Research Workshop in

Duality Theory in Algebra, Logic and Computer Science

August 15-17, 2012

On the Logic of Perfect MV-algebras: Projectivity, Unification, Structural Completeness

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INTRODUCTION

Introduction

- It is well known that MV -algebras are algebraic models of infinitely-valued Lukasiewicz logic L [C.C. Chang] and that the structure of non-equivalent formulas of L forms an ω-generated free MV-algebra, which is named Lindenbaum algebra. If we restrict the above structure to non-equivalent formulas with m propositional variables, then we will have the m-generated free MV -algebra.

Perfect *MV* -algebras are those *MV*–algebras generated by their infinitesimal elements or, equivalently, generated by their radical, where radical is the intersection of all maximal ideals. The radical of an MV -algebra, will be denoted by Rad(A).

Let we have any *MV* -algebra. The least integer for which $nx = 1$ is called *the order of* x. When such an integer exists, we denote it by $ord(x)$ and say that x has finite order, otherwise we say that x has infinite order and write $ord(x) = 1$.

An *MV* -algebra A is called *perfect* if for every nonzero element $x \in A$ $ord(x) = 1$ if and only if $ord(\neg x) < 1$.

An important example of a perfect *MV*-algebra is the subalgebra *S of the Lindenbaum algebra L of the first or*der Lukasiewicz logic generated by the classes of formulas which are valid when interpreted in [0, 1] *but* non-provable.

Hence perfect MV-algebras are directly connected with the very important phenomenon of incompleteness in Lukasiewicz first order logic .

PRELIMINARIES

Definition 1. *An algebra* $A = (A; \otimes, \oplus, *, 0, 1)$ *is said to be an MV-algebra iff it satisfies the following equations:*

1) $(x \oplus y) \oplus z = x \oplus (y \oplus z)$;

2) $x \oplus y = y \oplus x$;

3) $x \oplus 0 = x$;

4) $x \oplus 1 = 1$;

5) $0^* = 1$;

6) $1^* = 0$;

7) $x \otimes y = (x^* \oplus y^*)^*$;

8) $(x^* \oplus y)^* \oplus y = (y^* \oplus x)^* \oplus x$.

The unit interval of real numbers [0, 1] endowed with the following operations:

$x \oplus y = \min(1, x + y)$, $x \otimes y = \max(0, x + y - 1)$,

$x^* = 1 - x$, becomes an *MV*-algebra.

- **Definition 2**. *An algebra* $A = (A; \otimes, \oplus, *, 0, 1)$ *is said to be an MV(C)-algebra if it is MV-algebra and satisfies the following axiom*

