Dualities for N4-lattices and N4-lattices with modalities

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Outline

- Nelson's paraconsistent logics $N4$ and $N4^{\perp}$
- $N4$-lattices and $N4^{\perp}$-lattices
- Duality for $N4^{\perp}$-lattices
- A modal paraconsistent logic
- Modal $N4$ and $N4^{\perp}$-lattices
- Duality for modal $N4^{\perp}$-lattices

Nelson's logic of constructible falsity and its paraconsistent version

- Nelson's logic of constructible falsity, N3: Nelson in 1949
- Language: $\{\vee, \wedge, \to, \sim\}$
- AXIOMS:
  1. Axioms for positive logic (negation free fragment of IL)
  2. Axioms for a De Morgan negation:
     - $\sim\sim p \leftrightarrow p$
     - $\sim(p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
     - $\sim(p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
  3. $\sim(p \to q) \leftrightarrow (p \wedge \sim q)$
  4. $\sim p \to (p \to q)$
- RULE: Modus Ponens

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  4. $\sim p \rightarrow (p \rightarrow q)$
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Nelson's logic of constructible falsity and its paraconsistent version

- The logic N4; the paraconsistent version of N3: Almukdad and Nelson 1984
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  3. $\sim(p \rightarrow q) \leftrightarrow (p \wedge \sim q)$
  4. $\sim p \rightarrow (p \rightarrow q)$ N4 lacks this axiom.
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- $N4^{\perp}$ is the expansion of $N4$ with a propositional constant $\perp$ and the axioms

$$\perp \to \varphi \qquad\qquad \varphi \to \sim\perp.$$

Intuitionistic negation is then defined as usual: $\neg\varphi := \varphi \to \perp$.
We can define $\top := \sim\perp$.

Algebraic semantics for $N4$ and $N4^{\perp}$

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So they are of similarity type $\{\wedge, \vee, \rightarrow, \sim, \perp\}$.
- We introduce $N4$-lattices and $N4^{\perp}$-lattices respectively using twist algebras over Brouwerian lattices and over Heyting algebras. (Vakarelov, Fidel represented $N3$ lattices as twist algebras. This inspired Odintsov ...)

Algebraic semantics for $N4$ and $N4^{\perp}$

- A Brouwerian (or implicative) lattice is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, 1 \rangle$ where
  - $\langle A, \wedge, \vee, 1 \rangle$ is a distributive lattice with top element,
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- A bounded Brouwerian lattice (or Heyting algebra) is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, 1, 0 \rangle$ where
  - $\langle A, \wedge, \vee, \rightarrow, 1 \rangle$ is a Brouwerian lattice with a least element in the lattice order, and 0 is this least element.

Let $\mathbf{A} = \langle A, \wedge, \vee, \rightarrow, 1\rangle$ be a Brouwerian lattice.

The full twist-algebra over $\mathbf{A}$ is the algebra

$$\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \rightarrow, \sim\rangle$$

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- $\langle a, b\rangle \wedge \langle c, d\rangle := \langle a \wedge c, b \vee d\rangle$
- $\langle a, b\rangle \vee \langle c, d\rangle := \langle a \vee c, b \wedge d\rangle$
- $\langle a, b\rangle \to \langle c, d\rangle := \langle a \to c, a \wedge d\rangle$
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- $\langle a, b\rangle \to \langle c, d\rangle := \langle a \to c, a \wedge d\rangle$
- $\sim\langle a, b\rangle := \langle b, a\rangle$.

A twist-algebra over $\mathbf{A}$ is a subalgebra $\mathbf{C}$ of $\mathbf{A}^{\bowtie}$ such that

$$A = \pi_1[C](= \{a : \exists b\langle, a, b\rangle \in C\})$$

Definition

An $N4$-lattice is an algebra isomorphic to a twist-algebra over some Brouwerian lattice.

- $N4$-lattices form a variety (Odintsov).
- $\mathbf{N4}$ denotes the category of $N4$-lattices.

If $\mathbf{A} = \langle A, \wedge, \vee, \to, 1, 0\rangle$ is a Heyting algebra, then in the lattice order of $\mathbf{A}^{\bowtie}$

- $\langle a, b\rangle \leq \langle 1, 0\rangle$
- $\langle 0, 1\rangle \leq \langle a, b\rangle$

We set,

$$\mathbf{A}^{\bowtie} = \langle A \times A, \wedge, \vee, \to, \sim, \langle 0, 1\rangle\rangle$$

We call this algebra the bounded and full twist-algebra of $\mathbf{A}$.

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Definition

An $N4^{\perp}$-lattice is an algebra isomorphic to a twist-algebra over some Heyting algebra.

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Definition

An $N4^{\perp}$-lattice is an algebra isomorphic to a twist-algebra over some Heyting algebra.

- $\mathbf{N4}^{\perp}$ denotes the category of $N4^{\perp}$-lattices.

- $N4$-lattices produce Brouwerian lattices.

