Admissibility in Finite Algebras

George Metcalfe

Mathematics Institute
University of Bern

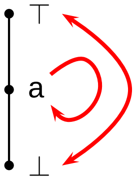
Joint work with Leonardo Cabrer and Christoph Röthlisberger

Workshop on Duality Theory in Algebra, Logic and Computer Science

Mathematical Institute, Oxford, 15-17 August, 2012

A First Example

Consider $\mathbf{C_3} = \langle \{\bot, \mathrm{a}, \top\}, \wedge, \vee, \neg \rangle$ described by:



There are **$\mathbf{C_3}$-valid** quasiequations such as

$$\{\neg x \approx y\} \;\Rightarrow\; x \approx \neg y,$$

and **$\mathbf{C_3}$-admissible** (perhaps not $\mathbf{C_3}$-valid) quasiequations like

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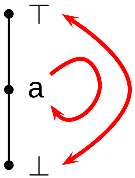

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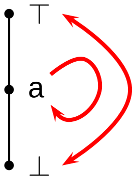

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Two Challenges

1. How can we *check* **A**-admissibility when **A** is finite?

- **A**-admissibility corresponds to validity in the *finite* **free algebra** $\mathbf{F_A}(|A|)$ and is hence **decidable**.
- But even $\mathbf{F_{C_3}}(2)$ has 82 elements... We look instead for (small) subalgebras of $\mathbf{F_A}(|A|)$ where validity matches **A**-admissibility.

2. How can we *axiomatize* **A**-admissibility in this case?

- We seek characterizations of the finite members of $\mathbb{Q}(\mathbf{A})$ that can be embedded into (powers of) $\mathbf{F_A}(\omega)$.
- We obtain these characterizations via **natural dualities**.

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Validity

Fix a finite algebra **A** for a language $\mathcal{L}$ with term algebra $\mathbf{Tm}_{\mathcal{L}}$.

An $\mathcal{L}$-**quasiequation** $\Sigma \Rightarrow \varphi \approx \psi$ is **A-valid**, written

$$\Sigma \models_{\mathbf{A}} \varphi \approx \psi,$$

if for every homomorphism $g \colon \mathbf{Tm}_{\mathcal{L}} \to \mathbf{A}$,

$$\begin{array}{c} g(\varphi') = g(\psi') \\ \text{for all } \varphi' \approx \psi' \in \Sigma \end{array} \quad \Longrightarrow \quad g(\varphi) = g(\psi).$$

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An **A-unifier** of a set of $\mathcal{L}$-equations Σ is a homomorphism (substitution) $\sigma\colon \mathbf{Tm}_{\mathcal{L}} \to \mathbf{Tm}_{\mathcal{L}}$ satisfying

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(If **A** is non-trivial, then given variables x, y not occurring in Σ:

$$\Sigma \text{ is } \mathbf{A}\text{-unifiable} \quad \Longleftrightarrow \quad \Sigma \Rightarrow x \approx y \text{ is not } \mathbf{A}\text{-admissible.})$$

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Free Algebras and Admissibility

Recall that the **free algebra** $\mathbf{F_A}(\kappa)$ on $\kappa \leq \omega$ generators may be taken to consist of equivalence classes of terms with respect to the congruence defined on $\mathbf{Tm}_{\mathcal{L}}(\kappa) \times \mathbf{Tm}_{\mathcal{L}}(\kappa)$ by $\varphi \sim \psi$ iff $\models_{\mathbf{A}} \varphi \approx \psi$.

For any finite algebra **A**:

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathbf{A}\text{-admissible} \quad \Longleftrightarrow \quad \Sigma \models_{\mathbf{F_A}(|A|)} \varphi \approx \psi.$$

Moreover, $\mathbf{F_A}(|A|)$ is **finite**, so checking **A**-admissibility is **decidable**.

But $\mathbf{F_A}(n)$ can be big even for small $|A|$ and n... Hence we seek (small) algebras **B** such that

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Checking Admissibility

For a **subalgebra B** of $\mathbf{F_A}(|A|)$ with **A** a **homomorphic image** of **B**:

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Recall that $\mathbb{V}(\mathbf{A}) = \mathbb{HSP}(\mathbf{A})$ and $\mathbb{Q}(\mathbf{A}) = \mathbb{ISP}(\mathbf{A})$ for finite **A**, where $\mathbb{H}$, $\mathbb{I}$, $\mathbb{S}$, and $\mathbb{P}$ denote closure under homomorphic and isomorphic images, subalgebras, and direct products, respectively.

Theorem

The following are equivalent:

(1) $\Sigma \Rightarrow \varphi \approx \psi$ *is* **A***-admissible* $\quad \Longleftrightarrow \quad \Sigma \models_{\mathbf{B}} \varphi \approx \psi$.