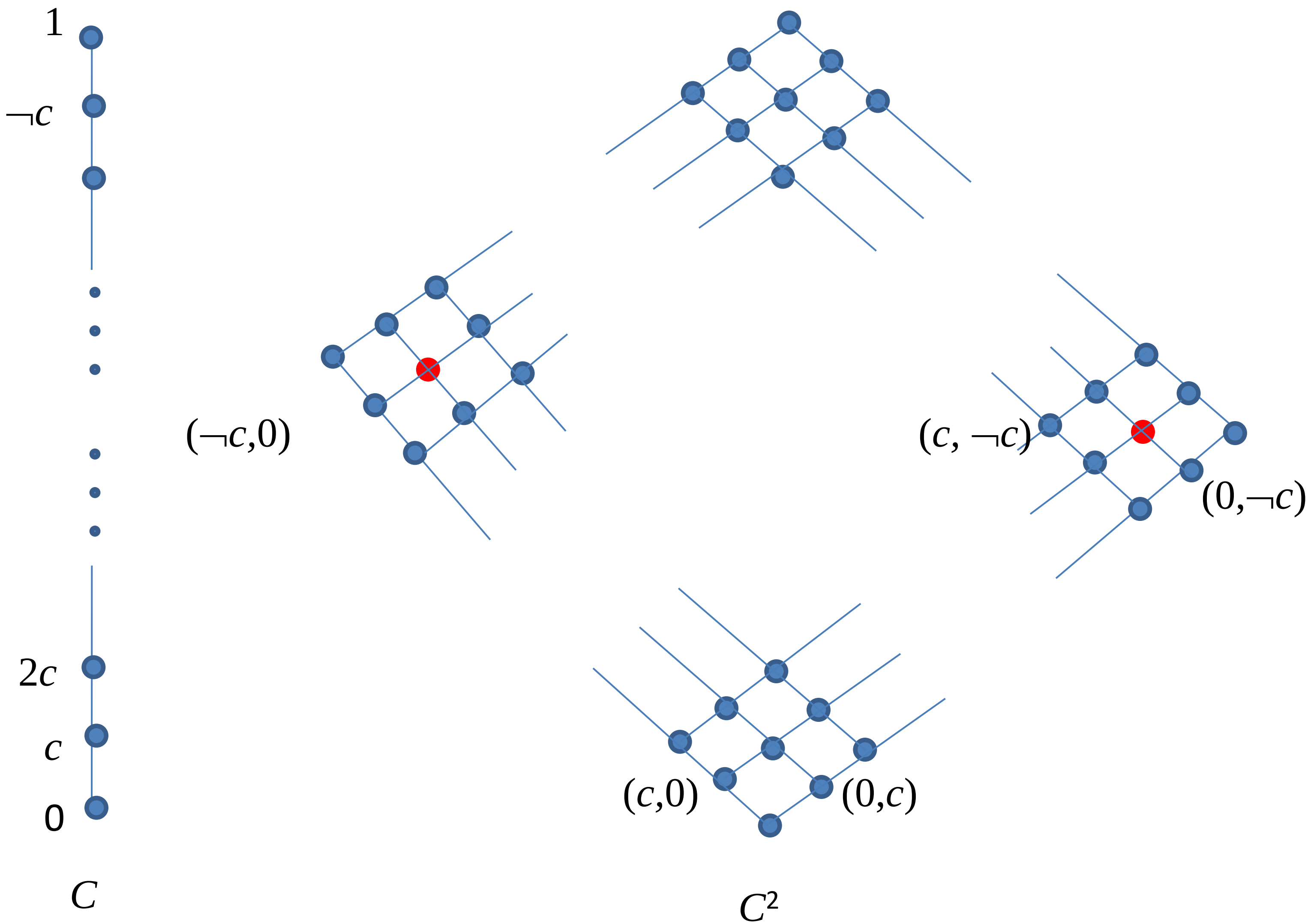
$$2(x^2) = (2x)^2$$

An algebra A is said to be *a free algebra in a variety* **K**, if there exists a set $A_0 \subset A$ *such that* A_0 *generates* A *and every mapping* f from A_0 *to any algebra* $B \in$ **K** *is extended to a homomorphism* h *from* A *to* B*. In this case* A_0 *is said to be the set of free generators of* A*.*

If the set of free generators is finite then A *is said to be a finitely generated free algebra .*

Perfect *MV* -algebras do not form a variety and contains non-simple subdirectly irreducible *MV*–algebras. The variety generated by all perfect *MV* –algebras is also generated by a single *MV* -chain, actually the *MV* –algebra C, defined by Chang. The algebra C, with generator $c \in C$, is isomorphic to $\Gamma(Z \times Z,(1, 0))$, with generator (0, 1). Let **MV(C)** be the variety generated by perfect algebras.

Theorem 3. *An* 1-*generated free* $MV(\mathrm{C})$-*algebra* $F_{\mathbf{MV(C)}}(1)$ *is isomorphic to* C^2 *with free generator* $(c,\neg c)$.



1
¬c
2c
c
0
C
(¬c,0)
(c, ¬c)
(0,¬c)
(c,0)
(0,c)
C²

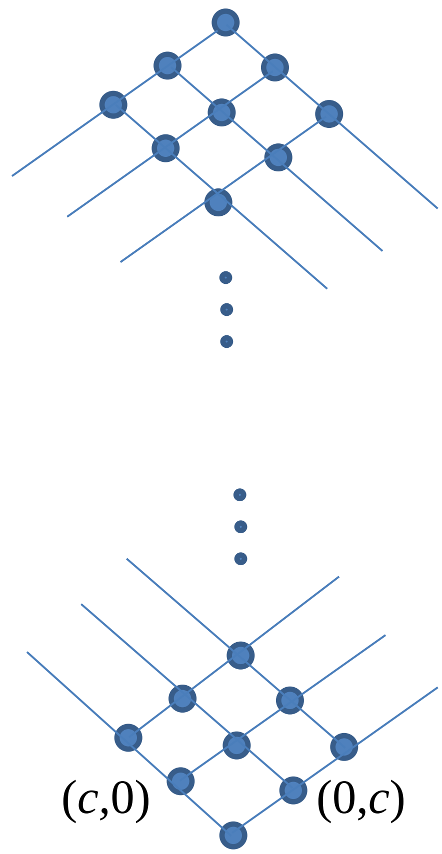
Let us introduce some notations:

let $C_1 = C = \Gamma(\mathrm{Z} \times \mathrm{Z}, (1, 0))$ with generator $c_1 (= c)$, $C_m = \Gamma(\mathrm{Z} \times \ldots \times \mathrm{Z}, (1,0, \ldots, 0))$ with generators

$$c_1 (= (0, \ldots, 1)), \ldots, c_m (= (0, 1, 0, \ldots, 0)),$$

where the number of factors Z is equal to $m + 1$ and $m > 1$. Let us denote

$\mathrm{Rad}(A) \cup \neg\mathrm{Rad}(A)$ through $\mathrm{R}^*(A)$.



