style spaces

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Hilbert algebras (I)

• $\{\rightarrow\}$ -fragment of Intuitionistic logic

⇒ H_S - category of Hilbert algebras and semi-homomorphisms
 ⇒ H_H - category of Hilbert algebras and homomorphisms
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 $\begin{array}{l} \mathbf{A} = \langle A; \rightarrow, 1 \rangle \text{ such that} \\ (1) \ a \rightarrow (b \rightarrow c) = 1, \\ (2) \ (a \rightarrow (b \rightarrow b)) \rightarrow ((a \rightarrow b) \rightarrow (a \rightarrow c)) = 1, \end{array} \end{array}$ Natural ordering on \mathbf{A} $a \leq b \text{ iff } a \rightarrow b = 1 \\ (3) \ a \rightarrow b = 1 = b \rightarrow a \text{ implies } a = b. \end{array}$

HomomorphismSemi-homomorphism $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ such that $(1) \ h(1_1) = 1_2,$ $(1) \ h(1_1) = 1_2,$ $(2) \ h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$ $(2) \ h(a \rightarrow_1 b) \le h(a) \rightarrow_2 h(b)$

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► Spectral-like dualities for H_S and H_H in Celani, Cabrer, and Montangie [2009]

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H-space $\langle X, \tau_{\kappa} \rangle$:

- $\kappa \subseteq \mathcal{KO}(X)$ basis,
- $\forall U, V \in \kappa$, sat $(U \cap V^c) \in \kappa$,
- $\langle X, \tau_{\kappa} \rangle$ is sober.

H-relation $R \subseteq X_1 \times X_2$: (duals of semi-hom.)

- $R^{-1}(U) \in \kappa_1$ for every $U \in \kappa_2$
- R(x) closed of X_2 , for all $x \in X_2$

H-functional relation when moreover: (duals of hom.)

• $(x, y) \in R$ implies $\exists z \in X_1 (x \leq z \text{ and } R(z) = cl(y))$

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*-augmented Priestley space $\langle X, \tau, \leq, S \rangle$:

• $\langle X, \tau, \leq \rangle$ Priestley space

■ S family of clopen upsets, that satisfies some conditions [...]

• $\forall U, V \in S, (\downarrow (U \cap V^c))^c \in S.$

*-augmented Priestley semi-morphism $R \subseteq X_1 \times X_2$:

• $(x_1, x_2) \notin R$ implies $\exists U \in S_2(x_2 \notin U \text{ and } R(x_1) \subseteq U)$

• $\{x_1 \in X_1 : R(x_1) \subseteq U\} \in S_2$ for all $U \in S_1$

*-augmented Priestley morphism when moreover:

$$\forall x_1 \in X_1, x_2 \in X_{S_2}, \\ (x_1, x_2) \in R \text{ implies } \exists x'_1 \in X_{S_1} (x_1 \le x'_1 \text{ and } R(x'_1) = \uparrow x_2)$$

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Hilbert algebras + supremum

Hilbert algebras with Supremum

- $\mathbf{A} = \langle A; \rightarrow, \lor, 1 \rangle$ such that
 - (1) $\langle A; \rightarrow, 1 \rangle$ Hilbert algebra
 - (2) $\langle A; \vee, 1 \rangle$ join semilattice with top element
 - (3) $a \lor b = b$ iff $a \to b = 1$

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Natural ordering on **A** $a \le b$ iff $a \to b = 1$ iff $a \lor b = b$

- $\{\rightarrow, \lor\}$ -fragment of Intuitionistic logic
- Subclass of BCK-algebras with lattice operations

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Hilbert algebras + infimum (II)

Hilbert algebras with Infimum (H^{\wedge} -algebras)

- $oldsymbol{\mathsf{A}}=\langle extsf{A};
 ightarrow,\wedge,1
 angle$ such that
 - (1) $\langle A;
 ightarrow, 1
 angle$ Hilbert algebra
 - (2) $\langle A; \wedge, 1 \rangle$ meet semilattice with top element
 - (3) $a \rightarrow b = 1$ iff $a = a \wedge b$

- Subclass of BCK-algebras with lattice operations
- Correspond to a logic studied in Figallo, Jr., Ramón, and Saad [2006]

- (2) $h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$
- (3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

Spectral-like duality

Priestley-style duality

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Natural ordering on **A** $a \leq b$ iff $a \rightarrow b = 1$ iff $a = a \wedge b$