(2) $\mathbb{Q}(\mathbf{F_A}(|A|)) = \mathbb{Q}(\mathbf{B})$.

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For a finite algebra **A**:

(i) Find the smallest $m \leq |A|$ such that $\mathbf{A} \in \mathbb{H}(\mathbf{F_A}(m))$.

(ii) Compute the set $\mathbb{S}(\mathbf{F_A}(m))$ of subalgebras of $\mathbf{F_A}(m)$.

(iii) Construct the set $\mathrm{Adm}(\mathbf{A}) = \{\mathbf{B} \in \mathbb{S}(\mathbf{F_A}(m)) \mid \mathbf{A} \in \mathbb{H}(\mathbf{B})\}$.

(iv) Find a proof system to check validity in a smallest $\mathbf{B} \in \mathrm{Adm}(\mathbf{A})$.

Steps (i)-(iii) have been implemented using the Algebra Workbench; step (iv) can be implemented using, e.g., MUltlog/MUltseq or $_3T^AP$.

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Structural Completeness

In some cases, **A**-admissibility coincides with **A**-validity; that is

$$\Sigma \Rightarrow \varphi \approx \psi \text{ is } \mathbf{A}\text{-admissible} \quad \Longleftrightarrow \quad \Sigma \models_{\mathbf{A}} \varphi \approx \psi$$

and **A** is called **structurally complete**.

Consider, e.g., $\mathbf{S_3^{\rightarrow}} = \langle \{-1, 0, 1\}, \rightarrow \rangle$ with operation table:

$\rightarrow$	-1	0	1
-1	1	1	1
0	-1	0	1
1	-1	-1	1

The procedure discovers a subalgebra of the 60-element free algebra $\mathbf{F}_{\mathbf{S_3^{\rightarrow}}}(2)$ isomorphic to $\mathbf{S_3^{\rightarrow}}$, and hence that $\mathbf{S_3^{\rightarrow}}$ is structurally complete.

Structural completeness has also been confirmed for the 3-element implicational Łukasiewicz algebra, Gödel algebra, and Stone algebra.

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Consider, e.g., $\mathbf{S_3^{\rightarrow}} = \langle \{-1, 0, 1\}, \rightarrow \rangle$ with operation table:

$\rightarrow$	-1	0	1
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The procedure discovers a subalgebra of the 60-element free algebra $\mathbf{F}_{\mathbf{S_3^{\rightarrow}}}(2)$ isomorphic to $\mathbf{S_3^{\rightarrow}}$, and hence that $\mathbf{S_3^{\rightarrow}}$ is structurally complete.

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For any finite algebra **A** *and subalgebra* **B** *of* $\mathbf{F}_{\mathbf{A}}(\omega)$*:*

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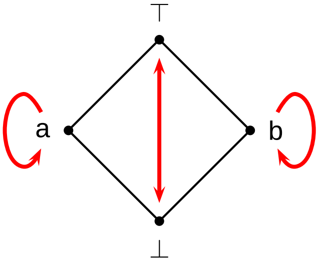
Example: A De Morgan Lattice

For the De Morgan lattice $\mathbf{D_4} = \langle \{\bot, \mathrm{a}, \mathrm{b}, \top\}, \wedge, \vee, \neg \rangle$ described by



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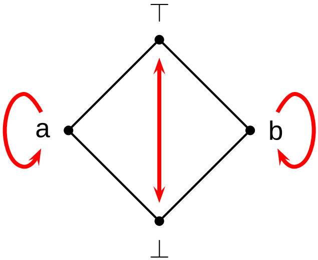

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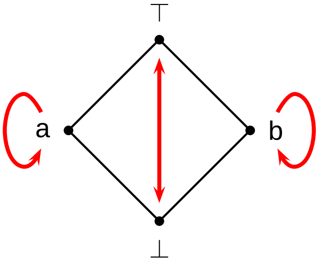

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Other Examples

- For the De Morgan algebra

$$\mathbf{D_4^b} = \langle \{\bot, a, b, \top\}, \wedge, \vee, \neg, \bot, \top \rangle$$

the procedure finds a smallest 10-element algebra in $\mathrm{Adm}(\mathbf{D_4^b})$.

