A general duality theory for clones

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What is clone theory?

Notation

Let $O_A^{(n)}$ be the set of all n-ary operations $A^n \to A$.
$O_A := \bigcup_{n \in \mathbb{N}_+} O_A^{(n)}$ (all non-nullary operations over A).

Definition

A subset $C \subseteq O_A$ is a clone (of operations) if

- it contains all projections $\pi_i^n \colon A^n \to A : (x_1, \ldots, x_n) \mapsto x_i$,
- for all $f \in C^{(n)}$, $f_1, \ldots, f_n \in C^{(k)}$, the k-ary operation

  $$f(f_1, \ldots, f_n)(x_1, \ldots, x_k) := f(f_1(x_1, \ldots, x_k), \ldots, f_n(x_1, \ldots, x_k))$$

  is also in C.

(think: term operations)

Why bother?

Clones describe possible behaviours of algebras.
$\Longrightarrow$ Understanding all clones on A means understanding all algebras with base set A.

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However...
...as soon as $|A| \geq 3$, it seems totally out of reach to understand the lattice of all clones on A completely.

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Let $R_A^{(n)}$ be the set of all n-ary relations on A.
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Definition

An operation $f \in O_A^{(n)}$ preserves a relation $\sigma \in R_A^{(k)}$, written $f \rhd \sigma$, whenever

$$\begin{pmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{1k} \end{pmatrix}, \ldots, \begin{pmatrix} a_{n1} \\ a_{n2} \\ \vdots \\ a_{nk} \end{pmatrix} \in \sigma \Longrightarrow \begin{pmatrix} f(a_{11}, a_{21} \ldots, a_{n1}) \\ f(a_{12}, a_{22} \ldots, a_{n2}) \\ \vdots \\ f(a_{1k}, a_{2k} \ldots, a_{nk}) \end{pmatrix} \in \sigma.$$

That is, σ forms a subalgebra of $(A, f)^k$.

The "most basic Galois connection in algebra" (1/2).

Definition

Let $F \subseteq O_A$, $R \subseteq R_A$.

$$\operatorname{Inv} F := \{\sigma \in R_A \mid \forall f \in F : f \rhd \sigma\},$$
$$\operatorname{Pol} R := \{f \in O_A \mid \forall \sigma \in R : f \rhd \sigma\}.$$

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A well-known result...

The Galois-closed classes of Pol-Inv are local closures of clones of operations and local closures of so-called clones of relations.

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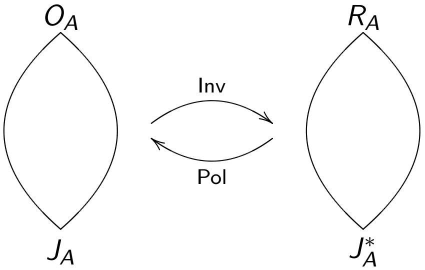
Let A be a finite set.



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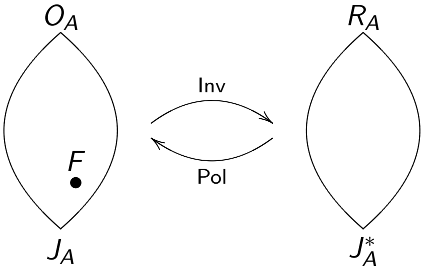

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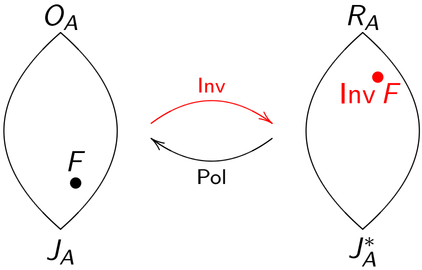

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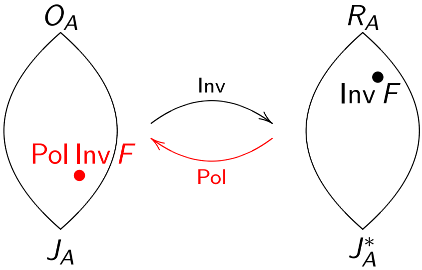

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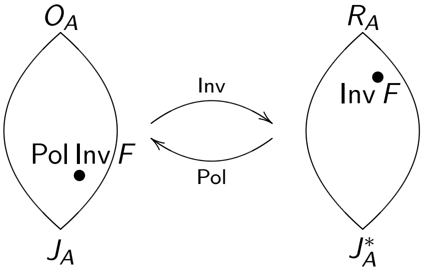

$F \subseteq O_A, R \subseteq R_A$

$\mathrm{Pol}\,\mathrm{Inv}\, F = \mathrm{Clo}(F)$

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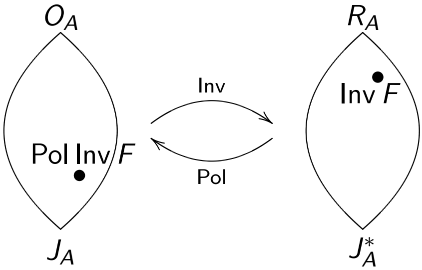

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- Usual approach: A clone C is interpreted as the set of term functions of an algebra and it is tried to dualize the algebra.
  Works for some clones.
- A new approach was suggested by D. Mašulović in 2006: Clones are dualized by treating them as sets of homomorphisms in a quasi-variety of algebras (understood as a category).
  Only works for centralizer clones with finite base set and dismisses Pol-Inv.


