Sheaf representations of MV-algebras via Priestley duality

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Definition

An MV-algebra is an algebra $(A, \oplus, \neg, 0)$ such that

- $(A, \oplus, 0)$ is a commutative monoid,
- $\neg\neg x = x$, that is, $\neg$ is an involution,
- $x \oplus 1 = 1$ where $1 := \neg 0$,
- $\neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$.

- MV-algebras are bounded distributive lattices in the term-definable operations:

$$a \vee b := \neg(\neg a \oplus b) \oplus b,$$
$$a \wedge b := \neg(\neg a \vee \neg b).$$

Examples

- Boolean algebras;
- The unit interval MV-algebra: $[0,1]$ with
  $$a \oplus b := \min\{a+b, 1\}, \quad \neg a := 1 - a;$$
  The associated $\leq$ is the natural order on $[0,1]$
- For a space X, the MV-algebra $C(X,[0,1])$ of continuous functions from X to $[0,1]$, operations & order given pointwise;
- Ultrapowers of $[0,1]$ — which have infinitessimals

The dual space of an MV-algebra

An MV-algebra A is in particular a bounded distributive lattice. As such it has a dual space X.

$$X(\cong Hom_{DL}(A,2) \cong pFilt(A) \cong pIdl(A))$$

with the order

$$x \leq y \iff F_x \supseteq F_y \iff I_x \subseteq I_y$$

and

$$\widehat{a} = \{x \in X \mid a \in F_x\}$$

$\sigma^{\downarrow} = <\widehat{a} \mid a \in A>$	$\sigma^{\uparrow} = <\widehat{a}^c \mid a \in A>$	$\sigma^p = \sigma^{\downarrow} \vee \sigma^{\uparrow}$
spectral topology	dual spectral topology	Priestley topology

Extended duality for MV-algebras

The lattice structure alone does not uniquely determine the MV-algebra in general. Consequently additional structure on X is needed to account for the MV-algebraic structure.

Such structure has been identified in work of N. G. Martínez [1996]; N. G. Martínez & Priestley [1998]; G & Priestley [2008], but has not interacted much with the representation theory for MV-algebras in general and geometric duality theory for special classes, both developed on the basis of MV-spectra.

Principal congruences of an MV-algebra

A simple but important fact in the representation theory of MV-algebras is that

$$\begin{aligned} \theta : A &\longrightarrow Con(A) \\ a &\longmapsto \theta(a) = <(0, a)>_{Con(A)} \end{aligned}$$

is a bounded lattice homomorphism.

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The MV-spectrum of A, is the dual space, Y, of $Con_{fin}(A)$

The MV-spectrum as a subspace of the dual space

Since $A \twoheadrightarrow Con_{fin}(A)$ is a bounded distributive lattice quotient, by duality, $Y \hookrightarrow X$ may be seen as a closed subspace of X:

$$Y = \{y \in X \mid I_y \text{ is closed under } \oplus\}$$

We will mainly consider Y in its spectral topology and its dual spectral topology. These are equal to the subspace topologies for the spectral and dual spectral topologies on X, respectively.

The MV-spectrum directly from the MV-algebra

The congruences of an MV-algebra are in 1-to-1 correspondence with MV-ideals: non-empty downsets closed under $\oplus$.

The MV-spectrum may also be seen as the set of those MV-ideals that are prime in the sense that one of $a \ominus b (:= \neg(\neg a \oplus b))$ and $b \ominus a$ is a member for all $a, b \in A$. This is the same set $Y \subseteq X$.

The spectral topology on Y as determined on the previous slide is also the hull-kernel or spectral topology corresponding to the MV-ideals of A.

The maximal MV-spectrum

Given an MV-algebra, A, the subspace Z of Y of maximal MV-ideals of A is called the maximal MV-spectrum. It is compact Hausdorff, but not in general spectral.

Examples

- If $A =$ the free n-generated MV-algebra, then Z is homeomorphic to the cube $[0, 1]^n$ with the Euclidean topology.
  - Free_n embeds in $C([0, 1]^n, [0, 1])$ but the embedding is not unique.
- If A is a Boolean algebra, then Z is its Stone dual space.
- If A is any chain, then Z is the one-point space.
- If A has infinitesimals, then we do not have $A \hookrightarrow C(Z, [0, 1])$.