Subclass of BCK-algebras with lattice operations

• Correspond to a logic studied in *Figallo, Jr., Ramón, and Saad* [2006]

∧-homomorphism

$$h: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$$
 such that

(1)
$$h(1_1) = 1_2$$
,

(2)
$$h(a \rightarrow_1 b) = h(a) \rightarrow_2 h(b)$$

(3) $h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

\-semi-homomorphism

$$(1) \quad I(1) \quad 1$$

(1)
$$h(1_1) = 1_2$$
,

$$(2) h(a \rightarrow_1 b) \leq h(a) \rightarrow_2 h(b)$$

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 $\land -\text{semi-homomorphism}$ $h : \mathbf{A}_1 \longrightarrow \mathbf{A}_2 \text{ such that}$ $(1) h(1_1) = 1_2,$ $(2) h(a \rightarrow_1 b) \le h(a) \rightarrow_2 h(b)$ $(3) h(a \wedge_1 b) = h(a) \wedge_2 h(b)$

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 \Rightarrow DH $^{\wedge}_{S}$ - category of DH $^{\wedge}$ -algebras and \wedge -semi-homomorphisms.

 \Rightarrow DH $^{\wedge}_{H}$ - category of DH $^{\wedge}$ -algebras and \wedge -homomorphisms.



 \Rightarrow DH^{\wedge} - category of DH^{\wedge}-algebras and \wedge -semi-homomorphisms

 $\Rightarrow \; \mathsf{DH}^\wedge_H$ - category of DH^\wedge -algebras and \wedge -homomorphisms.



 \Rightarrow DH^{\wedge} - category of DH^{\wedge}-algebras and \wedge -semi-homomorphisms

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- \Rightarrow DH^{\wedge} category of DH^{\wedge}-algebras and \wedge -semi-homomorphisms
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- ► We present Spectral-like and Priestley-style dualities for the two categories DH[∧]_S and DH[∧]_H:
 - that are based on the ones for Hilbert algebras
 - from which we can recover the ones for implicative semilattices as a particular case



- \Rightarrow DH^{\wedge} category of DH^{\wedge}-algebras and \wedge -semi-homomorphisms
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- \Rightarrow DH^{\wedge}_{S} category of DH^-algebras and $\wedge\text{-semi-homomorphisms}$
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Definition

- $F \subseteq A$ is an **meet filter** (or \land -filter) when:
 - $a \in F$ and $a \leq b$ implies $b \in F$,
 - $a, b \in F$ implies $a \land b \in F$

 ${
m Fi}_{\wedge}({f A})$ distributive lattice - ${
m Irr}_{\wedge}({f A})$ meet-irreducible elements

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$$\operatorname{Fi}_{\rightarrow}(\mathbf{A})$$

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 $\operatorname{Fi}_{\rightarrow}(\mathbf{A})$ $\operatorname{Fi}_{\wedge}(\mathbf{A}) \subseteq \operatorname{Fi}_{\rightarrow}(\mathbf{A})$

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 Basic tools (III)

$$\sigma: \mathbf{A} \longrightarrow \mathcal{P}(\operatorname{Irr}_{\rightarrow}(\mathbf{A}))$$
$$\mathbf{a} \longmapsto \{F \in \operatorname{Irr}_{\rightarrow}(\mathbf{A}) : \mathbf{a} \in F\}$$

•
$$\sigma(a) \Rightarrow \sigma(b) := (\downarrow (\sigma(a) \cap \sigma(b)^c))^c \quad (= \sigma(a \to b))$$

• $\sigma(a) \sqcap \sigma(b) := \uparrow (\sigma(a) \cap \sigma(b) \cap \operatorname{Irr}_{\wedge}(\mathbf{A})) \quad (= \sigma(a \land b))$

Theorem

For any DH^{\wedge} -algebra,

$$\langle A, \rightarrow, \wedge, 1 \rangle \cong \langle \sigma[A], \Rightarrow, \sqcap, \sigma(1) \rangle$$

 $[\text{NOTATION: } \sigma(a)^c := \text{complement of } \sigma(a), \text{ i.e. } \{F \in \text{Irr}_{\rightarrow}(\mathbb{A}) : a \notin F\}$ $\uparrow (\downarrow) := \text{upset (downset) generated, w.r.t. the order <math>\textcircled{P}^{\ast} \land \textcircled{P} \land @$

