Sheaf representations of MV-algebras via Priestley duality

Mai Gehrke
(joint work with Sam van Gool and Vincenzo Marra)

15 August 2012
Duality Theory in Algebra, Logic and Computer Science, Oxford
Definition

An MV-algebra is an algebra \((A, \oplus, \neg, 0)\) such that

- \((A, \oplus, 0)\) is a commutative monoid,
- \(\neg\neg x = x\), that is, \(\neg\) is an involution,
- \(x \oplus 1 = 1\) where \(1 := \neg 0\),
- \(\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x\).

MV-algebras are bounded distributive lattices in the term-definable operations:

\[
a \lor b := \neg(\neg a \oplus b) \oplus b,
\]
\[
a \land b := \neg(\neg a \lor \neg b).
\]
Examples

- Boolean algebras;

- The unit interval MV-algebra: $[0, 1]$ with
  \[ a \oplus b := \min\{a + b, 1\}, \quad \neg a := 1 - a; \]
  
  The associated $\leq$ is the natural order on $[0, 1]$.

- For a space $X$, the MV-algebra $C(X, [0, 1])$ of continuous functions from $X$ to $[0, 1]$, operations & order given pointwise;

- Ultrapowers of $[0, 1]$ — which have infinitessimals.
The dual space of an MV-algebra

An MV-algebra $A$ is in particular a bounded distributive lattice. As such it has a dual space $X$.

$$X(\cong \text{Hom}_{DL}(A, 2) \cong \text{pFilt}(A) \cong \text{plIdl}(A))$$

with the order

$$x \leq y \iff F_x \supseteq F_y \iff I_x \subseteq I_y$$

and

$$\hat{a} = \{ x \in X \mid a \in F_x \}$$

$$\sigma^\downarrow = \langle \hat{a} \mid a \in A \rangle$$

spectral topology

$$\sigma^\uparrow = \langle \hat{a}^c \mid a \in A \rangle$$

dual spectral topology

$$\sigma^p = \sigma^\downarrow \vee \sigma^\uparrow$$

Priestley topology
Extended duality for MV-algebras

The lattice structure alone does not uniquely determine the MV-algebra in general. Consequently additional structure on $X$ is needed to account for the MV-algebraic structure.

Such structure has been identified in work of N. G. Martínez [1996]; N. G. Martínez & Priestley [1998]; G & Priestley [2008], but has not interacted much with the representation theory for MV-algebras in general and geometric duality theory for special classes, both developed on the basis of MV-spectra.
Principal congruences of an MV-algebra

A simple but important fact in the representation theory of MV-algebras is that

\[ \theta : A \rightarrow \text{Con}(A) \]
\[ a \mapsto \theta(a) = \langle (0, a) \rangle_{\text{Con}(A)} \]

is a bounded lattice homomorphism.
Principal congruences of an MV-algebra

A simple but important fact in the representation theory of MV-algebras is that

\[ \theta : A \rightarrow Con(A) \]

\[ a \mapsto \theta(a) = \langle 0, a \rangle_{Con(A)} \]

is a bounded lattice homomorphism.

The image of this map is the lattice \( Con_{fin}(A) \) of finitely generated MV-algebra congruences of \( A \).
Principal congruences of an MV-algebra

A simple but important fact in the representation theory of MV-algebras is that

\[
\theta : A \longrightarrow \text{Con}(A)
\]

\[
a \longmapsto \theta(a) = <(0, a) >_{\text{Con}(A)}
\]

is a bounded lattice homomorphism.

The image of this map is the lattice \( \text{Con}_{\text{fin}}(A) \) of finitely generated MV-algebra congruences of \( A \).

The \textbf{MV-spectrum} of \( A \), is the dual space, \( Y \), of \( \text{Con}_{\text{fin}}(A) \).
The MV-spectrum as a subspace of the dual space

Since $A \xrightarrow{\rightarrow} Con_{fin}(A)$ is a bounded distributive lattice quotient, by duality, $Y \hookrightarrow X$ may be seen as a closed subspace of $X$:

$$Y = \{ y \in X | I_y \text{ is closed under } \oplus \}$$

We will mainly consider $Y$ in its spectral topology and its dual spectral topology. These are equal to the subspace topologies for the spectral and dual spectral topologies on $X$, respectively.
The MV-spectrum directly from the MV-algebra

The congruences of an MV-algebra are in 1-to-1 correspondence with MV-ideals: non-empty downsets closed under $\oplus$.