$R^*(C^2)$

Taking into account an order of generators of C_n it is generated by

$$\mathbf{a}_i = (c_{\varphi i(1)}, c_{\varphi i(2)}, \ldots , c_{\varphi i(n)})$$

for any $i \in \{1, \ldots , n!\}$, where

$$\varphi_i : \{1, \ldots , n\} \to \{1, \ldots , n\}$$

is any bijection. We have $n!$ diferent ordered generators that generate C_n.

Let B_n be the subalgebra of the algebra $C_n{}^{n!}$ generated by n generators

$$\mathbf{b}_{\boldsymbol{i}} = (\pi_i(\mathbf{a}_1), \pi_i(\mathbf{a}_2), \dots , \pi_i(\mathbf{a}_{n!})),$$

$i = 1, \dots , n.$

Notice, that the generators $\mathbf{b}_1, \dots , \mathbf{b}_n$ belong to Rad($C_n{}^{n!}$).

Therefore the algebra B_n is perfect. If n = 2, then $B_2 = R^*(C^2)$

Let us consider subalgebra A_k of the algebra $\Pi_{i=1}^{\infty} D_i^{(k)}$, where $D_i^{(k)} = C_k$ $(1 \leq k < n)$, generated by

$$\mathbf{d}_j^{(\mathrm{k})} = (u_{1k}^{(j)}, u_{2k}^{(j)}, u_{3k}^{(j)}, \ldots , u_{ik}^{(j)} , \ldots),$$

$j = 1, \ldots , n$, where $u_{ik}^{(1)}, \ldots , u_{ik}^{(n)}$ generate $D_i^{(k)}$, $(u_{ik}^{(1)}, \ldots , u_{ik}^{(n)}) \neq (u_{jk}^{(1)}, \ldots , u_{jk}^{(n)})$ for $i \neq j$ and $u_{ik}^{(1)} < \ldots < u_{ik}^{(n)}$ for any i.

Let B be a subalgebra of

$$B_n \times A_1 \times \ldots \times A_{n-1}$$

generated by

$$\mathbf{g}_1 = (\mathbf{b}_1, \mathbf{d}_1^{(1)}, \ldots, \mathbf{d}_1^{(n-1)}),$$

$$\ldots$$

$$\mathbf{g}_n = (\mathbf{b}_n, \mathbf{d}_n^{(1)}, \ldots, \mathbf{d}_n^{(n-1)}) .$$

Theorem 4. *n-generated free MV (C)-algebra $F_{\mathbf{MV(C)}}(n)$ is isomorphic to B^{2^n} with free generators*

$$G_1 = (\mathbf{g}_1^{\varepsilon_{11}}, \mathbf{g}_1^{\varepsilon_{21}}, \ldots, \mathbf{g}_1^{\varepsilon_{2^n 1}}),$$
$$G_2 = (\mathbf{g}_2^{\varepsilon_{12}}, \mathbf{g}_2^{\varepsilon_{22}}, \ldots, \mathbf{g}_2^{\varepsilon_{2^n 2}}),$$
$$\ldots$$
$$G_n = (\mathbf{g}_n^{\varepsilon_{1n}}, \mathbf{g}_n^{\varepsilon_{2n}}, \ldots, \mathbf{g}_n^{\varepsilon_{2^n n}}),$$

where $\mathbf{g}_i^{\varepsilon} = \begin{cases} \mathbf{g}_i, & \varepsilon = 1 \\ \neg\, \mathbf{g}_i, & \varepsilon = 0 \end{cases}.$

DUALITY

Duality

- Priestley duality relates the category of bounded distributive lattices to the category of Priestley spaces by mapping each bounded distributive lattice L to its ordered space $\mathrm{F}(L)$ of prime filters, and mapping each Priestley space X to the bounded distributive lattice $L(X)$ of clopen up-sets of X.
- When restricted to Heyting algebras and Heyting spaces respectively, these mappings give the restricted Priestley duality for Heyting algebras.

A Heyting algebra is an algebra $(A, \vee, \wedge, \rightarrow, 0, 1)$ of type $(2, 2, 2, 0, 0)$, where $(A, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the binary operation $\rightarrow$, which is called implication, satisfies

$$(\forall a, b, x \in A) \;\; x \wedge a \leq b \Leftrightarrow x \leq a \rightarrow b.$$

A Boolean space is zero-dimensional, compact and Hausdorf topological space.