START

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- The method of dualizing clones as sets of morphisms in categories can be used much more generally.
- The Galois connection Pol-Inv can be generalized to categories and dualized together with the clones.
- This is useful.

Treating and dualizing clones categorically.
(a lot of triviality)

We define clones in categories.

Let $\mathbf{A}$ be an object in a category $\mathcal{C}$ in which the non-empty finite powers of $\mathbf{A}$ exist.

Definition

An n-ary operation over $\mathbf{A}$ is a morphism from $\mathbf{A}^n$ to $\mathbf{A}$. Let $O_{\mathbf{A}}$ be the set of all finitary operations over $\mathbf{A}$.

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Definition

A set $C \subseteq O_{\mathbf{A}}$ is a clone of operations over $\mathbf{A}$ if it contains all the projection morphisms over $\mathbf{A}$ and, for $f \in C^{(n)}$, $f_1, \ldots, f_n \in C^{(k)}$, we also have

$$\underset{\mathbf{A}^n \to \mathbf{A}}{\overset{\uparrow}{f}} \circ \underbrace{\langle f_1, \ldots, f_n \rangle}_{\mathbf{A}^k \to \mathbf{A}^n} \in C.$$

Examples.

- If $\mathcal{C} = \mathcal{Set}$, this notion coincides with the usual notion of a clone.
- If $\mathbf{A} \in \mathcal{Top}$, then $O_{\mathbf{A}}$ is the clone of the topological space $\mathbf{A}$.
- Each clone C on a finite set A can be written as $O_{\mathbf{A}}$ for some algebraic structure $\mathbf{A}$ with carrier set A.
  (for instance, take $\mathbf{A} = \langle A, \mathrm{Inv}\, C \rangle$).

What's the connection to Lawvere theories?

A Lawvere theory is a small category with countably many objects $\mathbf{t_0}, \mathbf{t}_1, \mathbf{t}_2, \dots$ such that $\mathbf{t}_n$ is the n-th power of $\mathbf{t}_1$.

Fact

A set of operations C over $\mathbf{A} \in \mathcal{C}$ is a clone iff there exists a Lawvere theory $\mathcal{L}$ and a product-preserving functor $M\colon \mathcal{L} \to \mathcal{C}$ such that $M(\mathbf{t}_1) = \mathbf{A}$ and

$$C = \{M(f) \mid f \in \mathcal{L}(\mathbf{t}_n, \mathbf{t}_1), n \in \mathbb{N}\}.$$

Many definitions can be extended to this setting.

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- identities (~~$f(x,x,y) \approx x$~~ $f \circ \langle \pi^2_1, \pi^2_1, \pi^2_2 \rangle = \pi^2_1$)
- types of operations (nu, idempotency, semiprojection...)
  $f \in O^{(n)}_{\mathbf{A}}$ idempotent $:\Longleftrightarrow f \circ \langle id_{\mathbf{A}}, \dots, id_{\mathbf{A}} \rangle = id_{\mathbf{A}}$.

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  $f \in O^{(n)}_{\mathbf{A}}$ idempotent $:\Longleftrightarrow f \circ \langle id_{\mathbf{A}}, \dots, id_{\mathbf{A}} \rangle = id_{\mathbf{A}}$.
- essential arity
  i-th variable of $f \in O^{(n)}_{\mathbf{A}}$ nonessential $:\Longleftrightarrow$
  $f \circ \langle \pi^{n+1}_1, \dots, \pi^{n+1}_n \rangle = f \circ \langle \pi^{n+1}_1, \dots, \pi^{n+1}_{i-1}, \pi^{n+1}_{n+1}, \pi^{n+1}_{i+1}, \dots, \pi^{n+1}_n \rangle$.

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- types of operations (nu, idempotency, semiprojection...)
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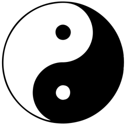
- minimality of a clone,
- ...

We can now dualize all notions.

operations	dual operations
clones of operations	clones of dual operations
essential variables of operations	essential variables of dual operations
nu operations	dual nu operations
⋮	⋮



Dual operations and dual clones.

Let $\mathbf{X}$ be an object in a category $\mathcal{X}$ in which the non-empty finite ~~powers~~ copowers of $\mathbf{X}$ exist.

Definition

An n-ary dual operation over $\mathbf{X}$ is a morphism from ~~$\mathbf{A}^n$ to $\mathbf{A}$~~ $\mathbf{X}$ to $n \cdot \mathbf{X}$. Let $\overline{O}_{\mathbf{X}}$ be the set of all finitary dual operations over $\mathbf{X}$.

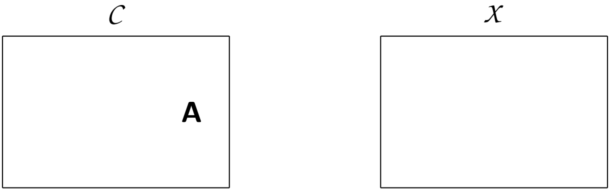
Definition

A set $C \subseteq \overline{O}_{\mathbf{X}}$ is a clone of dual operations over $\mathbf{X}$ if it contains all the ~~projection~~ injection morphisms over $\mathbf{X}$ and, for $f \in C^{(n)}$, $f_1, \ldots, f_n \in C^{(k)}$, we also have

$$\cancel{f \circ \langle f_1, \ldots, f_n \rangle \in C} \quad \underbrace{[f_1, \ldots, f_n]}_{n\cdot\mathbf{X} \to k\cdot\mathbf{X}} \circ \underset{\mathbf{X}\to n\cdot\mathbf{X}}{\underset{\uparrow}{f}} \in C.$$

The clone-duality.

