Sahlqvist correspondence for intuitionistic modal mu-calculus

Alessandra Palmigiano

Duality Theory in Algebra, Logic and Computer Science

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Correspondence via Duality

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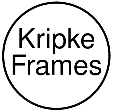

Correspondence via Duality

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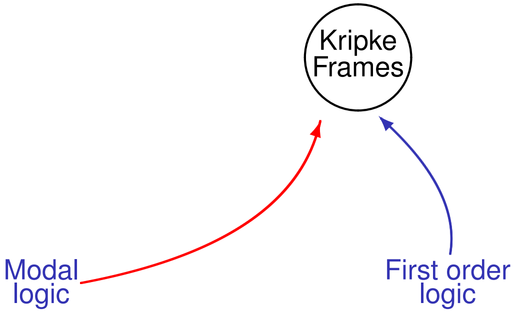

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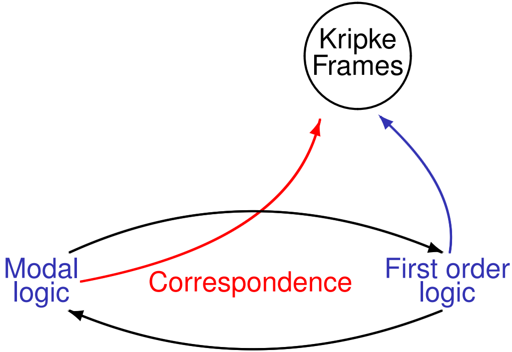
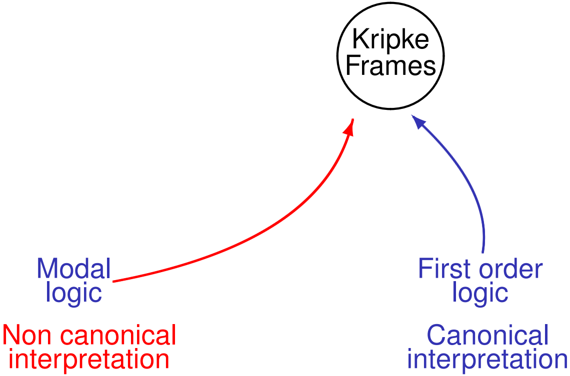


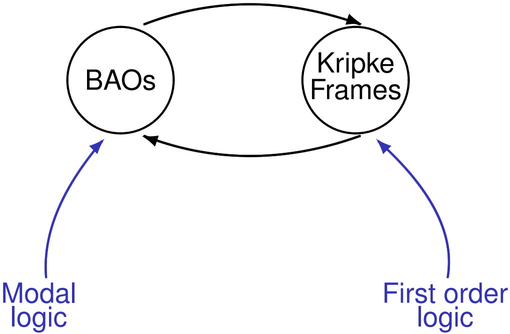
Correspondence via Duality

An asymmetry:

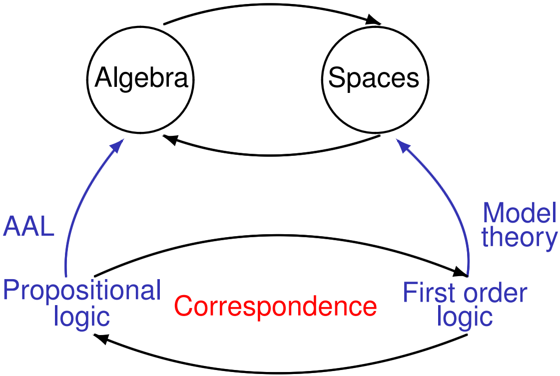


Correspondence via Duality

Symmetry re-established via duality:



Correspondence available not just for modal logic:



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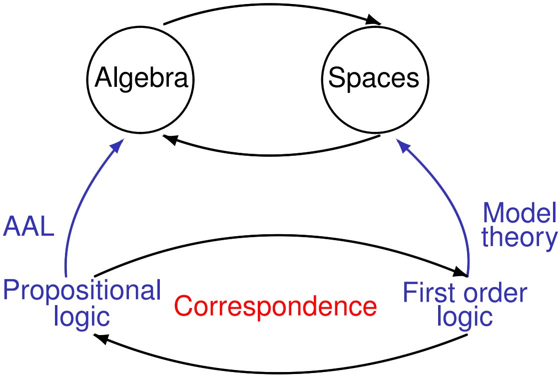

- specific correspondences as logical reflections of dualities
- dual characterizations as instances of Generalized Correspondence

The phenomenon of correspondence

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$$\mathcal{F}, w \Vdash \Diamond\Diamond p \to \Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall y, z(xRy \& yRz \to xRz)[w]$$

The phenomenon of correspondence

$$\mathcal{F}, w \Vdash \Diamond\Diamond p \to \Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall y, z(xRy \& yRz \to xRz)[w]$$