A Priestley space is a triple (X,R,Ω), where (X, Ω) is a Boolean space and R is an order relation on X such that, for all $x, y \in X$ with $x\overline{R}y$, there exists a clopen up-set V with $x \in V$ and $y \notin V$.

A morphism between Priestley spaces is a continuous order-preserving map.

We denote the category of Priestley spaces plus continuous order-preserving maps by PS.

A Heyting space X is a Priestley space such that $R^{-1}(U)$ is open for every open subset U of X.

A morphism between Heyting spaces, called a *strongly isotone map* (or *Heyting morphism* in other terminology), is a continuous map $\varphi\colon X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

We denote the category of Heyting spaces plus Heyting morphisms by $\textsc{HS}$.

For any Priestley space (X,R) we define $\mathrm{P}(X)$ as the set of all clopen up-sets of X.

For any $U, V \in \mathrm{P}(X)$ define:

$$U \vee V = U \cup V \text{ and } U \wedge V = U \cap V.$$

Then the algebra

$$\mathrm{P}((X,R)) = (P(X), \cup, \cap, \emptyset, X)$$

is a bounded distributive lattice.

On the other hand, for each bounded distributive lattice L, the set $\mathrm{F}(L)$ of all prime filters of L with the binary relation R on it, which is the inclusion between prime filters, and topologised by taking the family of

$$supp(a) = \{F \in \mathrm{F}(L) : a \in F\},$$

for $a \in L$, and their complements as a subbase, is an object of PS.

Therefore we have two contravariant functors

$$\mathrm{F} : \mathbf{D} \rightarrow \mathrm{PS} \text{ and } \mathrm{P} : \mathrm{PS} \rightarrow \mathbf{D}.$$

These functors establish a dual equivalence between the categories of bounded distributive lattices $\mathbf{D}$ and Priestley spaces PS.

For any Heyting space (X,R) and $U,V \in \mathcal{H}(X)$ (= the set of all clopen up-sets of X) define:

$$U \to V = X - (R^{-1}(U - V))$$

Then the algebra

$$\mathrm{H}((X,R)) = (\mathrm{H}(X), \cup, \cap, \to, \varnothing, X)$$

is a Heyting algebra.

We have two contravariant functors

$$\mathrm{F} : \mathbf{HA} \rightarrow \mathrm{HS} \text{ and } \mathrm{H} : \mathrm{HS} \rightarrow \mathbf{HA}.$$

These functors establish a dual equivalence between the categories **HA** and HS.

A Heyting algebra A is said to be *Göodel algebra* if it satisfies the linearity condition:

$$(a \to b) \vee (b \to a) = 1$$

for all $a, b \in A$.

It is well known that the Heyting spaces for Göodel algebras form root systems.

So we can define Gödel space X as such kind Heyting space that $R(x)$ is a chain for any $x \in X$.

The category of Gödel spaces and strongly isotone maps denote by GS.

An MV-space is Priestley space X such that $R(x)$ is a chain for any $x \in X$ and a morphism between MV-spaces is a strongly isotone map (or an MV- morphism), i. e. a continuous map $\varphi : X \to Y$ such that $\varphi(R(x)) = R(\varphi(x))$ for all $x \in X$.

Hence MV-space forms a root system.

We denote the category of MV-spaces plus MV-morphisms by MVS.

We are interested by subcategory MVSC of the category MVS, the objects of which are such kind of MV-spaces X for which there exist $MV(C)$-algebras A such that $\mathrm{M}(A) = X$, where $\mathrm{M}(A)$ is the set of all prime MV–filters of A.

We define binary relation $\equiv^*$ on MV-algebra A by the following stipulation:

$$x \equiv^* y \text{ iff } supp(x) = supp(y),$$

where *supp*(x) is defined as the set of all prime filters of A containing the element x. Then, it is a congruence with respect to $\otimes$ and $\vee$. The resulting set $\beta^*(A)(= A/\equiv^*)$ of equivalence classes is a bounded distributive lattice (which we call also the Belluce lattice of A).

We stress that β^* defines a covariant functor from the category of MV-algebras to the category of bounded distributive lattices. $\mathrm{M}(A)$ and $\mathrm{P}(\beta^*(A))$ are homeomorphic.

So, in the sequel we will use notation $\mathrm{P}(A)$ instead of $\mathrm{M}(A)$.

Theorem 5. *$\beta^*(F_{\mathbf{MV(C)}}(n))$ is a Gödel algebra*.