Let $\mathbf{B} = \langle B, \wedge, \vee, \to, \sim \rangle$ be an $N4$-lattice.

The relation $\preccurlyeq$ in B given by

$$a \preccurlyeq b \quad \Leftrightarrow \quad a \to b = (a \to b) \to (a \to b)$$

is a preorder. Its associated equivalence relation $\equiv$

$$a \equiv b \quad \Leftrightarrow \quad a \preccurlyeq b \text{ and } b \preccurlyeq a$$

is a congruence w.r.t. $\wedge, \vee, \to$.

The algebra $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \to \rangle / \equiv$ is a Brouwerian lattice. Back

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The algebra $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \rightarrow, \perp \rangle / \equiv$ is a Heyting algebra, where $[\sim\perp]$ is the top element.

In fact $(.)^{\bowtie}$ and $(.)_{\bowtie}$ can be respectively expanded to functors from the category of Brouwerian lattices to the category of $N4$-lattices and from this category to the first one.

Moreover, $(.)^{\bowtie}$ is a left adjoint to $(.)_{\bowtie}$.

Odintsov published (2010) a topological duality for $N4^{\perp}$-lattices based on

- Esakia duality for Heyting algebras
- Cornish and Fowler duality for De Morgan algebras.

We exploit Odintsov's representation of $N4^{\perp}$-lattices by H-twist structures (twist-algebras over a Heyting algebra and with some more strictre) to obtain a duality based only on Esakia duality for Heyting algebras.

Odintsov's representation of $N4^{\perp}$-lattices is a restriction of his representation of $N4$-lattices.
We extend this last one to a full categorical equivalence.

The category of twist-structures

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Let $\mathbf{A}$ be a Brouwerian lattice.
The set of dense elements of $\mathbf{A}$ is

$$D(\mathbf{A}) := \{a \vee (a \to b) : a, b \in A\}.$$

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Definition

A twist-structure is a triple $(\mathbf{A}, \nabla, \Delta)$ where

- $\mathbf{A}$ is a Brouwerian lattice,
- ∇ a lattice filter of $\mathbf{A}$ such that $D(\mathbf{A}) \subseteq \nabla$,
- Δ is an ideal of $\mathbf{A}$.

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- From twist-structures to $N4$-lattices.

Let $(\mathbf{A}, \nabla, \Delta)$ be a twist-structure.
The set

$$A_{(\nabla,\Delta)} := \{\langle a, b\rangle \in A \times A : a \vee b \in \nabla, a \wedge b \in \Delta\}$$

is closed under the operations $\wedge, \vee, \rightarrow, \sim$ of $\mathbf{A}^{\bowtie}$ and $\pi_1[A_{(\nabla,\Delta)}] = A$.
Hence, $\langle A_{(\nabla,\Delta)}, \wedge, \vee, \rightarrow, \sim\rangle$ is an $N4$-lattice.

Following Odintsov we denote this algebra by $Tw(\mathbf{A}, \nabla, \Delta)$.

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- From H-twist-structures to $N4^{\perp}$-lattices.

If $(\mathbf{A}, \nabla, \Delta)$ is an H-twist-structure, then $\langle 1, 0\rangle, \langle 0, 1\rangle \in A_{(\nabla,\Delta)}$ because $1 \wedge 0 \in \Delta$ and $1 \vee 0 \in \nabla$.
Hence, $\langle A_{(\nabla,\Delta)}, \wedge, \vee, \rightarrow, \sim, \langle 0, 1\rangle\rangle$ is an $N4^{\perp}$-lattice.

Representation of N4-lattices by twist-structures (Odintsov)

Any $N4$-lattice $\mathbf{B}$ can be obtained as $Tw(\mathbf{A}, \nabla, \Delta)$ for a suitable choice of $(\mathbf{A}, \nabla, \Delta)$.

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We define

$$\nabla(\mathbf{B}) := \{[a \vee \sim a] : a \in B\}$$

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where for $b \in B$, $[b]$ denotes the equivalence class modulo $\equiv$.

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- $\nabla(\mathbf{B})$ is a lattice filter of $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \rightarrow \rangle / \equiv$ (or $\mathbf{B}_{\bowtie} = \langle B, \wedge, \vee, \rightarrow, \perp \rangle / \equiv$) and contains the dense elements of $\mathbf{B}_{\bowtie}$

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- $\Delta(\mathbf{B})$ is an ideal of $\mathbf{B}_{\bowtie}$.

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- $\Delta(\mathbf{B})$ is an ideal of $\mathbf{B}_{\bowtie}$.
- The map $j_{\mathbf{B}} : B \rightarrow B/\equiv \times B/\equiv$ defined by

$$j_{\mathbf{B}}(a) := \langle [a], [\sim a] \rangle$$

is an isomorphism between $\mathbf{B}$ and $Tw(\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B}))$.

The category **Twist**

• OBJECTS: Twist-structures.