Let $\mathcal{C}$ and $\mathcal{X}$ be dually equivalent categories.



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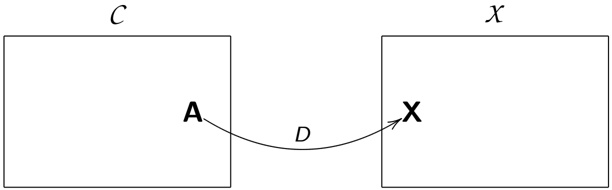

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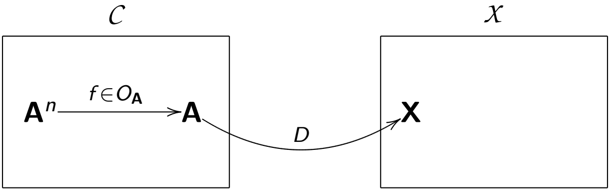

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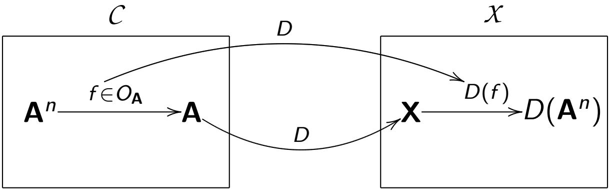

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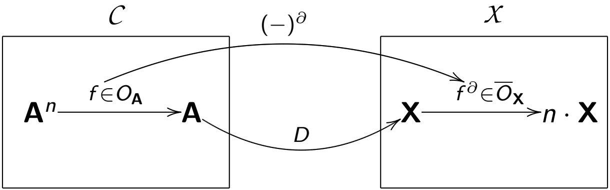

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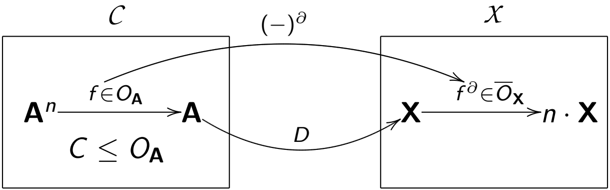

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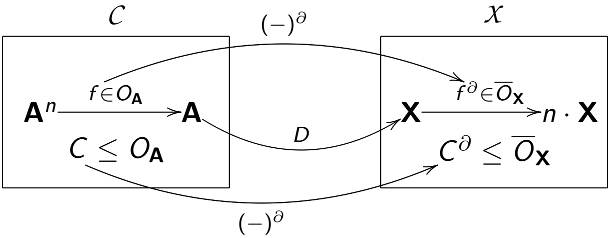
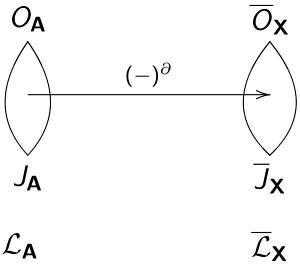


- The set of dual operations $C^\partial = \{f^\partial \mid f \in C\}$ is a clone of dual operations over $\mathbf{X}$ iff C is a clone of operations over $\mathbf{A}$.

The duality on the clone lattices.

Consequence.

The lattice of clones of operations over **A** and that of clones of dual operations over **X** are isomorphic.



2

Generalizing and dualizing Pol-Inv.

Generalizing relations.

Rewrite relations as mappings...

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This will be our starting point for generalizing relations.

The idea.

We do not take sets of the form $\{1, \ldots, k\}$, but objects from the category $\mathcal{C}$.

Generalized relations.

Definition

Let $\{1, \ldots, k\} \in \mathcal{S}et$.

A k-ary relation is a subset of $\mathcal{S}et(\{1, \ldots, k\}, A)$.

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Definition

For $f \in O_{\mathbf{A}}^{(n)}$ and a relation σ of type $\mathbf{B}$:

$$f \triangleright \sigma :\Longleftrightarrow \forall r_1, \ldots, r_n \in \sigma : \underset{\mathbf{A}^n \to \mathbf{A}}{\underset{\uparrow}{f}} \circ \underbrace{\langle r_1, \ldots, r_n \rangle}_{\mathbf{B} \to \mathbf{A}^n} \in \sigma.$$

Generalized relations (cont'd.).

Definition

Let $\mathbb{T}$ be an non-empty class of objects from (a skeleton of) $\mathcal{C}$.

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For $F \subseteq O_{\mathbf{A}}$ and $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, define

$$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F := \{\sigma \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}} \mid \forall f \in F : f \rhd \sigma\},$$
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Remark

For $\mathcal{C} = \mathcal{S}et$ and $\mathbb{T} = \{\{1, \ldots, k\} \mid k \in \mathbb{N}\}$, $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ coincides with Pol-Inv.

The main result for $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ (verbally).

The Galois-closed classes are what you expect.

Generalized local closures of clones of operations & generalized local closures of clones of relations.

Clones of relations.

Definition

A class R of relations is called a clone of relations on $\mathbf{A}$ if

- $\emptyset \in R$,
- R is closed under general superposition, i.e. the following holds: Let I be an index class, $\sigma_i \in R^{(\mathbf{B}_i)}$ $(i \in I)$ and let $\varphi : \mathbf{B} \to \mathbf{C}$ and $\varphi_i : \mathbf{B}_i \to \mathbf{C}$ be morphisms where $\mathbf{C} \in \mathcal{C}$ and $\mathbf{B} \in \mathbb{T}$. Then we also have $\bigwedge^{\varphi}_{(\varphi_i)}(\sigma_i) \in R$, where

$$\bigwedge\nolimits^{\varphi}_{(\varphi_i)}(\sigma_i) := \{r \circ \varphi \mid \forall\, i \in I : r \circ \varphi_i \in \sigma_i,\ r \in \mathcal{C}(\mathbf{C}, \mathbf{A})\}.$$

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This is a very natural definition. Really.