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Definition (Celani and Jansana [2012a])

 $I \in Id_F(\mathbf{A})$ is strong Frink ideal when for all $B \subseteq A$ and $I' \subseteq I$ finite subsets:

 $\bigcap_{a\in I'} \uparrow a \subseteq \langle B \rangle \text{ implies } \langle B \rangle \cap I \neq \emptyset$

[NOTATION: $\langle B \rangle :=$ implicative filter generated by B]



Definition (*Frink* [1954])

 $I \subseteq A$ is a **Frink ideal** when for all $b \in A$ and $I' \subseteq I$ finite subset:

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$$\left(\bigcap\{\downarrow b: I'\subseteq \downarrow b\}\subseteq I\right)$$

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Definition (Frink [1954])

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Definition (G. Bezhanishvili and Jansana [2011a])

 $F \in Fi_{\wedge}(\mathbf{A})$ is a (\wedge -)optimal when there is a Frink ideal I such that:

- F maximal element of $\{G \in Fi_{\wedge}(\mathbf{A}) : G \cap I \neq \emptyset\}.$
- I maximal element of $\{J \in \mathrm{Id}_F(\mathbf{A}) : F \cap J \neq \emptyset\}$.

Definition (*Celani and Jansana [2012a]*)

 $F \in Fi_{\rightarrow}(\mathbf{A})$ is $(\rightarrow -)$ optimal when there is a strong Frink ideal *I* such that:

- F maximal element of $\{G \in \operatorname{Fi}_{\rightarrow}(\mathbf{A}) : G \cap I \neq \emptyset\}$.
- I maximal element of $\{J \in \mathrm{Id}_{sF}(\mathbf{A}) : F \cap J \neq \emptyset\}$.

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 $\operatorname{Irr}_{\rightarrow}(\mathbf{A})$ $\operatorname{Irr}_{\wedge}(\mathbf{A})$ $\operatorname{Fi}_{\rightarrow}(\mathbf{A}) \operatorname{Fi}_{\wedge}(\mathbf{A})$




 $Op_{\rightarrow}(\mathbf{A}) \subseteq Fi_{\rightarrow}(\mathbf{A})$

 $\mathrm{Irr}_{\rightarrow}(\textbf{A})\subseteq \mathrm{Op}_{\rightarrow}(\textbf{A})$





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$$\varphi : \mathbf{A} \longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}))$$
$$\mathbf{a} \longmapsto \{ F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : \mathbf{a} \in F \}$$

$$\begin{aligned} \varphi(a) \Rightarrow \varphi(b) &:= \left(\downarrow (\varphi(a) \cap \varphi(b)^c) \right)^c \quad (= \varphi(a \to b)) \\ \varphi(a) \sqcap \varphi(b) &:= \uparrow (\varphi(a) \cap \varphi(b) \cap \operatorname{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \land b)) \end{aligned}$$

Theorem

For any DH^{\wedge} -algebra,

$$\langle A,
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 Priestley-style duality.
 Basic tools (IV)

$$\varphi : \mathbf{A} \longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}))$$
$$\mathbf{a} \longmapsto \{F \in \mathrm{Op}_{\rightarrow}(\mathbf{A}) : \mathbf{a} \in F\}$$

•
$$\varphi(a) \Rightarrow \varphi(b) := (\downarrow(\varphi(a) \cap \varphi(b)^c))^c \quad (= \varphi(a \to b))$$

• $\varphi(a) \sqcap \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap \operatorname{Op}_{\wedge}(\mathbf{A})) \quad (= \varphi(a \land b))$

Theorem

For any DH^{\wedge} -algebra,

$$\langle A,
ightarrow, \wedge, 1
angle \cong \langle \varphi[A], \Rightarrow, \sqcap, \varphi(1)
angle$$

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$$\varphi : \mathbf{A} \longrightarrow \mathcal{P}(\mathrm{Op}_{\rightarrow}(\mathbf{A}))$$
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• $\varphi(a) = \varphi(b) := \uparrow(\varphi(a) \cap \varphi(b) \cap Q_{ab} \land \varphi(b)) \quad (= \varphi(a \to b))$

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Theorem

For any DH^{\wedge} -algebra,

$$\langle \mathsf{A},
ightarrow, \wedge, 1
angle \cong \langle arphi[\mathsf{A}], \Rightarrow, \sqcap, arphi(\mathbf{1})
angle$$