• MORPHISMS: Let $(\mathbf{A}_1, \nabla_1, \Delta_1)$ and $(\mathbf{A}_2, \nabla_2, \Delta_2)$ twist-structures $(\mathbf{A}_1, \nabla_1, \Delta_1)$. A map $h\colon \mathbf{A}_1 \to \mathbf{A}_2$ is a morphism if it is a Brouwerian lattice homomorphism and

- $h[\nabla_1] \subseteq \nabla_2$
- $h[\Delta_1] \subseteq \Delta_2$.

The category **H-Twist**

• OBJECTS: Twist-structures whose algebra is a Heyting algebra. We call them *H-twist-structures*.

• MORPHISMS: The morphisms of **Twist** that preserve the bounds.

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by the functors $T\colon \mathbf{N4} \to \mathbf{Twist}$ and $N\colon \mathbf{Twist} \to \mathbf{N4}$ defined as follows.

- $T(\mathbf{B}) := (\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B}))$.
- $f\colon \mathbf{B_1} \to \mathbf{B_2}$

$T(f)\colon T(\mathbf{B_1}) \to T(\mathbf{B_2})$

is defined by

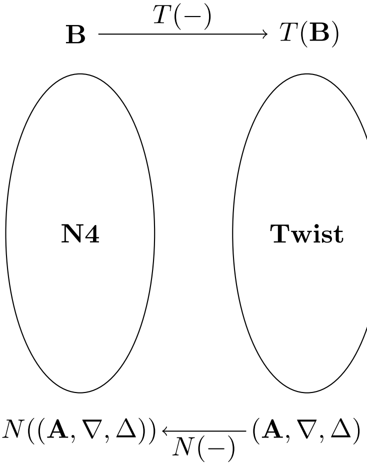
$T(f)([a]_{\equiv_1}) := [f(a)]_{\equiv_2}$

- $N((\mathbf{A}, \nabla, \Delta)) := Tw(\mathbf{A}, \nabla, \Delta)$
- $h\colon \mathcal{A}_1 \to \mathcal{A}_2$

$N(h)\colon N(\mathcal{A}_1) \to N(\mathcal{A}_1)$

defined by

$N(h)\langle a, b\rangle := \langle h(a), h(b)\rangle$.
for all $a, b \in A_1$.



Let $\mathbf{B}$ be an N4-lattice. Note that

$$N(T(\mathbf{B})) = Tw((\mathbf{B}_{\bowtie}, \nabla(\mathbf{B}), \Delta(\mathbf{B})))$$

Therefore

$$j_{\mathbf{B}} \colon \mathbf{B} \cong N(T(\mathbf{B})).$$

Conversely, given a twist-structure $(\mathbf{A}, \nabla, \Delta)$, let $\eta\colon (\mathbf{A}, \nabla, \Delta) \to T(N((\mathbf{A}, \nabla, \Delta)))$ be defined by:

$$\eta(a) := [\langle a, a' \rangle],$$

where $a' \in A$ is any element such that $\langle a, a' \rangle \in N(\mathbf{A}, \nabla, \Delta)$ and $[\langle a, a' \rangle]$ is the equivalence class of $\langle a, a' \rangle$ modulo the equivalence relation $\equiv$ of the $N4$-lattice $N((\mathbf{A}, \nabla, \Delta))$.
For all $a, b, a', b' \in A$, it holds that

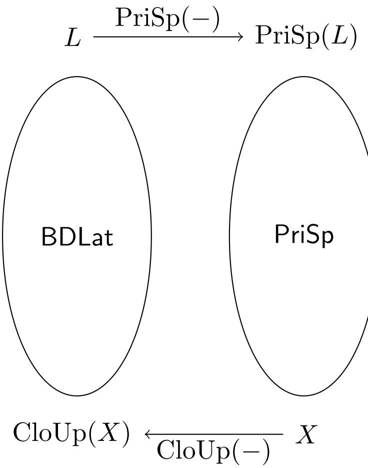
$$[\langle a, a' \rangle] = [\langle b, b' \rangle] \text{ iff } a = b.$$

The map $\eta\colon (\mathbf{A}, \nabla, \Delta) \to T(N(\mathbf{A}, \nabla, \Delta))$ is an isomorphism in the category $\mathbf{Twist}$.

Similarly,

The category $\mathbf{H\text{-}Twist}$ is equivalent to the category $\mathbf{N4}^{\perp}$ of bounded $N4$-lattices

Reminder of Esakia spaces and Esakia duality



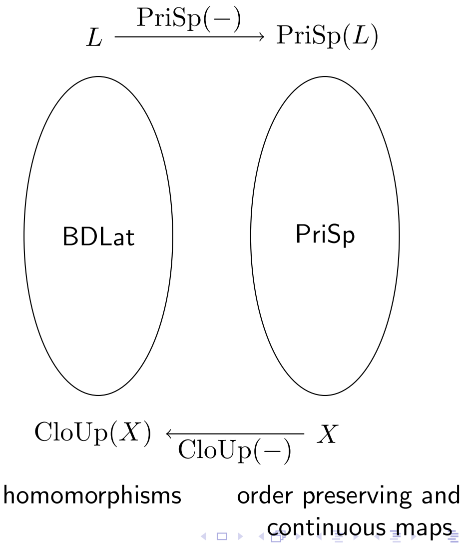
homomorphisms

order preserving and continuous maps

Reminder of Esakia spaces and Esakia duality

Priestley space: $\langle X, \leq, \tau \rangle$

- $\langle X, \tau \rangle$ is a Boolean space
- $\leq$ partial order s. t. if $x \not\leq y$, then there is a clopen up-set U with $x \in U$ and $y \notin U$.