Local closure operators.

Definition

Let $F \subseteq O_{\mathbf{A}}$, $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$, $s \geq 1$ and let $\mathbf{C} \in \mathbb{T}$. We define the following local closure operators:

$$\begin{aligned}\mathbf{C}\text{-Loc}\, F :=& \{f \in O_{\mathbf{A}}^{(n)} \mid n \geq 1, \forall\, r_1, \ldots, r_n \in \mathcal{C}(\mathbf{C}, \mathbf{A}) : \\ & \qquad \exists\, f' \in F : f \circ \langle r_1, \ldots, r_n \rangle = f' \circ \langle r_1, \ldots, r_n \rangle\}, \\ \text{s-LOC}^{\mathbb{T}}\, R :=& \{\sigma \in \mathrm{R}_{\mathbf{A}}^{\mathbb{T}} \mid \forall B \subseteq \sigma, |B| \leq s : \exists\, \sigma' \in R : B \subseteq \sigma' \subseteq \sigma\}.\end{aligned}$$

$$\mathrm{Loc}^{\mathbb{T}}\, F := \bigcap_{\mathbf{C} \in \mathbb{T}} \mathbf{C}\text{-Loc}\, F, \qquad \mathrm{LOC}^{\mathbb{T}}\, R := \bigcap_{s \geq 1} \text{s-LOC}^{\mathbb{T}}\, R.$$

The main result for $\mathrm{Pol}_{\mathbf{A}}$-$\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}}$ (formally).

Theorem

Let $F \subseteq O_{\mathbf{A}}$, $R \subseteq \overline{O}_{\mathbf{X}}$. Then,

- $\mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} F = \mathrm{Loc}^{\mathbb{T}} \mathrm{Clo}(F)$,
- $\mathrm{Inv}_{\mathbf{A}}^{\mathbb{T}} \mathrm{Pol}_{\mathbf{A}} R = \mathrm{LOC}^{\mathbb{T}} \mathrm{Clo}(R)$.

When can we forget the local closure operators?

Fact (from the results about the usual Pol-Inv)

If $\mathcal{C} = \mathcal{S}et$ and $\mathbb{T} = \{\{1, \ldots, k\} \mid k \geq 1\}$, then we can dismiss both local closure operators if A is a finite set.

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If $\mathcal{C} = \mathcal{S}et$ and $\mathbb{T} = \{\{1, \ldots, k\} \mid k \geq 1\}$, then we can dismiss both local closure operators if A is a finite set.

Question

What is the category-theoretic property behind this?

When can we forget the local closure operators (cont'd.)?

Proposition

We have $\mathrm{LOC}^{\mathbb{T}} R = R$ *for all* $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ *iff* $\mathcal{C}(\mathbf{B}, \mathbf{A})$ *is finite for all* $\mathbf{B} \in \mathbb{T}$.

When can we forget the local closure operators (cont'd.)?

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We have $\mathrm{LOC}^{\mathbb{T}} R = R$ *for all* $R \subseteq \mathrm{R}_{\mathbf{A}}^{\mathbb{T}}$ *iff* $\mathcal{C}(\mathbf{B}, \mathbf{A})$ *is finite for all* $\mathbf{B} \in \mathbb{T}$.

Proposition

We have $\mathrm{Loc}^{\mathbb{T}} C = C$ *for all* $C \leq O_{\mathbf{A}}$ *if, for each* $k \in \mathbb{N}$*, there exists* $n \geq k$ *and some* $\mathbf{B} \in \mathbb{T}$ *such that there exists an epimorphism from* $\mathbf{B}$ *to* $\mathbf{A}^n$.

Do we still have a Baker-Pixley type result?

Theorem (Baker, Pixley)

Assume that $F \subseteq O_A$ *contains a* $(d+1)$*-ary near-unanimity operation and that* A *is finite.*
Then $\mathrm{Clo}(F) = \mathrm{Pol}\,\mathrm{Inv}^{(d)} F$.

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Theorem (Baker, Pixley)

Assume that $F \subseteq O_{\mathbf{A}}$ contains a $(d+1)$-ary near-unanimity operation and that $\mathbf{A} \in \mathcal{S}et$ is finite.
Then $\mathrm{Clo}(F) = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(d \cdot \{1\})} F$.

Question

What is the category-theoretic property behind this?
More precisely: Which category theoretic property of $\{1\}$ and which category-theoretic consequence of the finiteness of $\mathbf{A}$ is needed in order to make this work?

Do we still have a Baker-Pixley type result? (cont'd.)

Generalized Baker-Pixely Theorem

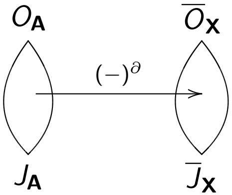
Assume that $F \subseteq O_{\mathbf{A}}$ contains a $(d+1)$-ary near-unanimity operation and that $\mathbf{B} \in \mathcal{C}$ such that

- $d \cdot \mathbf{B} \in \mathcal{C}$,
- *For all $n \in \mathbb{N}$, there exists a finite subset $\mathcal{F} \subseteq \mathcal{C}(\mathbf{B}, \mathbf{A})$ s.t. for all $f, g \in O_{\mathbf{A}}^{(n)}$ we have $f = g$ whenever $f \circ \langle \alpha_1, \ldots, \alpha_n \rangle = g \circ \langle \alpha_1, \ldots, \alpha_n \rangle$ for all $\alpha_1, \ldots, \alpha_n \in \mathcal{F}$.*

Then $\mathrm{Clo}(F) = \mathrm{Pol}_{\mathbf{A}} \mathrm{Inv}_{\mathbf{A}}^{(\mathrm{d} \cdot \mathbf{B})} F$.