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DH[^]-Priestley spaces

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DH[^]-Spectral spaces

DH[^]-Priestley spaces

$$\mathfrak{X} = \langle X, au_{\kappa}, \hat{X} \rangle$$

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DH[^]-Spectral spaces

DH[^]-Priestley spaces

• H-space
$$\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$$

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DH[^]-Spectral spaces

 $\mathfrak{X} = \langle \mathbf{X}, \tau_{\kappa}, \mathbf{\hat{X}} \rangle$

 Subset of X[/] that generates a Spectral space

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DH[^]-Spectral spaces

• H-space

 $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

- Subset of X / that generates a Spectral space
- $\forall U, V \in \kappa$:
 - (1) $U^c = \operatorname{cl}(U^c \cap \hat{X})$
 - (2) $\operatorname{cl}(U^c \cap V^c \cap \hat{X}) \in \kappa$

(3)
$$\forall \mathcal{W} \subseteq \kappa,$$

 $\operatorname{cl}\left(\bigcap_{\substack{W \in \mathcal{W} \\ c \in (\mathcal{W}_{0}^{c} \cap \cdots \cap \mathcal{W}_{n}^{c} \cap \hat{X})} \subseteq U^{c} \Rightarrow$

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DH[^]-Spectral spaces

• *H*-space

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•
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:

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(2)
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(3)
$$\forall \mathcal{W} \subseteq \kappa,$$

 $\operatorname{cl}\left(\bigcap_{W \in \mathcal{W}} W^{c} \cap \hat{X}\right) \subseteq U^{c} \Rightarrow$
 $\operatorname{cl}(W_{0}^{c} \cap \cdots \cap W_{n}^{c} \cap \hat{X}) \subseteq U^{c}$

DH[^]-Priestley spaces

$$\mathfrak{X} = \langle X, au, \leq, S, \hat{X} \rangle$$

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Characterization of the dual spaces of DH^{\wedge} -algebras

DH[^]-Spectral spaces

- H-space $\mathfrak{X} = \langle \mathbf{X}, \tau_{\kappa}, \mathbf{\hat{X}} \rangle$
- Subset of X / that generates a Spectral space
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$$\begin{array}{ll} \textbf{(3)} & \forall \mathcal{W} \subseteq \kappa, \\ & \operatorname{cl} \Big(\bigcap_{W \in \mathcal{W}} W^c \cap \hat{X} \Big) \subseteq U^c \Rightarrow \\ & & \operatorname{cl} (W_0^c \cap \cdots \cap W_n^c \cap \hat{X}) \subseteq U^c \end{array}$$

DH[^]-Priestley spaces

*-generalized Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$

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DH[^]-Spectral spaces

- *H*-space $\mathfrak{X} = \langle \mathbf{X}, \tau_{\kappa}, \hat{\mathbf{X}} \rangle$
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DH[^]-Spectral spaces

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DH[^]-Priestley spaces

- *-generalized Priestley space $\mathfrak{X} = \langle \mathbf{X}, \tau, \leq, \mathbf{S}, \mathbf{\hat{X}} \rangle$
- Subset of X that generates a Priestley space
- $\forall U, V \in S$:
 - (1) $U = \uparrow (U \cap \hat{X})$
 - (2) $\uparrow (U \cap V \cap \hat{X}) \in S$
 - (3) $\forall W$ clopen upset, $w^c \cap \hat{x} \subset \downarrow (w^c \cap \hat{x} \cap X_S)$ iff

 $W = U \cap \hat{X}$ for some $U \in S$,

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From DH^{\wedge} -algebras to their dual spaces

For a DH^{\wedge} -algebra $\mathbf{A} = \{A; \rightarrow, \land, 1\}$

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 From DH^{\wedge} -algebras to their dual spaces

 For a DH^{\wedge} -algebra $\mathbf{A} = \{A; \rightarrow, \wedge, 1\}$

 (Irr_ \rightarrow (A), $\tau_{\kappa_{\mathbf{A}}}$, (Irr_ \wedge (A)) is a DH^{\wedge} -Spectral space

 • For each $a \in A$, $\sigma(a) = \{F \in Irr_{\rightarrow}(\mathbf{A}) : a \in F\}$

• $au_{\kappa_{\mathbf{A}}}$ on $\mathrm{Irr}_{
ightarrow}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}}:=\{\sigma(a)^{c}:a\in F\}$

