Adjunctions induced by the Congruence Lattices of the Free Algebras in a Variety

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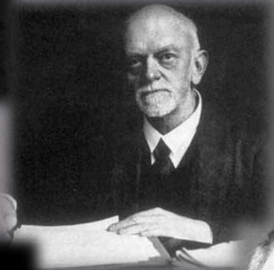
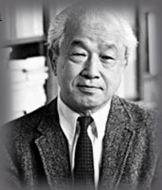
L. Pontryagin



D. Hilbert

K. Yosida



S. Kakutani

M. Stone

G. Birkhoff

General adjunction

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(i) The *associated congruence* (on $\mathscr{F}_\kappa$) of $S \subseteq A^\kappa$ is

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(ii) The *solution set* (over A) of $R \subseteq \mathscr{F}_\kappa^2$ is

$$\mathbb{V}(R) := \{a \in A^\kappa \mid s(a) = t(a) \text{ holds in } A, \text{ for all } (s,t) \in R\}.$$

Remark

(i) For any $R \subseteq \mathscr{F}_{\mathsf{K}}^{2}$, $\mathbb{V}(R) = \mathbb{V}(\langle R \rangle)$, where $\langle R \rangle$ is the congruence on $\mathscr{F}_{\mathsf{K}}$ generated by R.

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$$\mathbb{I}: \; 2^{A^\kappa} \longrightarrow 2^{\mathscr{F}_\kappa^2}$$
$$\mathbb{V}: \; 2^{\mathscr{F}_\kappa^2} \longrightarrow 2^{A^\kappa}$$

The pair $(\mathbb{I}, \mathbb{V})$ is a Galois connection between the powersets of $\mathscr{F}_\kappa^2$ and A^κ, *i.e.*

$$R \subseteq \mathbb{I}(S) \quad \textit{if, and only if,} \quad S \subseteq \mathbb{V}(R).$$

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This can be made functorial.

- Category A: A-algebras and homomorphisms.
- Category $\mathsf{Sub}_{\mathrm{d}}^{A}$: Subsets of A^κ and *definable maps*.

Definition

Given $S \subseteq A^\kappa$ and $T \subseteq A^\mu$, a function $\lambda\colon S \to T$ is *definable* if there exists a μ-tuple of terms $(l_\beta)_{\beta<\mu}$, with $l_\beta \in \mathscr{F}_\kappa$, such that

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$$\forall S \subseteq A^{\kappa}: \qquad \forall A \in \mathsf{A} \text{ with } A \cong \mathscr{F}_\kappa/\theta:$$

$$\mathscr{I}(S) = \mathscr{F}_\kappa/\mathbb{I}(S) \qquad \mathscr{V}(\mathscr{F}_\kappa/\theta) = \mathbb{V}(\theta)$$

The functor $\mathscr{I}$: Arrows.

Given $S \subseteq A^\kappa$, $T \subseteq A^\mu$, and definable map $\lambda\colon S \to T$ with $(l_\beta)_{\beta<\mu}$ be a μ-tuple of defining terms for λ. There is an induced homomorphism

$$\mathscr{I}(\lambda)\colon \mathscr{I}(T) \to \mathscr{I}(S)$$

which acts on each $s \in \mathscr{F}_\mu$ by substitution as follows:

$$\frac{s\big((X_\alpha)_{\beta<\mu}\big)}{\mathbb{I}(T)} \in \mathscr{I}(T) \quad \overset{\mathscr{I}(\lambda)}{\longmapsto} \quad \frac{s\big([X_\beta \backslash l_\beta]_{\beta<\mu}\big)}{\mathbb{I}(S)} \in \mathscr{I}(S)\,.$$

The functor $\mathscr{V}$: Arrows.

Given homomorphism $h\colon \mathscr{F}_\kappa/\theta_1 \to \mathscr{F}_\mu/\theta_2$. For each $\alpha < \kappa$, let π_α be the projection term on the α^{th} coordinate, and let π_α/θ_1 denote the equivalence class of π_α modulo θ_1. Fix, for each α, an arbitrary $f_\alpha \in h(\pi_\alpha/\theta_1)$. For any $(p_\beta)_{\beta<\mu} \in \mathbb{V}(\theta_2)$, set

$$\mathscr{V}(h)((p_\beta)_{\beta<\mu}) = \Big(f_\alpha\big((p_\beta)_{\beta<\mu}\big)\Big)_{\alpha<\kappa}.$$

Then $\mathscr{V}(h)\colon \mathbb{V}(\theta_2) \to \mathbb{V}(\theta_1)$ is a definable map.