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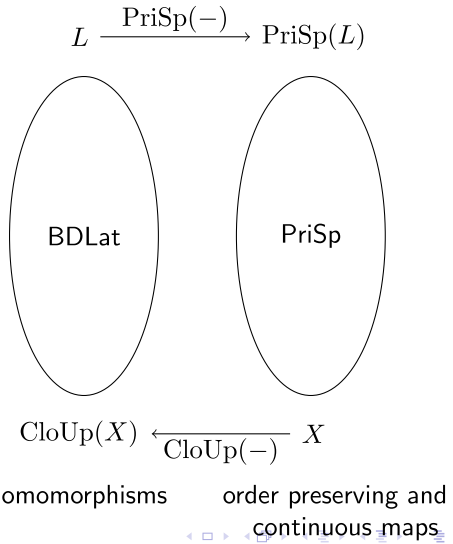

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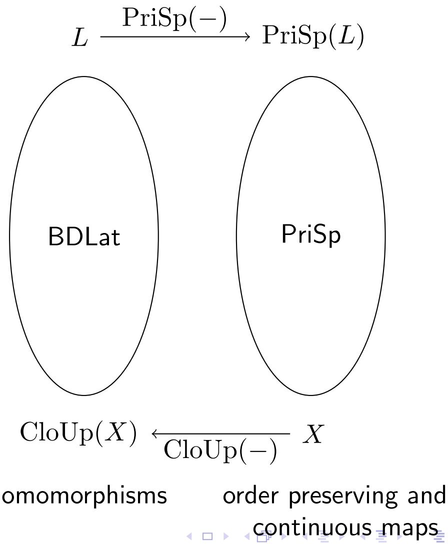
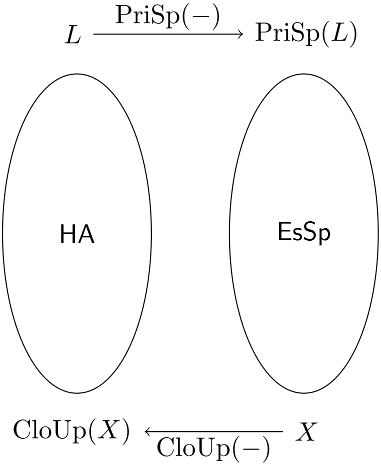
- the $\sigma(a)$'s together with the $\sigma^{c(a)}$'s form a subbase for a topology



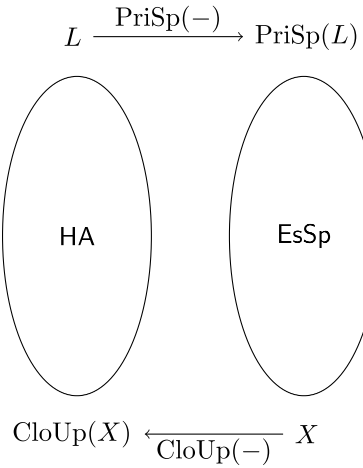


homomorphisms

order preserving and continuous maps $f : X_1 \to X_2$ that $(\forall x \in X_1) \uparrow f(x) \subseteq f[\uparrow x]$.

Esakia space: $\langle X, \leq, \tau \rangle$

- Priestley space
- $\leq$ satisfies that $\downarrow U$ is clopen for every clopen set U



homomorphisms

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Let $\boldsymbol{A}$ be a Heyting algebra and let $\mathrm{PriSp}(\boldsymbol{A})$ be its Esakia space.

- Filters of $\boldsymbol{A}$ correspond to closed up-sets of $\mathrm{PriSp}(\boldsymbol{A})$ by the map

$$C_F = \bigcap_{a \in F} \sigma(a),$$

- Ideals of $\boldsymbol{A}$ correspond to open up-sets of $\mathrm{PriSp}(\boldsymbol{A})$ by the map

$$O_I = \bigcup_{a \in I} \sigma(a).$$

The category of $N4^{\perp}$-Esakia spaces

Definition

An $N4^{\perp}$-Esakia space is a structure $\langle X, \leq, \tau, C, O \rangle$ where

1. $\langle X, \leq, \tau \rangle$ is an Esakia space,
2. C is a closed (up-set) such that $C \subseteq \max(X)$,
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Definition

Let $\langle X_1, \leq_1, \tau_1, C_1, O_1 \rangle$ and $\langle X_2, \leq_2, \tau_2, C_2, O_2 \rangle$ be $N4^{\perp}$-Esakia spaces. A morphism f from the first to the second is a map $f : X_1 \to X_2$ that satisfies:

1. f is a monotone and continuous function,
2. $f[C_1] \subseteq C_2$,
3. $f^{-1}[O_2] \subseteq O_1$.