($\Rightarrow$) Assume wRy and yRz. To show: $w \in R^{-1}[z]$.
Consider the minimal valuation making the antecedent true at w:

$$V^*(p) = \{z\}.$$

Indeed, because wRy and yRz, then $\mathcal{F}, V^*, w \Vdash \Diamond\Diamond p$. Therefore, $\mathcal{F}, V^*, w \Vdash \Diamond p$, i.e.

$$w \in V^*(\Diamond p) = R^{-1}[V^*(p)] = R^{-1}[z].$$

The minimal valuation argument

$$\mathcal{F}, w \Vdash \varphi \rightarrow \psi$$

Heuristics

Find the minimal valuation V^* that makes φ true at w and plug in its description in $ST_x(\psi)$.

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The success of this heuristics rests on two conditions:

- such a minimal valuation exists;
- for every p,

  $V^*(p)$ is definable in the first order language of $\mathcal{F}$.

For instance: $V^*(p) := \varnothing, W, \{z\}, R[w], R^{-1}[w], \ldots$

Minimal valuation argument, algebraically

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Right Ackermann Lemma

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Right Ackermann Lemma Let $\alpha, \beta, \gamma \in \mathcal{L}^{+}$.
$p \notin \mathrm{PROP}(\alpha)$, $\beta(p)$ positive in p, $\gamma(p)$ negative in p.

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1. $\mathbb{C}, V \Vdash \beta(\alpha/p) \leq \gamma(\alpha/p)$;
2. there exists some $V' \sim_p V$ such that $\mathbb{C}, V' \Vdash \alpha \leq p$ and $\mathbb{C}, V' \Vdash \beta(p) \leq \gamma(p)$.

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Proof. (2 $\Rightarrow$ 1): By assumption, $V(\alpha) = V'(\alpha) \leq V'(p)$.

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Proof. (2 ⇒ 1): By assumption, $V(\alpha) = V'(\alpha) \leq V'(p)$. Hence,

$$V(\beta(\alpha/p)) \leq V'(\beta(p)) \leq V'(\gamma(p)) \leq V(\gamma(\alpha/p)).$$

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(1 $\Rightarrow$ 2): let $V'(p) := V(\alpha)$.

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$$V(\beta(\alpha/p)) \leq V'(\beta(p)) \leq V'(\gamma(p)) \leq V(\gamma(\alpha/p)).$$

(1 $\Rightarrow$ 2): let $V'(p) := V(\alpha)$. Since α does not contain p, we have: $V'(\alpha) = V(\alpha) \leq V'(p)$. Moreover:

$$V'(\beta(p)) = V(\beta(\alpha/p)) \leq V(\gamma(\alpha/p)) = V'(\gamma(p)).$$

Ackermann Lemma Based Algorithm

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- engined by the Ackermann's lemma.

Algorithmic correspondence (on algebras)

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- Residuation, and approximation rules.
- Soundness in perfect lattice environment:

Algorithmic correspondence (on algebras)

Ackermann Lemma Based Algorithm

- engined by the Ackermann's lemma.
- Reduction rules leading to the Ackermann elimination step.
- Residuation, and approximation rules.
- Soundness in perfect lattice environment:
  - approximation: both $\bigvee$-generated by the c. $\vee$-primes and $\bigwedge$-gen. by the c. $\wedge$-primes;
  - residuation: all the operations are either right or left adjoints or residuals.

A calculus mechanizing minimal valuation meta-arguments

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$$\forall p[\Diamond\Diamond p \leq \Diamond p]$$

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 & \text{iff} & \forall w[R^{-1}[R^{-1}[w]] \subseteq R^{-1}[w]] \\
 & \text{iff} & \forall w[R[R[w]] \subseteq R[w]].
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1. $\mathbb{C}, V \Vdash \beta(LFP(\alpha_V)/p) \leq \gamma(LFP(\alpha_V)/p)$;
2. there exists some $V' \sim_p V$ such that $\mathbb{C}, V' \Vdash \alpha(p) \leq p$ and $\mathbb{C}, V' \Vdash \beta(p) \leq \gamma(p)$.