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From DH^{\wedge} -algebras to their dual spaces

For a
$$DH^{\wedge}$$
-algebra $\mathbf{A} = \{A; \rightarrow, \wedge, 1\}$

$$\langle \operatorname{Irr}_{d}(\mathbf{A}), \tau_{\kappa_{\mathbf{A}}}, \operatorname{Irr}_{d}(\mathbf{A}) \rangle$$
 is a DH^{\wedge} -Spectral space

- For each $a \in A$, $\sigma(a) = \{F \in \operatorname{Irr}_{\to}(A) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\operatorname{Irr}_{\rightarrow}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^{c} : a \in F\}$

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From DH^{\wedge} -algebras to their dual spaces

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- For each $a \in A$, $\sigma(a) = \{F \in \operatorname{Irr}_{\to}(A) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\operatorname{Irr}_{\rightarrow}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^{c} : a \in F\}$

 $\langle Op_{\downarrow}(\mathbf{A}), \tau_{\mathbf{A}}, \subseteq, \varphi[A] \rangle, Op_{\wedge}(\mathbf{A}) \rangle$ is a DH^{\wedge} -Priestley space

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From DH^{\wedge} -algebras to their dual spaces

For a
$$DH^{\wedge}$$
-algebra $\mathbf{A} = \{A; \rightarrow, \wedge, 1\}$

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- For each $a \in A$, $\sigma(a) = \{F \in \operatorname{Irr}_{\to}(A) : a \in F\}$
- $\tau_{\kappa_{\mathbf{A}}}$ on $\operatorname{Irr}_{\rightarrow}(\mathbf{A})$ with **basis**: $\kappa_{\mathbf{A}} := \{\sigma(a)^{c} : a \in F\}$

$$\langle \operatorname{Op}_{\rightarrow}(\mathsf{A}), \tau_{\mathsf{A}}, \subseteq, \varphi[A] \rangle, \operatorname{Op}_{\wedge}(\mathsf{A}) \rangle$$
 is a DH^{\wedge} -Priestley space

• For each $a \in A$, $\varphi(a) := \{F \in Op_{\rightarrow}(A) : a \in F\}$ • $\tau_{\mathbf{A}}$ on $\operatorname{Op}_{\mathbf{A}}(\mathbf{A})$ with subbasis: $\{\varphi(a) : a \in A\} \cup \{\varphi(b)^c : b \in A\}$

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From the spaces to the algebras

For a DH^{\wedge} -Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}):=\{U^{c}:U\in\kappa\}$ the binary operations:

 $U \Rightarrow_{\mathfrak{X}} V := (\operatorname{sat}(U \cap V^c))^c$

 $U \sqcap_{\mathfrak{X}} V := \operatorname{cl}(U \cap V \cap \hat{X})$

• $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a DH^{\wedge} -algebra.

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From the spaces to the algebras

For a DH^{\wedge} -Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$ Define on $D(\mathfrak{X}) := \{ U^{c} : U \in \kappa \}$ the binary operations: $U \Rightarrow_{\mathfrak{X}} V := (\operatorname{sat}(U \cap V^{c}))^{c}$

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From the spaces to the algebras

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From the spaces to the algebras

For a DH^{\wedge} -Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{ U^{c} : U \in \kappa \}$ the binary operations:

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• $\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$ is a DH^{\wedge} -algebra.

For a DH^{\wedge} -Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$

Define on S the binary operations:

 $U \Rightarrow_{\mathfrak{X}} V := (\downarrow (U \cap V^c))^c$

 $U\sqcap_{\mathfrak{X}}V:={\uparrow}(U\cap V\cap \hat{X})$

• $\langle S, \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X
angle$ is a DH^{\wedge} -algebra.

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From the spaces to the algebras

For a DH^{\wedge} -Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{ U^c : U \in \kappa \}$ the binary operations:

 $U \Rightarrow_{\mathfrak{T}} V := (\operatorname{sat}(U \cap V^c))^c$

 $U \sqcap_{\mathfrak{X}} V := \operatorname{cl}(U \cap V \cap \hat{X})$

•
$$\langle D(\mathfrak{X}), \Rightarrow_{\mathfrak{X}}, \sqcap_{\mathfrak{X}}, X \rangle$$
 is a DH^{\wedge} -algebra.