We show:

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Let $\mathcal{A} = \langle \boldsymbol{A}, \nabla, \Delta \rangle$ be an H-twist-structure.
Let $X(\boldsymbol{A})$ be the Esakia space of $\boldsymbol{A}$ and $\sigma : \boldsymbol{A} \simeq \mathrm{CloUp} X(\boldsymbol{A})$.

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$$C_{\mathcal{A}} = \bigcap_{a \in \nabla} \sigma(a) \qquad \text{and} \qquad O_{\mathcal{A}} = \bigcup_{a \in \Delta} \sigma(a)$$

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which are respectively a a closed up-set and an open up-set.

Since $a \vee \neg a = a \vee (a \to 0) \in \nabla$, for every $a \in A$, then
$C_{\mathcal{A}} \subseteq \sigma(a) \cup \sigma(\neg a) = \sigma(a) \cup (\downarrow\sigma(a))^c$, for every $a \in A$.
Therefore $\langle X(\boldsymbol{A}), \subseteq, \tau_{\boldsymbol{A}}, C_{\mathcal{A}}, O_{\mathcal{A}} \rangle$ is an $N4^{\perp}$-Esakia space.
We denote it by $\mathcal{A}_+$.

Let $X = \langle X, \leq, \tau, C, O\rangle$ be an $N4^{\perp}$-Esakia space. We define the twist structure

$$X^{+} = (X^{+}, \nabla_C, \Delta_O)$$

where

- X^{+} is the Heyting algebra of the clopen up-sets of the Priestley space $\langle X, \leq, \tau\rangle$,
- ∇_C is the filter associated with C,
- Δ_O is the ideal associated with O.

They are respectively

$$\nabla_C = \{U \in \mathrm{CloUp}(X) : C \subseteq U\}$$

and

$$\Delta_O = \{U \in \mathrm{CloUp}(X) : U \subseteq O\}.$$

Let $X_1 = \langle X_1, \leq_1, \tau_1, C_1, O_1 \rangle$ and $X_2 = \langle X_2, \leq_2, \tau_2, C_2, O_2 \rangle$ be $N4^{\perp}$-Esakia spaces and f a morphism from X_1 to X_2. The Priestley dual map f^+ from $\mathrm{CloUp} X_2$ to $\mathrm{CloUp} X_1$ is indeed a morphism from $(X_2)^+$ to $(X_1)^+$, that is

- $f^+[\nabla_{C_2}] \subseteq \nabla_{C_1}$ and
- $f{+}[\Delta_{O_2}] \subseteq \Delta_{O_1}$.

Let $\mathcal{A}_1 = (\boldsymbol{A}_1, \nabla_1, \Delta_1)$ and $\mathcal{A}_2 = (\boldsymbol{A}_2, \nabla_2, \Delta_2)$ be twist structures an h a morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$. The Priestley dual map h_+ from $X(\boldsymbol{A}_2)$ to $X(\boldsymbol{A}_1)$ is indeed a morphism from $\mathcal{A}_2$ to $\mathcal{A}_1$, that is

- $h_+[C_{\mathcal{A}_2}] \subseteq C_{\mathcal{A}_1}$ and
- $h_+[O_{\mathcal{A}_2}] \subseteq O_{\mathcal{A}_1}$.

The category of $N4^{\perp}$-Esakia spaces is dually equivalent to the category of H-twist-structures

$(\langle X, \leq, \tau, C, O\rangle)^{+} =$
$(\mathrm{CloUp}X, \nabla_{C}, \varDelta_{O})$

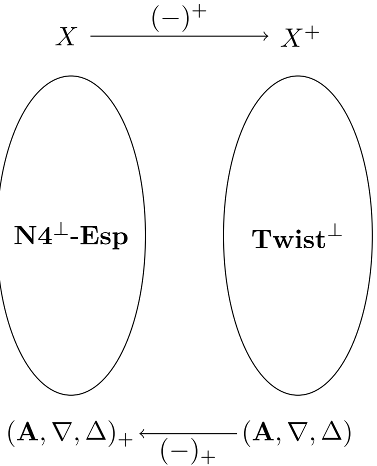
$f : X_1 \to X_2$

$f^{+} : \mathrm{CloUp}X_2 \to \mathrm{CloUp}X_1$
defined by $f^{+}(U) = f^{-1}[U]$

$(\boldsymbol{A}, \nabla, \varDelta)_{+} =$
$\langle X(\boldsymbol{A}), \subseteq, \tau_{\boldsymbol{A}}, C_{\mathcal{A}}, O_{\mathcal{A}}\rangle$