The full clone duality.

Let $\mathcal{C}$ and $\mathcal{X}$ be dually equivalent via $D\colon \mathcal{C} \to \mathcal{X}$.



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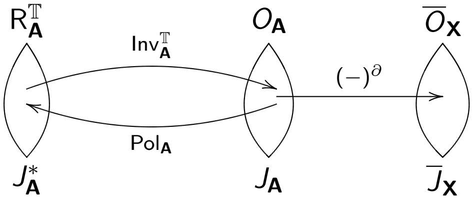

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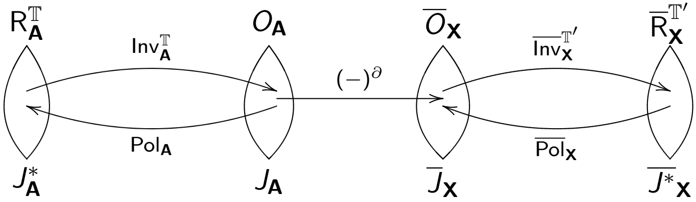

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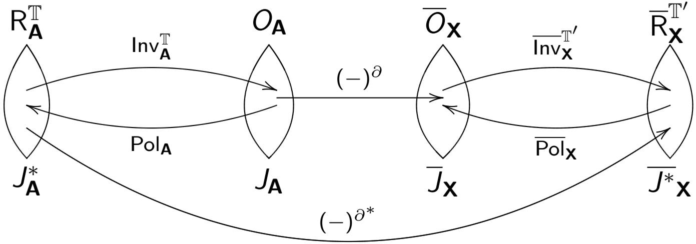
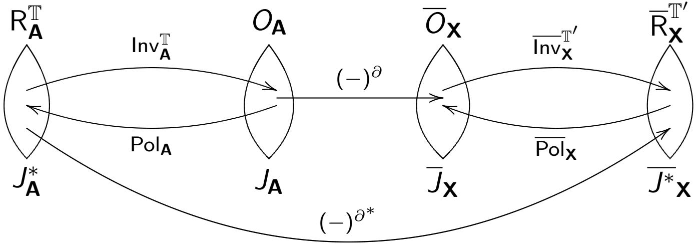


3

Using this framework.

There is some use in this.

In abstract categories:	Just a change of notation.
In concrete categories:	Different.



On the abstract level, its just pushing of symbols...

Let $f \in O_{\mathbf{A}}^{(n)}$.

f idempotent $\iff f \circ \langle id_{\mathbf{A}}, \dots, id_{\mathbf{A}} \rangle = id_{\mathbf{A}}$.

i-th variable of f nonessential $\iff$
$f \circ \langle \pi_1^{n+1}, \dots, \pi_n^{n+1} \rangle = f \circ \langle \pi_1^{n+1}, \dots, \pi_{i-1}^{n+1}, \pi_{n+1}^{n+1}, \pi_{i+1}^{n+1}, \dots, \pi_n^{n+1} \rangle$

On the abstract level, its just pushing of symbols...

Let $f \in O_{\mathbf{A}}^{(n)}$.

f idempotent $\iff [id_{\mathbf{X}}, \ldots, id_{\mathbf{X}}] \circ f^{\partial} = id_{\mathbf{X}}$.

i-th variable of f nonessential $\iff$
$[\iota_1^{n+1}, \ldots, \iota_n^{n+1}] \circ f^{\partial} = [\iota_1^{n+1}, \ldots, \iota_{i-1}^{n+1}, \iota_{n+1}^{n+1}, \iota_{i+1}^{n+1}, \ldots, \iota_n^{n+1}] \circ f^{\partial}$.

...but the viewpoint really changes in the concrete case.

Let $f \in O_{\mathbf{A}}^{(n)}$.

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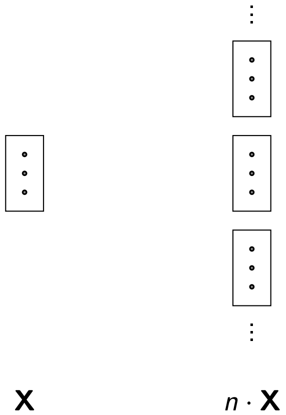
$$\iff \begin{cases} f^{\partial}[\mathbf{X}] \subseteq [\iota_1^n, \dots, \iota_{i-1}^n, \iota_i^n, \iota_{i+1}^n, \dots, \iota_n^n][(n-1) \cdot \mathbf{X}] & \text{if } n \geq 2, \\ \iota_1^2(x) = \iota_2^2(x) \text{ for all } x \in f^{\partial}[\mathbf{X}] & \text{if } n = 1. \end{cases}$$

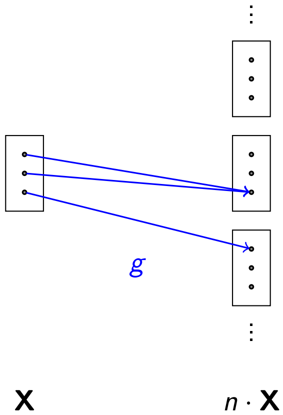
Let's try this for $\mathcal{T}op$.