Corollary 6. *If A is finitely generated $MV(C)$-algebra, then $\beta^*(A)$ is a Heyting lattice.*

Proposition 7. *Let T be a finite root system. Then there exists an MV (C)-algebra A such that* P*(A) is isomorphic to T.*

Proposition 8. *If T_1, T_2 are finite root systems and $f : T_1 \to T_2$ is a strongly isotone map, then there exist MV (C)-algebras A_1, A_2 and MV -homomorphism $h : A_2 \to A_1$ such that $P(A_i) = T_i$, $i = 1, 2$.*

- **Theorem 9.** *The category of* MV (C)*-algebras with* MV *–homomorphisms as a morphisms is dually equivalent to the category of Gödel spaces with strongly isotone maps as a morphisms.*

Theorem 10. *The contravariant functors*

$$P : \mathrm{MVSC} \to \mathbf{MV(C)}$$

and

$$H : \mathbf{MV(C)} \to \mathrm{MVSC}$$

establish categorical duality between the categories MVSC *and* $\mathbf{MV(C)}$.

Theorem 11. *The category* $\mathbf{MV(C)}$ *of* $MV\,(C)$*-algebras with* MV*–homomorphisms as morphisms is dually equivalent to the category* GS *of Göodel spaces with strongly isotone maps as morphisms.*

Let E be an equivalence relation on MV -space X. Then the partition of X induced by E is the family $\{E(x) : x \in X\}$ of the classes $E(x) = \{y : xEy\}$ for $x \in X$. We say that a set $Y \subset X$ is E-saturated (or simply saturated) if $Y = E(Y) = \bigcup\{E(x) : x \in Y\}$, that is, if Y is the union of equivalence classes.

According to the duality there is one-to-one correspondence between homomorphic images of an $MV(\mathrm{C})$-algebra A and closed up-sets of $\mathrm{H}(A)$, and between subalgebras of a $MV(\mathrm{C})$- algebra A and *correct partitions* of $\mathrm{H}(A)$, where *a correct partition* of $(X,R) \in \mathrm{MVSC}$ is a such equivalence relation E on X that

- E is a closed equivalence relation, i. e. E-saturation of any closed subset is closed;
- E-saturation of any upper cone is an upper cone.

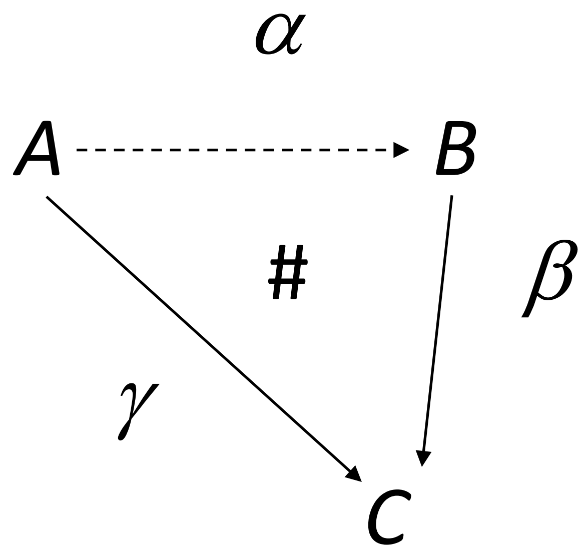
PROJECTIVITY AND UNIFICATION

Projectivity and Unification

Let $\mathbf{K}$ be a variety of algebras.

Definition 12. *An algebra* $A \in \mathbf{K}$ *is called* *projective*, *if for any* $B, C \in \mathbf{K}$, *any epimorphism (that is an onto homomorphism)* $\beta : B \to C$ *and any homomorphism* $\gamma : A \to C$, *there exists a homomorphism* $\alpha : A \to B$ *such that*

$$\beta\,\alpha = \gamma.$$


α
A
B
#
β
γ
C

- In varieties of algebras the projective algebras coincides with retracts of free algebras.

- An algebra A is a retract of an algebra B if there exist homomorphisms $\alpha : A \to B$ and $\beta : B \to A$ such that $\beta \alpha = \mathrm{Id}_A$.

Let us consider two-generated case.

Let A_1 be the subalgebra of the algebra $\Pi_{i=1}^{\infty} D_i^{(1)}$ where $D_i^{(1)} = C_1$, generated by

$$\mathbf{d}_j^{(1)} = (u_1^{(j)}, u_2^{(j)}, u_3^{(j)}, \dots , u_i^{(j)} , \dots),$$

$j = 1, 2$, where $u_i^{(1)}, u_i^{(2)}$ generate $D_i^{(1)}$ and $(u_i^{(1)}, u_i^{(2)}) \neq (u_j^{(1)}, u_j^{(2)})$ for $i \neq j$.