$h : (\boldsymbol{A}_1, \nabla_1, \varDelta_1) \to (\boldsymbol{A}_2, \nabla_2, \varDelta_2)$

$h_{+} : X(\boldsymbol{A}_2) \to X(\boldsymbol{A}_1)$
defined by $h_{+}(P) = h^{-1}[P]$



The modal paraconsistent logic $ModN4$

Introduced by U. Rivieccio (2011)

LANGUAGE: $\{\vee, \wedge, \rightarrow, \sim, \Box\}$
AXIOMATIZATION: Axioms and rule of $N4$ $+$ the modal monotonicity rules

$$\frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi} \qquad \frac{\sim\varphi \rightarrow \sim\psi}{\sim\Box\varphi \rightarrow \sim\Box\psi}$$

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Similarly, we have the modal paraconsistent logic $ModN4^{\perp}$ and modal $N4^{\perp}$-lattices.

We present modal twist-structures for modal $N4$-lattices. Everything extends in the obvious way to modal $N4^{\perp}$-lattices.

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The filter $\nabla(\mathbf{B})$ and the ideal $\Delta(\mathbf{B})$ satisfy:

- if $a \vee b \in \nabla(\mathbf{B})$, then $\Box a \vee \Diamond b \in \nabla(\mathbf{B})$.
- if $a \wedge b \in \Delta(\mathbf{B})$, then $\Box a \wedge \Diamond b \in \Delta(\mathbf{B})$.

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(Positive modal logic rules)

Definition

A bi-modal Brouwerian lattice is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \rightarrow, \Box, \Diamond \rangle$ where $\langle A, \wedge, \vee, \rightarrow \rangle$ is a Brouwerian lattice and $\Box$, $\Diamond$ are two monotone maps.

Definition

A modal twist-structure is a triple $\mathcal{A} = \langle \boldsymbol{A}, \nabla, \Delta \rangle$ where

1. $\boldsymbol{A}$ is a bi-modal Brouwerian lattice
2. ∇ is a filter that includes the dense elements of $\boldsymbol{A}$ and such that
   if $a \vee b \in \nabla$, then $\Box a \vee \Diamond b \in \nabla$, for every $a, b \in A$
3. Δ is an ideal of $\boldsymbol{A}$ such that
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3. $\varDelta$ is an ideal of $\boldsymbol{A}$ such that
   if $a \wedge b \in \varDelta$, then $\Box a \wedge \Diamond b \in \varDelta$, for every $a, b \in A$.

So, if $\mathbf{B} = \langle B, \wedge, \vee, \to, \sim, \Box \rangle$ is a modal $N4$-lattice, the structure $(\langle \mathbf{B}/{\equiv}, \wedge, \vee, \to, \Box, \Diamond \rangle, \nabla(\mathbf{B}), \varDelta(\mathbf{B}))$ is a modal-twist structure. We denote it by $T(\mathbf{B})$.

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Similarly we have modal H-twist-structures where $\boldsymbol{A}$ is a Heyting algebra.

Definition

Let $\mathcal{A}_1 = \langle \boldsymbol{A}_1, \nabla_1, \Delta_1 \rangle$ and $\mathcal{A}_2 = \langle \boldsymbol{A}_2, \nabla_2, \Delta_2 \rangle$ be modal twist structures. A morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$ is a homomorphism of bi-modal Brouwerian lattices $f : \boldsymbol{A}_1 \to \boldsymbol{A}_2$ that is also a morphism of twist-structures.

- The category $\mathbf{MTwist}$ ($\mathbf{MHTwist}$) of modal twist-structures (modal H-twist-structures):
  Objects: modal twist-structures (modal H-twist-structures)
  Morphisms: the maps just defined.
- The category $\mathbf{MN4}$ ($\mathbf{MN4}^{\perp}$) of modal $N4$-lattices (modal $N4^{\perp}$-lattices):
  Objects: $MN4$-lattices ($MN4^{\perp}$-lattices)
  Morphisms: The homomoprhisms

The equivalence between the category of $N4$-lattices and the category of twist structures can be expanded to an equivalence between the category $\mathbf{MN4}$ of modal $N4$-lattices and the category $\mathbf{MTwist}$.

Let $\mathcal{A} = (\boldsymbol{A}, \nabla, \Delta)$ be a modal twist structure.

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Recall: $Tw((\boldsymbol{A}, \nabla, \Delta))$ is an $N4$-lattice.

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The operation $\Box$ on $Tw((\boldsymbol{A}, \nabla, \Delta))$ defined by:

$$\Box\langle a, b\rangle = \langle \Box a, \Diamond b\rangle$$

satisfies

- if $\langle a, b\rangle \preccurlyeq \langle c, d\rangle$, then $\Box\langle a, b\rangle \preccurlyeq \Box\langle c, d\rangle$,
- if $\sim\langle a, b\rangle \preccurlyeq \sim\langle c, d\rangle$, then $\sim\Box\langle a, b\rangle \preccurlyeq \sim\Box\langle c, d\rangle$.