- For the Kleene lattice and Kleene algebra

$$\mathbf{C_3} = \langle \{\top, a, \bot\}, \wedge, \vee, \neg \rangle \quad \text{and} \quad \mathbf{C_3^b} = \langle \{\top, a, \bot\}, \wedge, \vee, \neg, \bot, \top \rangle$$

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Consider the algebra

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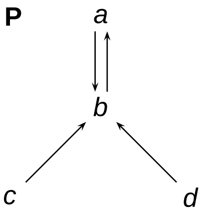
with $\star$ and the free algebras $\mathbf{F_P}(n)$ described by:



$\mathbf{F_P}(n)$

$$x_1 \longrightarrow \star(x_1) \rightleftarrows \star(\star(x_1))$$

$$\vdots$$

$$x_n \longrightarrow \star(x_n) \rightleftarrows \star(\star(x_n))$$

The smallest algebra in $\mathrm{Adm}(\mathbf{P})$ is $\mathbf{F_P}(2)$, but $\mathbf{P}$ can be embedded into $\mathbf{F_P}(1) \times \mathbf{F_P}(1)$, so $\mathbb{Q}(\mathbf{P}) = \mathbb{Q}(\mathbf{F_P}(4))$ and $\mathbf{P}$ is structurally complete.

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$$\mathbf{F_P}(n) \qquad \begin{tikzcd} x_1 \arrow[r] & \star(x_1) \arrow[r, shift left] & \star(\star(x_1)) \arrow[l, shift left] \\ & \vdots & \\ x_n \arrow[r] & \star(x_n) \arrow[r, shift left] & \star(\star(x_n)) \arrow[l, shift left] \end{tikzcd}$$

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For a finite algebra $\mathbf{A}$:

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Experiments in Admissibility

$\mathbf{A}$	$\lvert\mathbf{A}\rvert$	Quasivariety $\mathbb{Q}(\mathbf{A})$	Free algebra	$\lvert$Output Algebra$\rvert$
$\mathbf{Ł_3}$	3	algebras for $Ł_3$	$\lvert\mathbf{F_A}(1)\rvert = 12$	6
$\mathbf{Ł_3^{\rightarrow}}$	3	algebras for $Ł_3^{\rightarrow}$	$\lvert\mathbf{F_A}(2)\rvert = 40$	3
$\mathbf{B_1}$	3	Stone algebras	$\lvert\mathbf{F_A}(1)\rvert = 6$	3
$\mathbf{C_3^b}$	3	Kleene algebras	$\lvert\mathbf{F_A}(1)\rvert = 6$	4
$\mathbf{C_3}$	3	Kleene lattices	$\lvert\mathbf{F_A}(2)\rvert = 82$	4
$\mathbf{S}$	3	algebras for $\mathrm{RM}^{\rightarrow\neg}$	$\lvert\mathbf{F_A}(2)\rvert = 264$	6
$\mathbf{S_3^{\rightarrow}}$	3	algebras for $\mathrm{RM}^{\rightarrow}$	$\lvert\mathbf{F_A}(2)\rvert = 60$	3
$\mathbf{G_3}$	3	algebras for G_3	$\lvert\mathbf{F_A}(2)\rvert = 18$	3
$\mathbf{D_4}$	4	De Morgan lattices	$\lvert\mathbf{F_A}(2)\rvert = 166$	8
$\mathbf{D_4^b}$	4	De Morgan algebras	$\lvert\mathbf{F_A}(2)\rvert = 168$	10
$\mathbf{P}$	4	$\mathbb{Q}(\mathbf{P})$	$\lvert\mathbf{F_A}(2)\rvert = 6$	6
$\mathbf{S_4}$	4	$\mathbb{Q}(\mathbf{S_4})$	$\lvert\mathbf{F_A}(1)\rvert = 18$	6
$\mathbf{B_2}$	5	$\mathbb{Q}(\mathbf{B_2})$	$\lvert\mathbf{F_A}(1)\rvert = 7$	5

Axiomatizing Admissibility

- Can we **axiomatize A**-admissibility (i.e., $\mathbb{Q}(\mathbf{F_A}(\omega))$)?

- More generally, for a **quasivariety** $\mathcal{Q}$, can we axiomatize
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Characterizations of Admissibility

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(ii) $\Sigma \models_{\mathbf{F}_{\mathcal{Q}}(\omega)} \Delta$.

(iii) *For each finite set of equations* Π*:*

$$\models_{\mathcal{Q}} \Pi \quad \Leftrightarrow \quad \models_{\mathcal{U}} \Pi \qquad (\mathcal{U} = \{\mathbf{A} \in \mathcal{Q} \mid \Sigma \models_{\mathbf{A}} \Delta\}).$$

If $|\Delta| = 1$*, then the following is also equivalent to* (i)-(iii)*:*

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$$\models_{\mathcal{Q}} \varphi \approx \psi \quad \Leftrightarrow \quad \models_{\mathcal{Q}'} \varphi \approx \psi \qquad (\mathcal{Q}' = \{\mathbf{A} \in \mathcal{Q} \mid \Sigma \models_{\mathbf{A}} \Delta\}).$$

Characterizations of Admissibility

Theorem

For a quasivariety $\mathcal{Q}$ and clause $\Sigma \Rightarrow \Delta$, the following are equivalent:

(i) $\Sigma \Rightarrow \Delta$ *is* $\mathcal{Q}$*-admissible.*

(ii) $\Sigma \models_{\mathbf{F}_{\mathcal{Q}}(\omega)} \Delta$.