Theorem (V.M. and L. Spada, 2012)

The functor $\mathscr{V}\colon \mathsf{A} \longrightarrow \left(\mathsf{Sub}^{A}_{\mathrm{d}}\right)^{\mathrm{op}}$ *is left adjoint to the functor* $\mathscr{I}\colon \left(\mathsf{Sub}^{A}_{\mathrm{d}}\right)^{\mathrm{op}} \longrightarrow \mathsf{A}$. *In symbols,* $\mathscr{V} \dashv \mathscr{I}$.

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Corollary

The dual adjunction above restricts to a dual equivalence between FixA *and* $\mathsf{FixSub}^{A}_{\mathrm{d}}$*.*

Source.

1. V.M. and L. Spada, *The Dual Adjunction between MV-algebras and Tychonoff spaces*, Studia Logica 100, *in memoriam* Leo Esakia, 2012.

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Related constructions in the literature.

1. Algebraic geometry over groups. (*G. Baumslag, A. Myasnikov, B. Plotkin, V. Remeslennikov, etc.*)
2. Geometric categories (*Y. Diers.*)
3. Natural dualities (*D. Clark, B. Davey, M. Haviar, H. Priestley, etc.*)

Hilbert's *Nullstellensatz*

David Hilbert, circa 1900.

The classical setting for affine algebraic geometry.

- Study solutions of systems of polynomial equations $p_i(x_1, \ldots, x_n) = 0$, $i = 1, \ldots, m$, where both coefficients and solutions range in an algebraically closed ground field k, such as *e.g.* $\mathbb{C}$. Set $R = \{p_i(x_1, \ldots, x_n)\}_{i=1}^{m}$.

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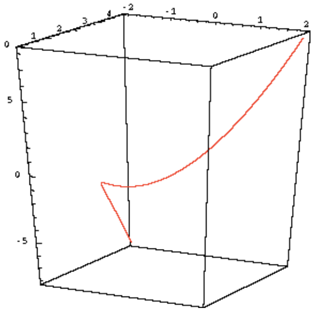
- $\mathbb{V}(R) = \mathbb{V}(\langle R \rangle)$, where $\langle R \rangle$ is the ideal generated by $R \subseteq k[x_1, \ldots, x_n]$.

The *twisted cubic*: $\mathbb{V}\left(\{y - x^2, z - x^3\}\right)$



(Parametrisation: $t \longmapsto (t, t^2, t^3)$.)

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- The pair $(\mathbb{V}, \mathbb{I})$ is a Galois connection between the powersets of $k[x_1, \ldots, x_n]$ and k^n.
- *Question*: What are the sets fixed by this Galois connection?

An ideal of $I \subseteq k[x_1, \ldots, x_n]$ is called:

- *Prime* if $k[x_1, \ldots, x_n]/I$ is an integral domain.
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Hilbert's *Nullstellensatz*

Let k be algebraically closed, let $R \subseteq k[x_1, \ldots, x_n]$, and set $I := \langle R \rangle$. Then:

$$\mathbb{I}\left(\mathbb{V}\left(R\right)\right) = I \iff I \text{ is radical.}$$

In other words, the subsets of $k[x_1, \ldots, x_n]$ fixed by $\mathbb{I} \circ \mathbb{V}$ are precisely the radical ideals.

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- The *radical* of $R \subseteq k[x_1, \ldots, x_n]$ is the ideal

$$\sqrt{R} := \bigcap \{P \subseteq k[x_1, \ldots, x_n] \mid P \text{ prime ideal containing } R\}.$$

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Corollary

For any $R \subseteq k[x_1, \ldots, x_n]$,

$$\mathbb{I}\left(\mathbb{V}\left(R\right)\right) = \sqrt{R}\,.$$

What about $\mathbb{V} \circ \mathbb{I}$? A subset $S \subseteq k^n$ is an *affine variety* if it is of the form $S = \mathbb{V}(R)$ for some $R \subseteq k[x_1, \ldots, x_n]$.

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Trivial *co-Nullstellensatz*

If k is algebraically closed, for any subset $S \subseteq k^n$ we have:

$$\mathbb{V}(\mathbb{I}(S)) = S \iff S \text{ is an affine variety.}$$

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Note. It is not trivial, however, that the Zariski-closed sets indeed are precisely the closed sets of a topology, *i.e.* that $\mathbb{V} \circ \mathbb{I} \colon 2^{k^n} \to 2^{k^n}$ satisfies Kuratowski's closure axioms.

Algebraic geometry	Universal algebra
Ground field k	A
$k[x_1, \ldots, x_n]$	$\mathscr{F}_n$
Affine space k^n	A^n
Ideal of $k[x_1, \ldots, x_n]$	Congruence on $\mathscr{F}_n$
Affine variety in k^n	Fixed subset of A^n
Coord. ring $k[x_i]/\mathbb{I}(\mathbb{V}(S))$	Quotient $\mathscr{F}_n/\mathbb{I}(\mathbb{V}(S))$
Homomorphism of k-alg.	A-homomorphism
Map of affine varieties	Term-definable map

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Map of affine varieties	Term-definable map
Nullstellensatz	?
co-*Nullstellensatz*	?
Maximal ideal	Maximal congruence
Prime ideal	?