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Thus, $\langle A_{(\nabla,\varDelta)}, \wedge, \vee, \rightarrow, \sim, \Box\rangle$ is a modal $N4$-lattice. We also denote it by $N((\boldsymbol{A}, \nabla, \varDelta))$.

- Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be modal twist structures and let $f : \mathcal{A}_1 \to \mathcal{A}_1$ be a morphism. The map $N(f) : N(\mathcal{A}_1) \to N(\mathcal{A}_2)$ between the twist structures satisfies:

$$N(f)(\Box\langle a, b\rangle) = \Box N(f)(\langle a, b\rangle).$$

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- Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two modal $N4$-lattices and let $h : \mathbf{B}_1 \to \mathbf{B}_2$ be a homomorphism. Then

$$T(h)(\Box[a]_\equiv) = \Box T(h)([a]_\equiv) \quad \text{and} \quad T(h)(\Diamond[a]_\equiv) = \Diamond T(h)([a]_\equiv).$$

Therefore $T(h)$ is a morphism from $T(\mathbf{B}_1)$ to $T(\mathbf{B}_2)$.

Extending $N4^{\perp}$-Esakia spaces to account for the modality

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A neighborhood operation on an Esakia space $\langle X, \leq, \tau \rangle$ is a map $\nu : X \to \mathcal{P}^{\uparrow}(X)$ such that

if $x \leq y$, then $\nu(x) \subseteq \nu(y)$.

Let $\langle X, \leq, \tau, \nu \rangle$ be an Esakia space with a neighborhood operation. We define the map $\Box_\nu : \mathrm{CloUp}(X) \to \mathrm{CloUp}(X)$ as follows:

$$\Box_\nu(U) = \{x \in X : U \in \nu(x)\}.$$

Then $\Box_\nu$ is monotone w.r.t. $\subseteq$ and $\langle \mathrm{CloUp}(X), \Box_\nu \rangle$ is a Heyting algebra with a monotone operation.

Let $\boldsymbol{A}$ be a Heyting algebra and let $\Box$ be a monotone operation on $\boldsymbol{A}$.

Let $\langle X(\boldsymbol{A}), \subseteq, \tau_{\boldsymbol{A}} \rangle$ be the Esakia space of $\boldsymbol{A}$.

We define the neighborhood operation $\nu_\Box$ on $\langle X(\boldsymbol{A}), \subseteq, \tau_{\boldsymbol{A}} \rangle$ as follows:

$$\nu_\Box(P) = \{U \in \mathrm{CloUp}X(\boldsymbol{A}) : (\exists F \in \mathrm{Fi}(\boldsymbol{A}))(F \subseteq \Box^{-1}[P]$$
$$\text{and } \{Q \in X(\boldsymbol{A}) : F \subseteq Q\} \subseteq U)\}$$

for every prime filter P of $\boldsymbol{A}$.

Then, for every $a \in A$,

$$\sigma(\Box a) = \Box_{\nu_\Box} \sigma(a).$$

For all clopen up-sets U of $X(\boldsymbol{A})$ and every $P \in X(\boldsymbol{A})$,

$U \in \nu_\Box(P)$ iff there is a closed up-set D such that $D \subseteq U$ and
for every clopen-upset V such that $D \subseteq V$ it holds that $V \in \nu_\Box(P)$.

Lemma

Let $\mathcal{A} = \langle \boldsymbol{A}, \nabla, \Delta \rangle$ be a modal H-twist-structure.

The neighborhood operations $\nu_\Box$ and $\nu_\Diamond$ on $\langle X(\boldsymbol{A}), C_\nabla, O_\Delta \rangle$ satisfy the following conditions:

1. *if $C_\nabla \subseteq U \cup V$, then $C_\nabla \subseteq (\Box_{\nu_\Box} U \cup \Box_{\nu_\Diamond} V)$,*
2. *if $U \cap V \subseteq O_\Delta$, then $(\Box_{\nu_\Box} U \cap \Box_{\nu_\Diamond} V) \subseteq O_\Delta$.*

for all clopen up-sets U, V.

Definition

A *modalized* $N4^{\perp}$*-Esakia space* is a structure $\langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ where

1. $\langle X, \leq, \tau, C, O \rangle$ is an $N4^{\perp}$-Esakia space,
2. ν_1, ν_2 are neighborhood operations on $\langle X, \leq, \tau \rangle$ such that
   1. for every $U \in \mathrm{CloUp}(X)$, $\Box_{\nu_1} U, \Box_{\nu_2} U \in \mathrm{CloUp}(X)$,
   2. if $C \subseteq U \cup V$, then $C \subseteq \Box_{\nu_1} U \cup \Box_{\nu_2} V$,
   3. if $U \cap V \subseteq O$, then $\Box_{\nu_1} U \cup \Box_{\nu_2} V \subseteq O$,
   4. $U \in \nu_1(x)$ iff there exist a closed up-set D such that $D \subseteq U$ and for every clopen-upset V such that $D \subseteq V$ it holds that $V \in \nu_1(x)$,
   5. $U \in \nu_2(x)$ iff there exist a closed up-set D such that $D \subseteq U$ and for every clopen-upset V such that $D \subseteq V$ it holds that $V \in \nu_2(x)$.