Well-known facts from the literature:

- The following are equivalent:
  - A bounded distributive lattice D is normal: For all $a, b \in D$, if $a \vee b = 1$ then there are $c, d \in A$ with $c \wedge d = 0$ and $a \vee d = 1$ and $c \vee b = 1$.
  - Each point in the dual space of D is below a unique maximal point
  - The inclusion of the maximal points of the dual space of D admits a continuous retraction
- For any MV-algebra A, the lattice $Con_{fin}(A)$ is relatively normal (that is, each interval $[a, b]$ is a normal lattice).

As a consequence Y is always a root-system, that is, $\uparrow y$ is a chain for each $y \in Y$. The space Z is always compact Hausdorff, and the map

$$m : Y \longrightarrow Z, \ y \mapsto \text{ unique maximal point above } y$$

is a continuous retraction

The map k

There is a continuous retraction $k : (X, \sigma^p) \longrightarrow (Y, \sigma^{\downarrow})$ (already present in the work of Martínez)

This map may be given a simple description:

$$k(x) = \max\{z \in X \mid I_x \oplus I_z \subseteq I_x\}$$

yielding

(Interpolation Lemma) If $x \leq x'$ then there is x'' with

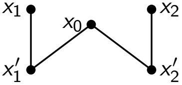
$$x \leq x'' \leq x' \quad \text{and} \quad k(x'') \geq k(x) \quad \text{and} \quad k(x'') \geq k(x')$$

From X to Z without using the MV structure

Combining the two earlier retractions we get

$$m \circ k : (X, \sigma^p) \longrightarrow (Z, \sigma^{\downarrow})$$

The kernel of this map is given by the relation $x_1 W x_2$ iff there are $x_1', x_2', x_0 \in X$ with



Proof: If $mk(x_1) = mk(x_2)$, then take $x_i' = k(x_i)$ and $x_0 = mk(x_i)$.

For the converse note that if $x \leq x'$, then by (Int) there is x'' between with greater k-image than both, but then $mk(x) = mk(x'') = mk(x')$. So all the elements of X in one order component have the same mk-image

Kaplansky's theorem

[Kaplansky 1947]
Let Z_1, Z_2 be compact Hausdorff spaces such that the lattices $C(Z_1, [0,1])$ and $C(Z_2, [0,1])$ are isomorphic. Then Z_1 and Z_2 are homeomorphic spaces.

Kaplansky theorem for arbitrary MV-algebras

Theorem

If A_1 and A_2 are MV-algebras having isomorphic lattice reducts, then the max MV-spectra of A_1 and A_2 are homeomorphic.

- Note that the max MV-spectrum of an MV-algebra of the form $C(Z,[0,1])$ is Z so that our result generalizes Kaplansky's result.

Proof (sketch).

The maximal MV-spectrum can be reconstructed from the lattice spectrum using the relation W. □

Sheaf representations over a spectral space

$F : CompOp(Y) \longrightarrow MV$ functor satisfying some unicity and gluing properties and $A \cong F(Y)$

alternatively

$e : E \longrightarrow Y$ local homeomorphism with MV-algebra stalks and $A \cong GlobSec(e)$

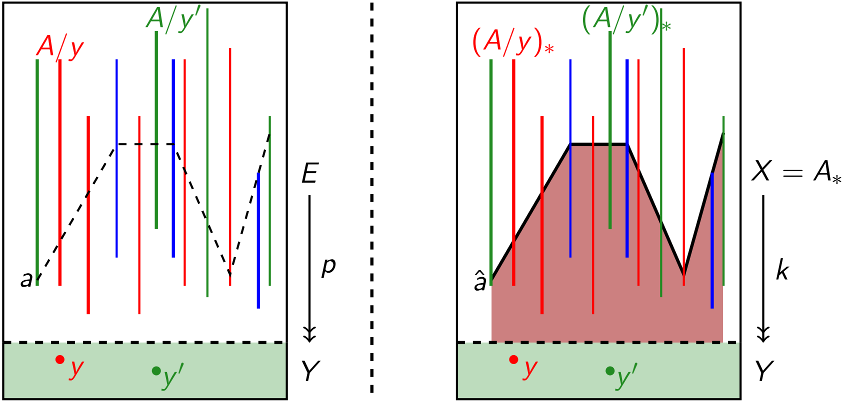
alternatively

$A \hookrightarrow \Pi_{y \in Y} A_y$ subdirect product with

- open equalizers
- a finite patching property

(weak spectral product)

Spectral products and duality



Spectral sum

Let X (with $\sigma^{\downarrow}$) be the dual space of A, and let Y be a spectral space (with $\sigma^{\uparrow}$), then X is a spectral sum over Y provided there is a continuous surjection $k : (X, \sigma^{p}) \longrightarrow (Y, \sigma^{\downarrow})$ satisfying:

- For any $a, b \in A$, the equalizer

$$\|a = b\| := \{y \in Y \mid \widehat{a} \cap X_y = \widehat{b} \cap X_y\}$$

  is compact open where $X_y = k^{-1}(\uparrow y)$
- Let $(U_i)_{i=1}^{n}$ be a compact open cover of Y, and $(a_i)_{i=1}^{n} \subseteq A$ with $U_i \cap U_j \subseteq \|a_i = a_j\|$. Then there exists a unique element $b \in A$ such that $U_i \subseteq \|a_i = b\|$

Theorem

If X is a spectral sum over Y, then A is a spectral product over Y

The dual space of an MV-algebra as a spectral sum

Let A be an MV-algebra, and X its dual space. Let Y be the MV-spectrum of A in its dual spectral topology (that is, with $\sigma^{\uparrow}$)

Theorem (G, van Gool, Marra)

X is a spectral sum over Y, and thus, in particular, A has a sheaf representation over Y.

- The ensuing sheaf representation is the one of [Dubuc & Poveda, 2010]
- The map $k : X \longrightarrow Y$ is used and $X_y = k^{-1}(\uparrow y)$ is the dual of the quotient $A_y = A/I_y$ for each $y \in Y$
- Our proof uses only elementary facts about MV-algebras and, crucially, the Interpolation Lemma

Equalizers are compact open

For any $a, b \in A$, the equalizer

$$\begin{aligned} \|a = b\| &= \{y \in Y \mid \widehat{a} \cap X_y = \widehat{b} \cap X_y\} \\ &= \{y \in Y \mid [a]_y = [b]_y\} \\ &= \{y \in Y \mid (a, b) \in \theta_y\} \end{aligned}$$

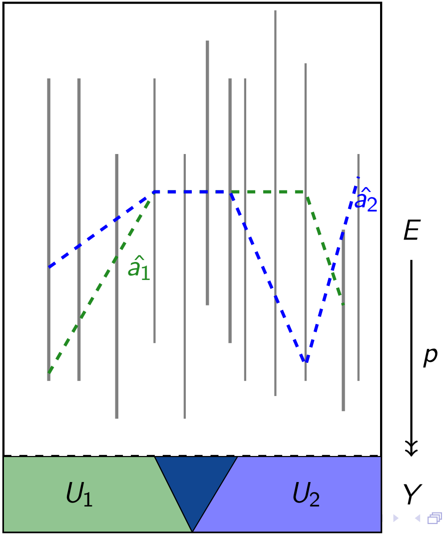
where $[a]_y$ is the equivalence class of a in the quotient A/I_y and θ_y is the corresponding congruence relation on A.
It is well known that

$$(a, b) \in \theta_y \iff d(a, b) = (a \ominus b) \oplus (b \ominus a) \in I_y$$

Thus we have

$$\|a = b\| = (\widehat{d(a, b)})^c$$

Patching property



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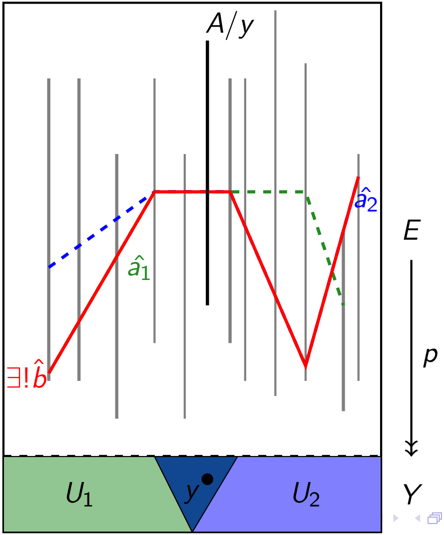

Transparent solution via duality

- There is at most one $b \in A$,
- It must satisfy $\hat{b} = \bigcup_{i=1}^{n}(\hat{a}_i \cap k^{-1}(U_i)) =: K$, and K is closed
- Prove that K is a open: $K^c = \bigcup_{i=1}^{n}(\hat{a}_i{}^c \cap k^{-1}(U_i))$
- Prove that K is a downset: Interpolation Lemma!!!
- This also yields a formula for b by compact approximation:

$$b = \bigvee_{i=1}^{n}(a_i \odot \neg m u_i),$$

where $m, n \in \mathbb{N}$ and $u_i \in A$ such that $\hat{u}_i{}^c = U_i$.