The MV-spectrum may also be seen as the set of those MV-ideals that are prime in the sense that one of $a \ominus b (:= \neg(\neg a \oplus b))$ and $b \ominus a$ is a member for all $a, b \in A$. This is the same set $Y \subseteq X$.

The spectral topology on $Y$ as determined on the previous slide is also the hull-kernel or spectral topology corresponding to the MV-ideals of $A$. 
The maximal MV-spectrum

Given an MV-algebra, $A$, the subspace $Z$ of $Y$ of maximal MV-ideals of $A$ is called the maximal MV-spectrum. It is compact Hausdorff, but not in general spectral.

Examples

- If $A =$ the free $n$-generated MV-algebra, then $Z$ is homeomorphic to the cube $[0, 1]^n$ with the Euclidean topology.
  
  - Free$_n$ embeds in $C([0, 1]^n, [0, 1])$ but the embedding is not unique.

- If $A$ is a Boolean algebra, then $Z$ is its Stone dual space.

- If $A$ is any chain, then $Z$ is the one-point space.

- If $A$ has infinitesimals, then we do not have $A \hookrightarrow C(Z, [0, 1])$. 
Well-known facts from the literature:

- The following are equivalent:
  - A bounded distributive lattice $D$ is normal: For all $a, b \in D$, if $a \lor b = 1$ then there are $c, d \in A$ with $c \land d = 0$ and $a \lor d = 1$ and $c \lor b = 1$.
  - Each point in the dual space of $D$ is below a unique maximal point
  - The inclusion of the maximal points of the dual space of $D$ admits a continuous retraction
  - For any MV-algebra $A$, the lattice $\text{Con}_{\text{fin}}(A)$ is relatively normal (that is, each interval $[a, b]$ is a normal lattice).

As a consequence $Y$ is always a root-system, that is, $\uparrow y$ is a chain for each $y \in Y$. The space $Z$ is always compact Hausdorff, and the map

$$m : Y \to Z, \quad y \mapsto \text{unique maximal point above } y$$

is a continuous retraction
The map $k$

There is a continuous retraction $k : (X, \sigma^p) \longrightarrow (Y, \sigma^\downarrow)$ (already present in the work of Martínez)

This map may be given a simple description:

$$k(x) = \max\{z \in X \mid l_x \oplus l_z \subseteq l_x\}$$

yielding

(Interpolation Lemma) If $x \leq x'$ then there is $x''$ with

$$x \leq x'' \leq x' \quad \text{and} \quad k(x'') \geq k(x) \quad \text{and} \quad k(x'') \geq k(x')$$
From $X$ to $Z$ without using the MV structure

Combining the two earlier retractions we get

$$m \circ k : (X, \sigma^p) \to (Z, \sigma^\downarrow)$$

The kernel of this map is given by the relation $x_1 W x_2$ iff there are $x'_1, x'_2, x_0 \in X$ with

Proof: If $mk(x_1) = mk(x_2)$, then take $x'_i = k(x_i)$ and $x_0 = mk(x_i)$.

For the converse note that if $x \leq x'$, then by (Int) there is $x''$ between with greater $k$-image than both, but then $mk(x) = mk(x'') = mk(x')$. So all the elements of $X$ in one order component have the same $mk$-image.
Kaplansky’s theorem

[Kaplansky 1947]
Let $Z_1, Z_2$ be compact Hausdorff spaces such that the lattices $C(Z_1, [0, 1])$ and $C(Z_2, [0, 1])$ are isomorphic. Then $Z_1$ and $Z_2$ are homeomorphic spaces.
Kaplansky theorem for arbitrary MV-algebras

**Theorem**

If $A_1$ and $A_2$ are MV-algebras having isomorphic lattice reducts, then the max MV-spectra of $A_1$ and $A_2$ are homeomorphic.

- Note that the max MV-spectrum of an MV-algebra of the form $C(Z, [0, 1])$ is $Z$ so that our result generalizes Kaplansky's result.

**Proof (sketch).**

The maximal MV-spectrum can be reconstructed from the lattice spectrum using the relation $W$. 