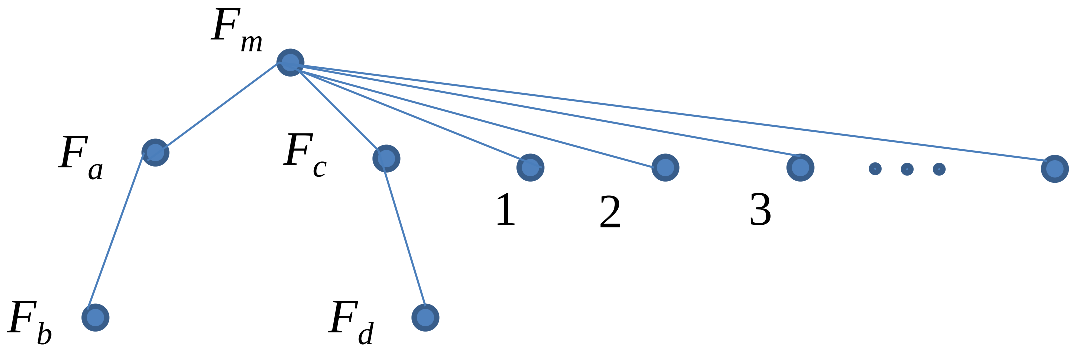
Let B_2 be the subalgebra of $C_2{}^2$ generated by $\mathbf{b}_1 = (c_1, c_2)$ and $\mathbf{b}_2 = (c_2, c_1)$ and B subalgebra of $C_2{}^2 \times A_1$ generated by

$g_1 = (c_1, c_2, u_1{}^{(1)}, u_2{}^{(1)}, u_3{}^{(1)}, \dots, u_i{}^{(1)}, \dots)$ and

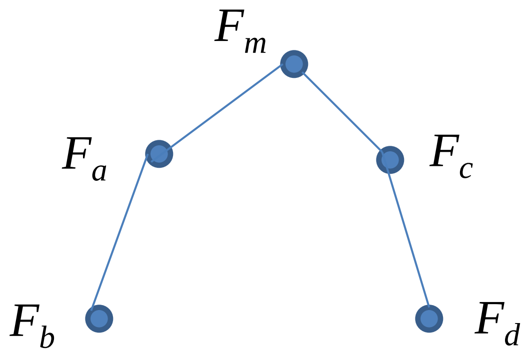
$g_2 = (c_2, c_1, u_1{}^{(2)}, u_2{}^{(2)}, u_3{}^{(2)}, \dots, u_i{}^{(2)}, \dots)$,

which is perfect $MV(C)$-algebra.

Notice that 2-generated free $MV(C)$-algebra $F_{\mathbf{MV(C)}}(2)$ is isomorphic to B^2 with free generators $G_1=(g_1,\neg g_1,g_1,\neg g_1)$, $G_2 = (g_2, g_2,\neg g_2,\neg g_2)$.



$\text{H}(B_2)$



$H(R^*(C_2^{\,2}))$

$F_m = [(\neg c_2, \neg c_2))$, $F_a = [(\neg c_2, 1))$, $F_b = [(\neg c_1, 1))$, $F_c = [(1, \neg c_2))$,
$F_d = [(1, \neg c_1))$.

Theorem 13. *Any two-generated finitely presented* $MV(C)$*-algebra* A *is projective.*

Theorem 13. *Any two-generated finitely presented $MV(C)$-algebra A is projective.*

Theorem 14. *Any n-generated finitely presented $MV(C)$-algebra A is projective.*

Theorem 13. *Any two-generated finitely presented* $MV(C)$*-algebra* A *is projective.*

Theorem 14. *Any* n*-generated finitely presented* $MV(C)$*-algebra* A *is projective.*

Theorem 15. *Any* m*-generated chain* $MV(C)$*-algebra* C_k $(k \leq m)$ *is projective.*

Projective formulas

A formula $\alpha \in F(m)$ is called ***projective*** if there exists a substitution $\sigma : P_m \to F(m)$ such that $\vdash \sigma(\alpha)$ and $\alpha \vdash \beta \leftrightarrow \sigma(\beta)$, for all $\beta \in F(m)$.

Sometimes we say that σ above *witnesses* the projectivety of α.

Proposition 16. *If* α *is a projective formula of m variables and* σ *witnesses its projectivity, then* $\sigma^2 = \sigma$.

Despite the rather syntactical definition, projective formulas are indeed in tightly connected to projective algebras.

Proposition 17. *Let* α *be a projective formula in m variables, then F(m)/*$[\alpha)$ *is a projective algebra.*

Unification

Notice, that if we have a formula α of a logic *L*, then its algebraic translation will be the equation $\alpha = 1$ understanding logical connective as a algebraic operations.