Semisimple duality

We shall presently state an analogue of the *Nullstellensatz* for abstract varieties, in the case of maximal congruences.

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- A congruence is *maximal* if it is proper and inclusion-maximal.
- An A-algebra is *simple* if it has no non-trivial (=proper, non-identity) congruences.
- A congruence is *max-radical* if it is proper, and an intersection of maximal congruences.
- The *max-radical* of a congruence θ, written $\sqrt{\theta}$, is the intersection of all maximal congruences extending θ.
- An A-algebra is *semisimple* if it is a subdirect product of simple algebras.

Definition (Stone-Gel'fand-Kolmogorov condition)

The A-algebra A has *enough max-points* if, for each κ, the correspondence $p \in A^{\kappa} \longmapsto \mathbb{I}\,(p) := \mathbb{I}\,(\{p\})$ is a bijection between points of A^{κ} and maximal congruences on $\mathscr{F}_{\kappa}$.

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Motivation. For the ring of continuous real-valued functions $\mathrm{C}(X)$ on a compact Hausdorff space X, the S.-G.-K. lemma states that maximal ideals are in bijection with the points of X via the assignment $x \longmapsto \{f \in \mathrm{C}(X) \mid f(x) = 0\}$.

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Lemma (*Nullstellensatz* for max-points)

If A has enough max-points, then the objects of FixA *are precisely the semisimple* A*-algebras.*

Sketch of Proof. To prove: For any $S \subseteq \mathscr{F}_{\kappa}{}^{2}$,

$$\mathbb{I}\left(\mathbb{V}\left(S\right)\right) = \sqrt{\langle S \rangle} := \bigcap \left\{\theta' \text{ maximal } \mid S \subseteq \theta'\right\} .$$

$i.e.$ $\mathbb{I} \circ \mathbb{V}$ constructs the radical congruence generated by S. We have

$$\mathbb{I}\left(\mathbb{V}\left(S\right)\right) = \mathbb{I}\Big(\bigcup_{p \in \mathbb{V}\left(S\right)} \{p\}\Big) = \bigcap_{p \in \mathbb{V}\left(S\right)} \mathbb{I}\left(p\right),$$

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where the last equality is given by G.a.n. (Galois abstract nonsense).
Suffices to show: $\{\mathbb{I}\left(p\right) \mid p \in \mathbb{V}\left(S\right)\}$ is the set of all maximal congruences on $\mathscr{F}_{\kappa}$ containing S. Indeed, each $\mathbb{I}\left(p\right)$ is maximal by enough max-points, and $\mathbb{I}\left(p\right)$ extends S by G.a.n. *Vice versa*, if θ is maximal, then by enough max-points we have $\mathbb{V}(\theta) = \{q\}$ for some $q \in A^{\kappa}$. Then $\theta \subseteq \mathbb{I}\left(q\right)$ by G.a.n., and therefore $\theta = \mathbb{I}\left(q\right)$ by the maximality of θ and $\mathbb{I}\left(q\right)$. Finally, if θ extends S, then $\mathbb{V}(\theta) \subseteq \mathbb{V}(S)$ by G.a.n., so that $q \in \mathbb{V}(S)$. □

Theorem (V.M. and L. Spada, 2012)

The functor $\mathscr{V}\colon \mathsf{A} \longrightarrow \left(\mathsf{Sub}_{\mathrm{d}}^{A}\right)^{\mathrm{op}}$ *is left adjoint to the functor* $\mathscr{I}\colon \left(\mathsf{Sub}_{\mathrm{d}}^{A}\right)^{\mathrm{op}} \longrightarrow \mathsf{A}$. *In symbols,* $\mathscr{V} \dashv \mathscr{I}$.

Induced equivalence.

Category FixA: Full subcategory of A with objects the A-algebras $B = \mathscr{F}_{\kappa}/\theta$ such that $\mathbb{I} \circ \mathbb{V}(B) = \theta$.

Category $\mathsf{FixSub}_{\mathrm{d}}^{A}$: Full subcategory of $\mathsf{Sub}_{\mathrm{d}}^{A}$ with objects the subsets $S \subseteq A^{\kappa}$ such that $\mathbb{V} \circ \mathbb{I}(S) = S$.

Corollary

The dual adjunction above restricts to a dual equivalence between FixA *and* $\mathsf{FixSub}_{\mathrm{d}}^{A}$.

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Corollary

If A has enough max-points, the dual adjunction above restricts to a dual equivalence between $\mathsf{A}_{\mathrm{rad}}$ *and* $\mathsf{FixSub}_{\mathrm{d}}^{A}$.