□
Sheaf representations over a spectral space

\[ F : \text{CompOp}(Y) \to \text{MV} \text{ functor satisfying some unicity and gluing properties and } A \cong F(Y) \]

alternatively

\[ e : E \to Y \text{ local homeomorphism with MV-algebra stalks and } A \cong \text{GlobSec}(e) \]

alternatively

\[ A \hookrightarrow \prod_{y \in Y} A_y \text{ subdirect product with} \]
- open equalizers
- a finite patching property

(weak spectral product)
Spectral products and duality
Spectral sum

Let $X$ (with $\sigma^\downarrow$) be the dual space of $A$, and let $Y$ be a spectral space (with $\sigma^\uparrow$), then $X$ is a spectral sum over $Y$ provided there is a continuous surjection $k : (X, \sigma^p) \rightarrow (Y, \sigma^\downarrow)$ satisfying:

- For any $a, b \in A$, the equalizer

\[
\|a = b\| := \{y \in Y \mid \hat{a} \cap X_y = \hat{b} \cap X_y\}
\]

is compact open where $X_y = k^{-1}(\uparrow y)$

- Let $(U_i)_{i=1}^n$ be a compact open cover of $Y$, and $(a_i)_{i=1}^n \subseteq A$ with $U_i \cap U_j \subseteq \|a_i = a_j\|$. Then there exists a unique element $b \in A$ such that $U_i \subseteq \|a_i = b\|$

**Theorem**

If $X$ is a spectral sum over $Y$, then $A$ is a spectral product over $Y$
The dual space of an MV-algebra as a spectral sum

Let $A$ be an MV-algebra, and $X$ its dual space. Let $Y$ be the MV-spectrum of $A$ in its dual spectral topology (that is, with $\sigma^\uparrow$).

Theorem (G, van Gool, Marra)

$X$ is a spectral sum over $Y$, and thus, in particular, $A$ has a sheaf representation over $Y$.

- The ensuing sheaf representation is the one of [Dubuc & Poveda, 2010]
- The map $k : X \rightarrow Y$ is used and $X_y = k^{-1}(\uparrow y)$ is the dual of the quotient $A_y = A/I_y$ for each $y \in Y$
- Our proof uses only elementary facts about MV-algebras and, crucially, the Interpolation Lemma
Equalizers are compact open

For any \( a, b \in A \), the equalizer

\[
\|a = b\| = \{ y \in Y \mid \hat{a} \cap X_y = \hat{b} \cap X_y \}
\]

\[
= \{ y \in Y \mid \llbracket a \rrbracket_y = \llbracket b \rrbracket_y \}
\]

\[
= \{ y \in Y \mid (a, b) \in \theta_y \}
\]

where \( \llbracket a \rrbracket_y \) is the equivalence class of \( a \) in the quotient \( A/I_y \) and \( \theta_y \) is the corresponding congruence relation on \( A \).

It is well known that

\[
(a, b) \in \theta_y \iff d(a, b) = (a \ominus b) \oplus (b \ominus a) \in I_y
\]

Thus we have

\[
\|a = b\| = (\overline{d(a, b)})^c
\]
Sheaf representations of MV-algebras via Priestley duality

Patching property

\[
\begin{align*}
\exists \hat{b} & \quad U_1 \quad U_2 \\
E & \quad p \\
Y & \quad U_1 \\
\end{align*}
\]
Sheaf representations of MV-algebras via Priestley duality

Patching property

$$\exists! \hat{b}$$

$$E$$

$$p$$

$$A/y$$

$$U_1$$

$$U_2$$

$$Y$$

$$\hat{a}_1$$

$$\hat{a}_2$$
Transparent solution via duality

- There is at most one $b \in A$,
- It must satisfy $\hat{b} \in \bigcup_{i=1}^{n} (\hat{a}_i \cap k^{-1}(U_i)) =: K$, and $K$ is closed
- Prove that $K$ is a open: $K^c = \bigcup_{i=1}^{n} (\hat{a}_i^c \cap k^{-1}(U_i))$
- Prove that $K$ is a downset: Interpolation Lemma!!!
- This also yields a formula for $b$ by compact approximation:

$$b = \bigvee_{i=1}^{n} (a_i \odot \neg mu_i),$$

where $m, n \in \mathbb{N}$ and $u_i \in A$ such that $\hat{u}_i^c = U_i$. 