Definition

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Therefore,

Proposition

Let $\mathcal{A} = \langle \boldsymbol{A}, \nabla, \Delta \rangle$ *be a modal H-twist-structure. Then* $\mathcal{A}_+ := \langle X(\boldsymbol{A}), C_{\nabla}, O_{\Delta}, \nu_{\Box}, \nu_{\Diamond} \rangle$ *is a modalized* $N4^{\perp}$*-Esakia space.*

Let $X = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be a modalized $N4^{\perp}$-Esakia space. We define in the Heyting algebra $\mathrm{CloUp}(X)$ the operations $\Box_{\nu_1}$ and $\Box_{\nu_2}$ and consider the filter ∇_C and the ideal Δ_O. Then:

Lemma

Let $X = \langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be a modalized $N4^{\perp}$-Esakia space. Then $X^{+} = \langle \mathrm{CloUp}(X)^m, \nabla_C, \Delta_O \rangle$ is a modal H-twist-structure.

Definition

Let X and X' be two modalized $N4^{\perp}$-Esakia spaces. A map $f : X \to X'$ is a morphism if it is a morphism of $N4^{\perp}$-Esakia spaces and in addition satisfies,

1. for every $x \in X$ and every clopen up-set U of X',
   $U \in \nu_1'(f(x))$ if and only if $f^{-1}[U] \in \nu_1(x)$,
2. for every $x \in X$ and every clopen up-set U of X',
   $U \in \nu_2'(f(x))$ if and only if $f^{-1}[U] \in \nu_2(x)$.

Definition

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   $U \in \nu_2'(f(x))$ if and only if $f^{-1}[U] \in \nu_2(x)$.

Proposition

Let $\mathcal{A}_1 = \langle \boldsymbol{A}_1, \nabla_1, \Delta_1 \rangle$ and $\mathcal{A}_2 = \langle \boldsymbol{A}_2, \nabla_2, \Delta_2 \rangle$ be modal H-twist-structures and let h be a morphism from $\mathcal{A}_1$ to $\mathcal{A}_2$. Then h_+ is a morphism from the modalized $N4^{\perp}$-Esakia space $(\mathcal{A}_2)_+$ to the modalized $N4^{\perp}$-Esakia space $(\mathcal{A}_1)_+$.

Proposition

Let X and X' be modalized $N4^{\perp}$-Esakia spaces and let $f : X \to X'$ be a morphism from X to X'. Then the map $f^{+} : X'^{+} \to X^{+}$ is a morphism of modal H-twist-structures.

The above results allow to prove that the category of modalized $N4^{\perp}$-Esakia spaces is dually equivalent to the category of modal H-twist structures, and therefore to the category of $MN4^{\perp}$-lattices. We only need to deal with the natural transformations. This is straightforward once we have:

Lemma

Let $\langle X, \leq, \tau, C, O, \nu_1, \nu_2 \rangle$ be a modalized $N4^{\perp}$-Esakia space. Then the map $\varepsilon : X \to (X^{+})_{+}$ satisfies that

1. $\nu_{\Box_{\nu_1}}(\varepsilon(x)) = \{\varepsilon[U] : U \in \nu_1(x)\}$,
2. $\nu_{\Box_{\nu_2}}(\varepsilon(x)) = \{\varepsilon[U] : U \in \nu_2(x)\}$.

Conclusions and further work

- We have obtained a new duality for $N4^{\perp}$-lattices and we have extended it to $MN4^{\perp}$-lattices. In a more or less standard way they can be modified to obtain dualities for $N4$-lattices and $MN4$-lattices. From these complete “relational semantics” for the Paraconsitent Modal logic of U. Rivieccio can be easily obtained using the algebraic completeness theorem.

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• The technique of representing algebras with a lattice reduct by twist structures is used in the study of pre-bilattices and bilattices. The work done can be a source of inspiration for the study of modal expansions of the logics defined using classes of bilattices.

Suppose $x \in \max(X)$. Then, for every clopen up-set U, we have that $x \notin (\downarrow U)^c$ iff $x \in \downarrow U$ iff there is $y \in U$ such that $x \leq y$. By maximality of x, this means that $x = y$, so $x \in U$. Hence, $x \in U \cup (\downarrow U)^c$ for every clopen up-set U. Conversely, suppose $x \notin \max(X)$, i.e., there is $y \in X$ such that $x < y$. Since X is a Priestley space, we know that there is a clopen up-set V such that $x \notin V$ and $y \in V$. Moreover, $x \in \downarrow V$, i.e., $x \notin (\downarrow V)^c$. This means that $x \notin V \cup (\downarrow V)^c$, so $x \notin \bigcap\{U \cup (\downarrow U)^c : U \in \mathrm{CloUP}(X)\}$.