And vice versa, if we have an equation $\boldsymbol{p} = \boldsymbol{q}$, which is equivalent to $\boldsymbol{(p \to q) \wedge (q \to p)} = 1$, then its logical translation will be $(p \to q) \wedge (q \to p)$ understanding algebraic operations as logical connectives.

An *E-nification problem*

$$\Pi(x_1, \ldots, x_m)$$

is a pair given by a finite set of variables $x_1, \ldots, x_m$ and finite set of *m*-variable equations $\{r_i = q_i : i = 1, \ldots, n\}$.

A solution to it (or a *unifier* for it) is an endomorphism

$\sigma : F(m) \to F(m)$ such that $E \vdash \sigma(r_i) = \sigma(q_i)$,

where $F(m)$ is a free algebra with the free generators $x_1, \dots, x_m$ in the variety $\mathbf{V}_L$ corresponding to the logic *L*, and $E \vdash \psi$ (where ψ is a first-order formula, usually an equation) means a provability in the theory *E*.

- Let $U_E(\Pi)$ is the set of unifiers $\sigma : F(m) \to F(m)$ for the unification problem $\Pi(x_1, \ldots, x_m)$.
- Problem $P(x_1, \ldots, x_m)$ is *solvable* iff $U_E(P) \neq \varnothing$.

An endomorphism

$$\sigma_1 : F(m) \to F(m)$$

is *less general* than an endomorphism

$\sigma_2 : F(m) \to F(m)$ (in symbols $\sigma_1 \preccurlyeq \sigma_2$)

if and only if there is an endomorphism

$\tau : F(m) \to F(m)$ such that

$$E \vdash \tau(\sigma_2(x_i)) \leftrightarrow \sigma_1(x_i)$$

for all $i \in \{1, \ldots, m\}$

- Let $(\Sigma, \leq)$ be a poset, where $\leq$ is the ordering induced by the quasi-ordering identifying the equivalence classes with its elements.
- Max Σ is said to be *basis* of unifiers for P.

We say that a theory E *has unification type*

- 1, iff for every solvable unification problem Π, $|\text{Max } \Sigma| = 1$;
- ω, iff for every solvable unification problem Π, $|\text{Max } \Sigma| = n > 1$, $n \in \omega$;
- ∞, iff for every solvable unification problem Π, $|\text{Max } \Sigma|$ is infinite.

- We say that E has *finitary unification type* iff it has type 1 or ω.

Theorem 18. *Equational class of* $MV(C)$*-algebras has unitary unification type.*

STRUCTURAL COMPLETENESS

- In logic, a rule of inference is ***admissible*** in a formal system if the set of theorems of the system does not change when that rule is added to the existing rules of the system.

- The concept of an admissible rule was introduced by Paul Lorencen (1955).

- A Tarski-style ***consequence relation*** is a relation $\vdash$ between sets of formulas, and formulas, such that
- $\alpha \vdash \alpha$
- if $\Gamma \vdash \alpha$, then $\Gamma, \Delta \vdash \alpha$

A consequence relation such that

- if $\Gamma \vdash \alpha$, then $\sigma(\Gamma) \vdash \sigma(\alpha)$

for all substitutions σ is called ***structural***.

- A ***structural inference rule*** (or just ***rule*** for short) is given by a pair (Γ, β), usually written as

$$\Gamma / \beta \quad \text{or} \quad \alpha_1, \dots, \alpha_n / \beta,$$

where $\Gamma = \{\alpha_1, \dots, \alpha_n\}$ is a finite set of formulas, and β is a formula.

An ***instance*** of the rule is

$$\sigma(\alpha_1), \dots, \sigma(\alpha_n) / \sigma(\beta)$$

for a substitution σ.

- The rule Γ / β is ***derivable*** in a given logic, if

$$\Gamma \vdash \beta.$$

- It is ***admissible*** if for every instance of the rule, $\sigma(\beta)$ is a theorem whenever all formulas from $\sigma(\Gamma)$ are theorems.

- A logic is ***structurally complete*** if every rule that is admissible (preserves the set of theorems) should also be derivable.

  Let us formulate the following property for a logic L:

- $\alpha \vdash \beta$ iff $(\vdash \sigma(\alpha) \Rightarrow \vdash \sigma(\beta))$,

for every substitution σ.

- A logic is ***structurally complete*** if every rule that is admissible (preserves the set of theorems) should also be derivable.

  Let us formulate the following property for a logic L:

❖ $\alpha \vdash \beta$ iff $(\vdash \sigma(\alpha) \Rightarrow \vdash \sigma(\beta))$,

for every substitution σ.