$\mathcal{X} = \mathcal{T}op$



Let's try this for $\mathcal{Top}$.

$\mathcal{X} = \mathcal{Top}$



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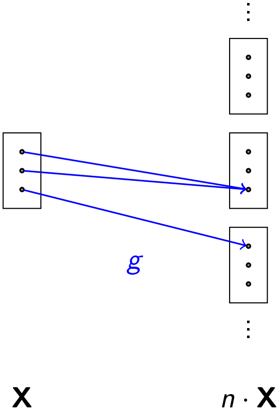

Essentially binary

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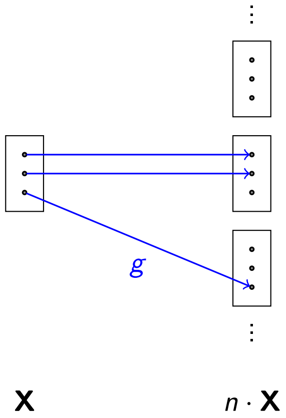

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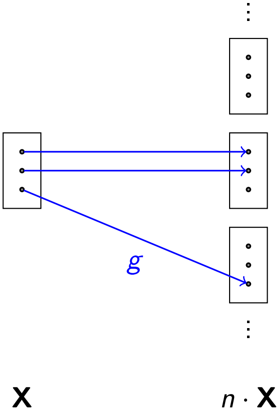

$$\forall x \in X \; \exists i \in \{1, \ldots, n\} : \\ g(x) = \iota_i^n(x)$$

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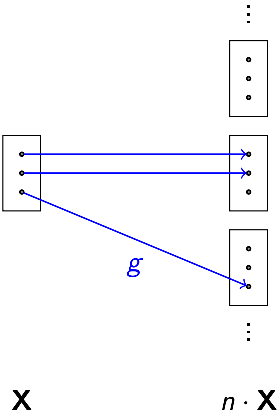


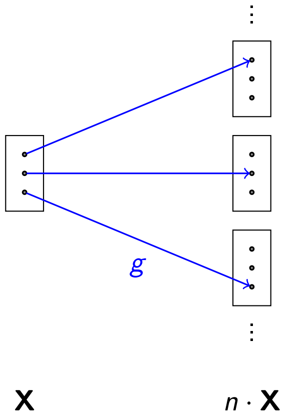
$$\forall x \in X \; \exists i \in \{1, \ldots, n\} : \\ g(x) = \iota_i^n(x)$$

Idempotent

Essentially binary

Let's try this for $\mathcal{Top}$.

$\mathcal{X} = \mathcal{Top}$



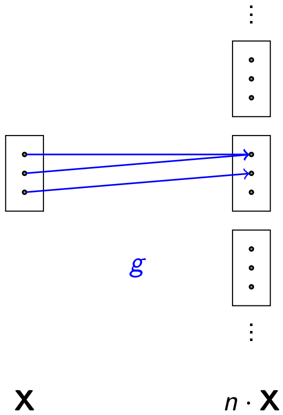
$$\forall x \in X \; \exists i \in \{1, \ldots, n\} : \\ g(x) = \iota_i^n(x)$$

Idempotent

Essentially ternary

Let's try this for $\mathcal{Top}$.

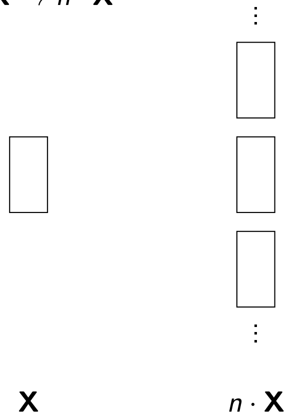
$\mathcal{X} = \mathcal{Top}$



Essentially unary

Continuity, please!

$g : \mathbf{X} \to n \cdot \mathbf{X}$



Continuity, please!

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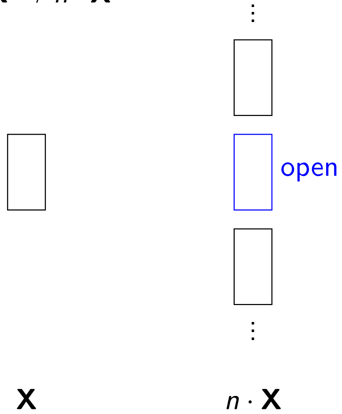

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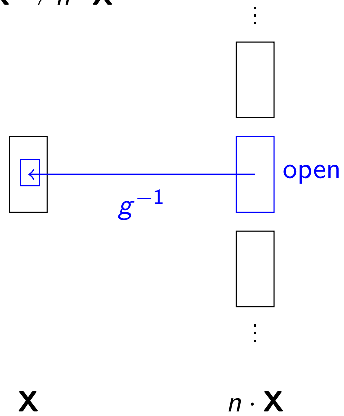

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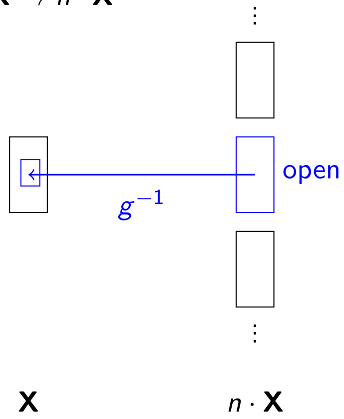


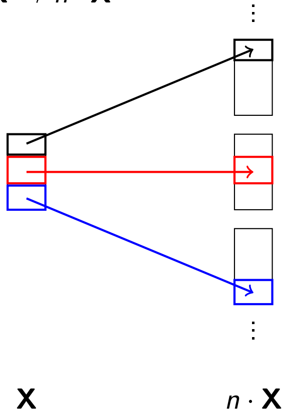
Essentiality of variables

$\longleftrightarrow$

Connectedness of $\mathbf{X}$

Continuity, please!