For a
$$DH^{\wedge}$$
-Priestley space $\mathfrak{X} = \langle X, \tau, \leq, S, \hat{X} \rangle$

Define on S the binary operations:

 $U \Rightarrow_{\mathfrak{T}} V := (\downarrow (U \cap V^c))^c$ $U \sqcap_{\mathfrak{X}} V := \uparrow (U \cap V \cap \hat{X})$

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From the spaces to the algebras

For a DH^{\wedge} -Spectral space $\mathfrak{X} = \langle X, \tau_{\kappa}, \hat{X} \rangle$

Define on $D(\mathfrak{X}) := \{ U^{c} : U \in \kappa \}$ the binary operations:

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 is a DH^{\wedge} -algebra.

 DH^{\wedge} -Spectral relations

DH[^]-Priestley relations

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• $\forall x \in \hat{X}_1$, $R(x) = \operatorname{cl}(R(x) \cap \hat{X}_2)$.

- semi-morphism
- $\forall x \in \hat{X}_1, \ R(x) = \uparrow (R(x) \cap \hat{X}_2).$

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 Characterization of the dual morphisms of
 $\wedge -(semi)$ -homomorphisms
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 DH^A-Spectral relations
 DH^A-Priestley relations
 DH^A-Priestley relations

 R $\subseteq X_1 \times X_2$ $R \subseteq X_1 \times X_2$ $R \subseteq X_1 \times X_2$
 \bullet H-relation
 \bullet \bullet

• $\forall x \in \hat{X}_1, R(x) = \operatorname{cl}(R(x) \cap \hat{X}_2).$

- *-augmented Priestley semi-morphism
- $\forall x \in \hat{X}_1, \ R(x) = \uparrow (R(x) \cap \hat{X}_2).$

Spectral-like duality Priestley-style duality Morphisms Implicative semilattices Preliminaries Characterization of the dual morphisms of \wedge -(semi)-homomorphisms DH^{\wedge} -Spectral relations DH[^]-Priestley relations Duals of A-semi-homomorphisms $R \subseteq X_1 \times X_2$ $R \subset X_1 \times X_2$ H-relation *-augmented Priestley • $\forall x \in \hat{X}_1, R(x) = \operatorname{cl}(R(x) \cap \hat{X}_2).$ semi-morphism • $\forall x \in \hat{X}_1, R(x) = \uparrow (R(x) \cap \hat{X}_2).$

Spectral-like duality Priestley-style duality Morphisms Implicative semilattices Conclusion Characterization of the dual morphisms of \wedge -(semi)-homomorphisms DH^{\wedge} -Spectral relations DH^{\wedge} -Priestley relations Duals of A-semi-homomorphisms $R \subseteq X_1 \times X_2$ $R \subseteq X_1 \times X_2$ *-augmented Priestley H-relation • $\forall x \in \hat{X}_1, R(x) = \operatorname{cl}(R(x) \cap \hat{X}_2).$ semi-morphism • $\forall x \in \hat{X}_1, R(x) = \uparrow (R(x) \cap \hat{X}_2).$

Duals of $\wedge\text{-homomorphisms}$

functional - when moreover

• $(x, y) \in R$ implies $\exists z \in X_1(x \le z \text{ and } R(z) = \operatorname{cl}(y))$

unctional - when moreover

$$\forall x_1 \in X_1, x_2 \in X_{S_2}, \\ (x_1, x_2) \in R \text{ implies } \exists x'_1 \in \\ X_{S_1} (x_1 \leq x'_1 \text{ and } R(x'_1) = \uparrow x_2)$$

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Spectral-like duality Priestley-style duality Morphisms Implicative semilattices Conclusion Characterization of the dual morphisms of \wedge -(semi)-homomorphisms DH^{\wedge} -Spectral relations DH^{\wedge} -Priestley relations Duals of \land -semi-homomorphisms $R \subseteq X_1 \times X_2$ $R \subset X_1 \times X_2$ *-augmented Priestley H-relation • $\forall x \in \hat{X}_1, R(x) = \operatorname{cl}(R(x) \cap \hat{X}_2).$ semi-morphism • $\forall x \in \hat{X}_1, R(x) = \uparrow (R(x) \cap \hat{X}_2).$

Duals of \wedge -homomorphisms

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From \wedge -(semi)-homomorphisms to their dual relations

For a \wedge -semi-homomorphism $h: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

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From \wedge -(semi)-homomorphisms to their dual relations