Proof. By the *Nullstellensatz* for max-points, $\mathsf{A}_{\mathrm{rad}} = \mathsf{FixA}$. □

Epilogue

An endofunction $\mathrm{Cl}\colon 2^X \to 2^X$ is a *closure operator* if it is:

1. *Extensive*: $S \subseteq \mathrm{Cl}(S)$,
2. *Isotone*: $S \subseteq T \Rightarrow \mathrm{Cl}(S) \subseteq \mathrm{Cl}(T)$, and
3. *Idempotent*: $\mathrm{Cl}(\mathrm{Cl}(S)) = \mathrm{Cl}(S)$.

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3. *Idempotent*: $\mathrm{Cl}(\mathrm{Cl}(S)) = \mathrm{Cl}(S)$.

For any cardinal κ,

$$\mathbb{V} \circ \mathbb{I} : \; 2^{A^\kappa} \longrightarrow 2^{A^\kappa} \quad \text{is a closure operator.}$$

$$\mathbb{I} \circ \mathbb{V} : \; 2^{\mathscr{F}_\kappa^2} \longrightarrow 2^{\mathscr{F}_\kappa^2} \quad \text{is a closure operator.}$$

Additional properties of closure operator Cl:

1. *Topological*: $\operatorname{Cl}\left(\bigcup_{i\in I} S_i\right) = \bigcup_{i\in I} \operatorname{Cl}(S_i)$, for each finite (poss. empty) index set I.
2. *Algebraic*: $\operatorname{Cl}(S) = \bigcup_{T\subseteq S} \operatorname{Cl}(S)$, where the union is restricted to finite subsets T.

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General questions.

Which conditions on the class A *and the algebra* A *imply that* $\mathbb{V} \circ \mathbb{I}$ *is topological, and* $\mathbb{I} \circ \mathbb{V}$ *is algebraic?*

	Enough max.	Enough primes	Cng. $\bigcap$ of primes
Boole $(A = \{0,1\})$	Yes	Yes	Yes
DL_{01} $(A = \{0,1\})$	No	Yes	Yes
MV $(A = [0,1])$	Yes	No	Yes
k-alg. $(A = k)$	Yes	No	No

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k-alg. $(A = k)$	Yes	No	No

	$\mathbb{V} \circ \mathbb{I}$ is topological	$\mathbb{I} \circ \mathbb{V}$ is algebraic
Boole $(A = \{0,1\})$	Yes (Stone)	Yes (Trivial *Nullstellensatz*)
DL_{01} $(A = \{0,1\})$	Yes (?)	Yes (Trivial *Nullstellensatz*)
MV $(A = [0,1])$	Yes (Tychonoff)	No (*Nullstellensatz* max.)
k-alg. $(A = k)$	Yes (Zariski)	No (*Nullstellensatz*)

	Enough max.	Enough primes	Cng. $\bigcap$ of primes
Boole $(A = \{0,1\})$	Yes	Yes	Yes
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MV $(A = [0,1])$	Yes	No	Yes
k-alg. $(A = k)$	Yes	No	No

	$\mathbb{V} \circ \mathbb{I}$ is topological	$\mathbb{I} \circ \mathbb{V}$ is algebraic
Boole $(A = \{0,1\})$	Yes (Stone)	Yes (Trivial *Nullstellensatz*)
DL_{01} $(A = \{0,1\})$	Yes (Priestley)	Yes (Trivial *Nullstellensatz*)
MV $(A = [0,1])$	Yes (Tychonoff)	No (*Nullstellensatz* max.)
k-alg. $(A = k)$	Yes (Zariski)	No (*Nullstellensatz*)

ORDERED TOPOLOGICAL SPACES AND THE REPRESENTATION OF DISTRIBUTIVE LATTICES

By H. A. PRIESTLEY

[Received 29 December 1970]

1. Introduction

In [**18**] it was shown that, by defining a topology and an order relation on the set of 2-valued homomorphisms of a distributive lattice, one could obtain a space dual to the given lattice in a natural way and so develop an analogue for distributive lattices of the standard duality theory for Boolean algebras.

This paper is a sequel to [**18**]. It falls into two parts: the first (§§ 2–5) deals with some aspects of the theory of ordered topological spaces; the second, concerning distributive lattices, takes as its starting point the basic representation theorem proved in [**18**] and proceeds, with the aid of the results of §§ 2–5, to relate properties of a lattice to properties of its dual space.

2. Ordered topological spaces

A subset E of a quasi-ordered set X is *increasing* if $x \in E$, $y \in X$, $x \leqslant y$ together imply $y \in E$. A *decreasing* set is defined dually. A set is decreasing if and only if its complement is increasing. For any subset E of X, $i(E)$ ($d(E)$) will denote the intersection of all increasing (decreasing) sets

Thank you for your attention.