$g : \mathbf{X} \to n \cdot \mathbf{X}$



Idempotent dual operations over $\mathbf{X}$
$\longleftrightarrow$
Connectedness of $\mathbf{X}$

Putting this together.

Theorem

Let $n \in \mathbb{N}$. The following statements are equivalent:

a) $\mathbf{X}$ *has exactly n connected components.*

Putting this together.

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Let $n \in \mathbb{N}$. The following statements are equivalent:

a) $\mathbf{X}$ *has exactly n connected components.*

b) *The essential arity of the dual operations among $\overline{O}_{\mathbf{X}}$ is strictly bounded by n.*

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c) *The lattice of idempotent clones of dual operations over $\mathbf{X}$ is isomorphic to the partition lattice $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$.*

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d) *For each $k \in \mathbb{N}$, there are exactly $k!S(n, k)$ essential k-ary dual idempotent operations over $\mathbf{X}$.*

Putting this together.

Theorem

Let $n \in \mathbb{N}$. The following statements are equivalent:

a) *$D(\mathbf{A})$ has exactly n connected components.*
b) *The essential arity of the operations among $O_{\mathbf{A}}$ is strictly bounded by n.*
c) *The lattice of idempotent clones of operations over $\mathbf{A}$ is isomorphic to the partition lattice $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$.*
d) *For each $k \in \mathbb{N}$, there are exactly $k!S(n, k)$ essential k-ary idempotent operations over $\mathbf{A}$.*

Putting this together.

Theorem

Let $n \in \mathbb{N}$. The following statements are equivalent:

a) $D(\mathbf{A})$ *is the coproduct of* n *coproduct-irreducible top. spaces.*

b) *The essential arity of the operations among* $O_{\mathbf{A}}$ *is strictly bounded by* n.

c) *The lattice of idempotent clones of operations over* $\mathbf{A}$ *is isomorphic to the partition lattice* $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$.

d) *For each* $k \in \mathbb{N}$, *there are exactly* $k!S(n, k)$ *essential* k*-ary idempotent operations over* $\mathbf{A}$.

Putting this together.

Theorem

Let $n \in \mathbb{N}$. The following statements are equivalent:

a) $\mathbf{A}$ *is the product of* n *product-irreducible objects from* $\mathcal{C}$.

b) *The essential arity of the operations among* $O_{\mathbf{A}}$ *is strictly bounded by* n.

c) *The lattice of idempotent clones of operations over* $\mathbf{A}$ *is isomorphic to the partition lattice* $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$.

d) *For each* $k \in \mathbb{N}$, *there are exactly* $k!S(n,k)$ *essential* k*-ary idempotent operations over* $\mathbf{A}$.

With the Stone duality $D\colon \mathcal{Bool} \to \mathcal{Stone}$.

Corollary

Let $\mathbf{A}$ *be a Boolean Algebra. For each* $n \in \mathbb{N}$, *TFAE:*

a) $D(\mathbf{A})$ *has exactly* n *connected components.*

b) *The essential arity of the polymorphisms of* $\mathbf{A}$ *is strictly bounded by* n.

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With the Stone duality $D\colon \mathcal{Bool} \to \mathcal{Stone}$.

Corollary

Let **A** *be a Boolean Algebra. For each* $n \in \mathbb{N}$, *TFAE:*

a) **A** *has exactly* 2^n *elements.*

b) *The essential arity of the polymorphisms of* **A** *is strictly bounded by* n.

c) *The lattice of idempotent clones of polymorphisms of* **A** *is isomorphic to the partition lattice* $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$.

d) *For each* $k \in \mathbb{N}$, *there are exactly* $k!S(n,k)$ *essential* k*-ary dual idempotent operations over* **X**.

With the Gelfand duality $D\colon \mathcal{C^*Alg} \to \mathcal{Comp_2}$.

Corollary

Let $\mathbf{A}$ *be a comm. unital* C^**-Algebra. For each* $n \in \mathbb{N}$*, TFAE:*

a) $D(\mathbf{A})$ *has exactly* n *connected components.*

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c) *The lattice of idempotent clones of polymorphisms of* $\mathbf{A}$ *is isomorphic to the partition lattice* $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$*.*

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With the Gelfand duality $D\colon \mathcal{C}^*\mathcal{Alg} \to \mathcal{Comp}_2$.

Corollary

Let **A** *be a comm. unital C^*-Algebra. For each $n \in \mathbb{N}$, TFAE:*

a) **A** *has exactly 2^n idempotent elements.*

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Let's look at similar, yet different result.

We take the Priestley duality.

Proposition

Let **A** *be a bounded distr. lattice. For each* $n \in \mathbb{N}$*, TFAE:*

a) $D(\mathbf{A})$ *is the coproduct of* n *coproduct-irreducible Priestley spaces.*
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Let's look at similar, yet different result.

We take the Priestley duality.