(iii) *For each finite set of equations* Π*:*

$$\models_{\mathcal{Q}} \Pi \quad \Leftrightarrow \quad \models_{\mathcal{U}} \Pi \qquad (\mathcal{U} = \{\mathbf{A} \in \mathcal{Q} \mid \Sigma \models_{\mathbf{A}} \Delta\}).$$

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Bases

A **basis for the admissible quasiequations** of a quasivariety $\mathcal{Q}$ is a set of quasiequations axiomatizing $\mathbb{Q}(\mathbf{F}_{\mathcal{Q}}(\omega))$ relative to $\mathcal{Q}$.

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A Notable Example

Iemhoff and Rozière established independently that the following "Visser rules" form a basis for the admissible quasiequations of **Heyting algebras** (equivalently, **intuitionistic logic**) ($n = 2, 3 \ldots$):

$$\{\top \approx z \vee (\bigwedge_{i=1}^{n}(y_i \to x_i) \to (y_{n+1} \vee y_{n+2}))\} \;\Rightarrow\; \top \approx z \vee \bigvee_{j=1}^{n+2}(\bigwedge_{i=1}^{n}(y_i \to x_i) \to y_j).$$

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Two Useful Lemmas

Suppose now that $\mathcal{Q}$ is a **locally finite** quasivariety (i.e., its finitely generated members are finite); e.g., $\mathcal{Q} = \mathbb{Q}(\mathbf{A})$ for $\mathbf{A}$ finite.

Lemma

The following are equivalent for any set of $\mathcal{L}$-clauses Λ:

(1) *Λ is a basis for the admissible clauses of $\mathcal{Q}$.*

(2) *For each finite $\mathbf{B} \in \mathcal{Q}$: $\mathbf{B} \in \mathbb{IS}(\mathbf{F}_{\mathcal{Q}}(\omega))$ iff $\mathbf{B}$ satisfies Λ.*

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How Can Natural Dualities Help?

- Suppose that $\underset{\sim}{A}$ yields a **strong natural duality** on $\mathbb{Q}(\mathbf{A})$ ($\mathbf{A}$ finite).
- A basis Λ for the admissible clauses of $\mathbb{Q}(\mathbf{A})$ characterizes the finite algebras of $\mathbb{Q}(\mathbf{A})$ that can be embedded into $\mathbf{F}_{\mathbb{Q}(\mathbf{A})}(\omega)$.
- These algebras correspond on the dual side to a class $\mathcal{C}$ of images of finite powers of $\underset{\sim}{A}$ under morphisms of the dual category.
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Example: Bounded Distributive Lattices

Lemma

The following are equivalent for any finite $\mathbf{A} \in \mathcal{BDL}$*:*

(i) $\mathbf{A} \in \mathbb{IS}(\mathbf{F}_{\mathcal{BDL}}(\omega))$.

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$$\{x \vee y \approx \top\} \quad \Rightarrow \quad \{x \approx \top, y \approx \top\} \tag{1}$$

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$\{(1), (2)\}$ *is a basis for the admissible clauses of* $\mathcal{BDL}$.

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$\Sigma \Rightarrow \varphi \approx \psi$ *is* $\mathcal{BDL}$*-admissible* *iff* $\Sigma \models_{\mathcal{BDL}} \varphi \approx \psi$.

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Consider the variety $\mathcal{DMA} = \mathbb{ISP}(\mathbf{D}_4^{\mathbf{b}})$ of **De Morgan algebras**.

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A Last Example

Consider $\mathbf{S} = \langle \{-1, 0, 1\}, \to, \neg \rangle$ with operations $\neg x = 1 - x$ and

$\to$	-1	0	1
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Let $0 \preceq \varphi$ denote $\varphi \to \varphi \approx \varphi$ and $\varphi \leftrightarrow \psi =_{\mathrm{df}} \neg((\varphi \to \psi) \to \neg(\psi \to \varphi))$.

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Concluding Remarks

- We have developed (efficient) procedures for checking unifiability, admissibility, and structural completeness in finite algebras.

  G. Metcalfe and C. Röthlisberger. Unifiability and admissibility in finite algebras. *Proceedings of Computability in Europe 2012*, LNCS 7318, Springer, 2012.

- We also have a strategy for axiomatizing admissible clauses and quasiequations for finite algebras via natural dualities.

  L. Cabrer and G. Metcalfe. Admissibility via natural dualities. Submitted.

- Can admissibility be useful for checking validity in finite algebras (e.g., speeding-up proof search or shortening derivations)?

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