- The property is equivalent to the notion of a structural completeness in the sense of Pogorzelski, i.e. any structural admissible rule of a logic is derivable.

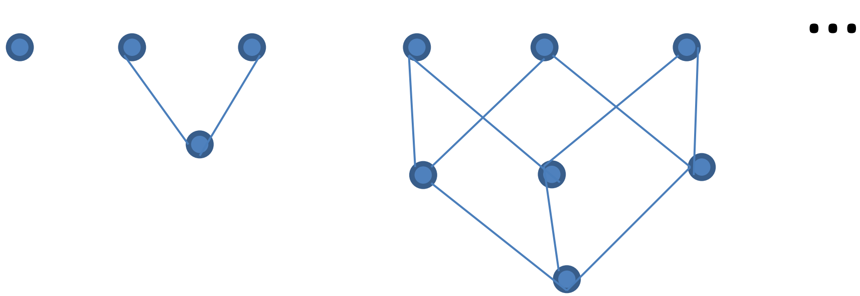
- **Proposition 19.** (T. Prucnal (1976), R. Grigolia (1995)). *Superintuitionistic logic – Medvedev logic is structurally complete.*

- **Proposition 20.** (W. Dzik and A. Wronski (1973)). *Gödel logic GD and its extensions are structurally complete.*

- Let L be a logic and $\mathbf{V_L}$ the variety corres-ponding to L.
- We say that $\mathbf{V_L}$ is *finitely approximated* if it generated by its finite members.
- An m-generated algebra A is *weakly projec-tive* if it is embedded into m-generated free algebra $F_{\mathbf{V_L}}(m)$ of the varietyt $\mathbf{V_L}$.

Theorem 21. *If* $\mathbf{V}_L$ *is finitely approximated and generated by a family of finite subdirectly irreducible weakly projective algebras, then the logic L, corresponding to the variety* $\mathbf{V}_L$ *, is structurally complete.*

Medvedev Logic *M*



Let $\boldsymbol{n} = \{1, \ldots, n\}$ and $\mathrm{M}_n = \{X \subset \boldsymbol{n} : X \neq \boldsymbol{n}\}$ be a poset. In the picture they are depicted M_1, M_2, and M_3.

Let A_n be the Heyting algebra of all up-sets of M_n.

- **Proposition 22.** *The variety* $\mathbf{V_M}$ *is generated by the family* $\{A_n\}_{n\in\omega}$.

- **Theorem 23.** A_n *is projective algebra in the variety* $\mathbf{V_M}$*, for any positive integer* n.

- **Corollary 24.** *Medvedev logic M is structurally complete.*

Gödel logic

The variety $\mathbf{V}_{GD}$, corresponding to Gödel logic G is generated by finite chain Heyting algebras, which are projective. So, we have

- **Proposition 25.** *Gödel logic* G *is structurally complete.*

Logic P

Let P be super intuitionistic logic, correspon-ding to the variety $\mathbf{V}_P$ generated by finite projective Heyting algebras. Then

- **Theorem 26.** *The logic* P *is structurally complete.*

Now we formulate structurally completeness for a logic L in the following way:

$$\alpha^n \to \beta \in \mathrm{T}, \text{ for some positive integer } n, \quad \Leftrightarrow$$
$$(\forall \varphi : F \to F)\,(\varphi(\alpha) \in \mathrm{T} \Rightarrow \varphi(\beta) \in \mathrm{T}),$$

where T is the set of all theorems of the logic L, φ is an endomorphism of the algebra $(F, \to, \neg, 0, 1)$ which is a free algebra in the class of algebras of the type (2,1,0,0).

Let us note that, since according to deduction theorem in Lukasiewicz logic:

$$\alpha \vdash \beta \quad \text{if and only if} \vdash \alpha^{n} \rightarrow \beta$$

for some positive integer n, the property is equivalent to the notion of a structural completeness in the sense of Pogorzelski, i.e. any structural admissible rule of a logic is derivable.

In algebraic terms the property has the following formulation:

$$\alpha^n \to \beta = 1, \text{ for some positive integer } n, \quad \Leftrightarrow$$
$$(\forall \varphi : F \to F)\ (\varphi(\alpha) = 1 \Rightarrow \varphi(\beta) = 1),$$

where φ is an endomorphism of the ω-generated free algebra $(F, \to, \neg, 0, 1)$ in the variety of MV–algebras.

Let L_P be a logic corresponding to variety $\mathbf{MV(C)}$, i. e. L_P is the set of Lukasiewicz formulas which are valid in all $MV(C)$-algebras.

Theorem 27. *The logic L_P is structurally complete.*

THANK YOU