For a \wedge -semi-homomorphism $h: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

$${\it R}_h \subseteq {
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ightarrow}({f A}_2) imes {
m Irr}_{
ightarrow}({f A}_1)$$
 is given by:

$$(P,Q)\in R_h ext{ iff } h^{-1}[P]\subseteq Q$$

is a DH^{\wedge} -Spectral relation
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Conclusion

From \wedge -(semi)-homomorphisms to their dual relations

For a \wedge -semi-homomorphism $h: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

 $\overline{R_h \subseteq \operatorname{Irr}_{\rightarrow}(\mathsf{A}_2) \times \operatorname{Irr}_{\rightarrow}(\mathsf{A}_1)}$ is given by:

 $\overline{(P,Q)} \in R_h \text{ iff } h^{-1}[P] \subseteq Q$

is a DH^{\wedge} -Spectral relation

 $R_h \subseteq \operatorname{Op}_{
ightarrow}(\mathbf{A}_2) imes \operatorname{Op}_{
ightarrow}(\mathbf{A}_1)$ is given by: $(P,Q) \in R_h ext{ iff } h^{-1}[P] \subseteq Q$ is a DH^{\wedge} -Priestley relation

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From \wedge -(semi)-homomorphisms to their dual relations

For a \wedge -semi-homomorphism $h: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$

 $\overline{R_h \subseteq \operatorname{Irr}_{\rightarrow}(\mathsf{A}_2) \times \operatorname{Irr}_{\rightarrow}(\mathsf{A}_1)}$ is given by:

 $(P,Q) \in R_h$ iff $h^{-1}[P] \subseteq Q$

is a DH^{\wedge} -Spectral relation

 $egin{aligned} R_h \subseteq \mathrm{Op}_{
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ightarrow}(\mathbf{A}_1) ext{ is given by:} \ & (P,Q) \in R_h ext{ iff } h^{-1}[P] \subseteq Q \end{aligned}$

is a DH^{\wedge} -Priestley relation

Similarly for \wedge -homomorphisms and functional relations







• \Box_R is a \wedge -semi-homomosphism (\wedge -homomorphism).

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For a DH^{\wedge} -Spectral (functional) relation $R \subseteq X_1 \times X_2$

$$\Box_R : D(\mathfrak{X}_2) \longrightarrow D(\mathfrak{X}_1)$$
$$U \longmapsto \{x \in X_1 : R(x) \subseteq U\}$$

• \Box_R is a \wedge -semi-homomosphism (\wedge -homomorphism).

For a *DH*^{\wedge}-Priestley (functional) relation $R \subseteq X_1 \times X_2$

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Dualities for Implicative semilattices as a particular case

$a \wedge b \leq c \text{ iff } c \leq a \rightarrow b$

Spectral-like duality

IS spaces (defined in *Celani* [2003])

 $DH^{\wedge} ext{-}\operatorname{Spectral spaces}\ \langle X, au_{\kappa},\hat{X}
angle$ such that $\hat{X}=X$

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Implicative semilattices Conclusion

Dualities for Implicative semilattices as a particular case

$a \wedge b \leq c \text{ iff } c \leq a \rightarrow b$

$\begin{array}{c} \textbf{Spectral-like duality} \\ \textit{IS spaces} & \Leftrightarrow & \textit{DH}^{\wedge}\text{-Spectral spaces} \\ (\text{defined in }\textit{Celani [2003]}) & & & & & & & \\ & & & \hat{X} = X \end{array}$

Priestley-style duality

∗-generalized Esakia spaces ⇔ (defined in *G. Bezhanishvili and* Jansana [2011b]) $\begin{array}{l} \mathcal{DH}^{\wedge}\text{-Priestley spaces}\\ \langle X,\tau,\leq,\mathcal{S},\hat{X}\rangle \text{ such that}\\ \hat{X}=X \end{array}$

Conclusion and further work

► We have considered as objects of our categories, DH[^]-algebras, ⟨A, →, ∧, 1⟩

 $a \wedge b = a$ iff $a \rightarrow b = 1$

- We use the existing dualities for Hilbert algebras
- We add a subset, to represent the conjunction
 - What about the dualities for H^{\wedge} -algebras?
 - May this strategy be followed for other classes of algebras?

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Preliminaries	Spectral-like duality	Priestley-style duality	Objects	Morphisms	Implicative semilattices	Conclusion
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Thanks!

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