Proposition

Let $\mathbf{A}$ *be a bounded distr. lattice. For each* $n \in \mathbb{N}$*, TFAE:*

a) *There exist* n *(but not more) elements* $a_1, \ldots, a_n \in A \setminus \{0\}$ *such that* $\bigvee a_i = 1$ *and* $a_i \wedge a_j = 0$ *for* $i \neq j$*.*

b) *The essential arity of the polymorphisms of* $\mathbf{A}$ *is strictly bounded by* n*.*

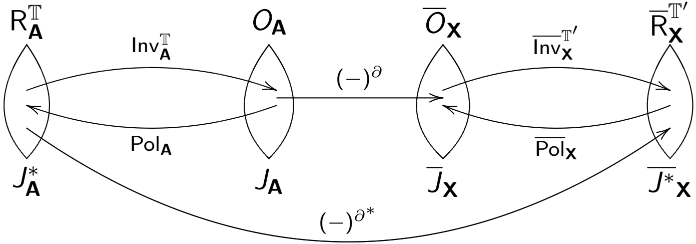
c) *The lattice of idempotent clones of polymorphisms of* $\mathbf{A}$ *is isomorphic to the partition lattice* $\langle \mathrm{Part}(\{1, \ldots, n\}), \preccurlyeq \rangle$*.*

d) *For each* $k \in \mathbb{N}$*, there are exactly* $k!S(n,k)$ *essential* k*-ary dual idempotent operations over* $\mathbf{X}$*.*

Every slide needs a title.

Why did this work?

The answer lies in the coproduct.



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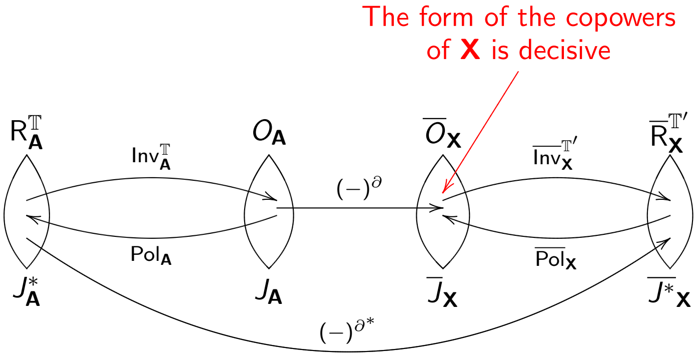

Looking at the copowers of the structures.

Definition (for concrete categories)

$\mathbf{X}$ has non-deformed copowers to the degree k, if, for any $n \geq k$, each $y \in n \cdot \mathbf{X}$ is in the image of $\iota\colon k \cdot \mathbf{X} \to n \cdot \mathbf{X}$ where ι is a cotupling of injection morphisms.

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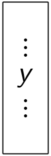

$n \cdot \mathbf{X}$

Looking at the copowers of the structures.

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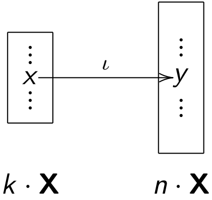
$\mathbf{X}$ has non-deformed copowers to the degree k, if, for any $n \geq k$, each $y \in n \cdot \mathbf{X}$ is in the image of $\iota \colon k \cdot \mathbf{X} \to n \cdot \mathbf{X}$ where ι is a cotupling of injection morphisms.



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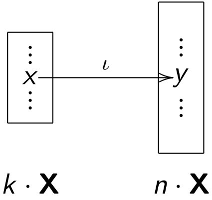

This property is rare, but there are many well-known categories in which all objects have non-deformed copowers to the degree 1: $\mathcal{Set}$, $\mathcal{Top}$, $p\mathcal{Set}$, $\mathcal{Graph}$, $\mathcal{Pries}$,...

An example of what easy copowers give us.

Let $\mathbf{X}$ be finite and let $\mathcal{X}$ be concrete.

Theorem

The copowers of $\mathbf{X}$ are non-deformed to some degree $k \in \mathbb{N}$.
$\Longrightarrow$ *essential arity of operations among $O_{\mathbf{A}}$ is bounded.*

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Let $\mathbf{X}$ be finite and let $\mathcal{X}$ arise via a concrete duality with the dualizing object $\mathbf{M}$ being a retract of $\mathbf{A}$.

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- ...the ideal of all idempotent clones $\langle \mathcal{I}_{\mathbf{A}}]$ is isomorphic to $\langle \mathrm{Part}(Y), \preccurlyeq \rangle$ for some set Y.
- ...we can fully characterize all minimal clones in $\mathcal{L}_{\mathbf{A}}$.

There are (many) examples in which we can use this.

If we dualize the following clones, then we obtain a clone of dual operations over **X** which has non-deformed copowers of degree 1:

- Clones over Boolean algebras,
- clones over De Morgan algebras,
- clones over Heyting algebras,
- clones over (bounded) distributive lattices,
- clones over median algebras,
- clones over commutative unital C^*-algebras,
- clones over M-spaces with unit,
- ...

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Is the clone generated by all unary and all idempotent operations over $\mathbf{A}$ the full clone $O_{\mathbf{A}}$?

Can we also get the concrete answer?

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Theorem

The following two statements are equivalent:

1. $\mathrm{Clo}\,(\mathcal{I}_{\mathbf{A}} \cup \mathrm{End}\,\mathbf{A}) = O_{\mathbf{A}}$.
2. *For each* $Y \in \mathrm{Con}(\mathbf{X})$ *and* $(Y_1, Y_2) \in \mathrm{Spl}(Y)$ *there exists* $Y' \in \mathrm{Con}(\mathbf{X}) \setminus \{Y\}$ *such that* Y_1 *or* Y_2 *can be order-embedded into* Y'.

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- Thank you!