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Proof. (2 ⇒ 1): $\mathcal{F}, V' \Vdash \alpha(p) \leq p$ implies that $V'(p)$ is a pre-fixpoint of $\alpha_{V'} = \alpha_V$, hence $LFP(\alpha_V) = \bigwedge\{b \in \mathbb{C} \mid \alpha_V(b) \leq b\} \leq V'(p)$. Hence

$$V(\beta(LFP(\alpha_V)/p) \leq V'(\beta(p)) \leq V'(\gamma(p)) \leq V(\gamma(LFP(\alpha_V)/p).$$

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$$V(\beta(LFP(\alpha_V)/p) \leq V'(\beta(p)) \leq V'(\gamma(p)) \leq V(\gamma(LFP(\alpha_V)/p).$$

(1 ⇒ 2): let $V'(p) := LFP(\alpha_V)$. Hence $V'(\alpha) = LFP(\alpha_V) = V'(p)$. Moreover:

$$V'(\beta(p)) = V(\beta(\alpha/p)) \leq V(\gamma(\alpha/p)) = V'(\gamma(p)).$$

The Löb inequality

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$$\begin{array}{ll} & \forall p[\Box(\Box p \to p) \leq \Box p] \\ \text{iff} & \forall p \forall i \forall m[(i \leq \Box(\Box p \to p) \ \& \ \Box p \leq m) \Rightarrow i \leq m] \end{array}$$

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\end{array}$$

This expresses transitivity + converse well foundedness of $R_\Box$ in FO + LFP.
If the target language is more expressive, then it makes sense to analogously enhance the expressivity of the source language

Applying ALBA to mu-formulas

$$\forall p[\nu X.\Box(p \wedge X) \leq p]$$

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\text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \qquad \text{(FA)}
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- If all the propositional variables occur inside the scope of fixpoint binders, no hope to reach the Ackermann's shape.
- ***Need***: define approximation & residuation/adjunction rules to display occurrences in the scope of fixpoint binders

Approximation rules for fixpoint binders

$$\frac{\mathbf{i} \leq \mu X.\varphi(X, \psi/!x)}{\exists \mathbf{j}[\mathbf{i} \leq \mu X.\varphi(X, \mathbf{j}/!x) \ \& \ \mathbf{j} \leq \psi]} \ (\mu\text{-A})$$

$$\frac{\nu X.\varphi(X, \psi/!x) \leq \mathbf{m}}{\exists \mathbf{n}[\nu X.\varphi(X, \mathbf{n}/!x) \leq \mathbf{m} \ \& \ \psi \leq \mathbf{n}]} \ (\nu\text{-A})$$

$[\varphi]$ is completely $\bigvee$-preserving in x in (ν-A) and completely $\bigwedge$-preserving in x in (ν-A).
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Approximation rules are sound on perfect lattices, and applicable beyond modal mu-calculus.

A case study

$$\forall p[\nu X.\Box(p \wedge X) \leq p]$$

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$$\begin{array}{ll} & \forall p[\nu X.\Box(p \wedge X) \leq p] \\ \text{iff} & \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}] \end{array}$$

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\bigvee_{\kappa \geq 1} \blacklozenge^{\kappa} & \dashv & \bigwedge_{\kappa \geq 1} \Box^{\kappa} &
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$$

Some adjunction rules

$$\frac{\mu X.(A(X) \vee B(p)) \leq \chi}{p \leq \nu X.(E(X) \wedge D(\chi/p))}\ (\mu\text{-Adj}) \qquad \frac{\chi \leq \nu X.(E(X) \wedge D(p))}{\mu X.(A(X) \vee B(\chi/p)) \leq p}\ (\nu\text{-Adj})$$

In each rule,

- $A(X) = \bigvee_{i \in I} \delta_i(X)$, $B(p) = \bigvee_{j \in J} \delta'_j(p)$, $E(X) = \bigwedge_{i \in I} \beta_i(X)$ and $D(p) = \bigwedge_{j \in J} \beta'_j(p)$, I and J are finite sets of indexes,
- each δ_i and δ'_j is a unary left adjoint,
- each β_i and β'_j is a unary right adjoint,
- $\delta_i \dashv \beta_i$ and $\delta'_j \dashv \beta'_j$ for each i and j.

Recursive inequalities and the enhanced ALBA

SLR	SRA	Skeleton	PIA
			$+ \quad \wedge$
$+ \quad \vee$		$+ \quad \vee$	$+ \quad \Box$
$+ \quad \wedge$	$+ \quad \wedge$	$+ \quad \wedge$	$+ \quad \vee$
$+ \quad \Diamond$	$+ \quad \Box$	$+ \quad \Diamond$	$+ \quad \rightarrow$
		$+ \quad \mu X$	$+ \quad \nu X$
$- \quad \vee$	$- \quad \vee$	$- \quad \vee$	$- \quad \vee$
$- \quad \wedge$	$- \quad \Diamond$	$- \quad \wedge$	$- \quad \Diamond$
$- \quad \Box$		$- \quad \Box$	$- \quad \wedge$
$- \quad \rightarrow$		$- \quad \rightarrow$	$- \quad \mu X$
		$- \quad \nu X$	

Three types of ingredients on either side of $\leq$:

- approximation-friendly *skeleton* $\varphi'(\overrightarrow{!x_i}) \leq \psi'(\overrightarrow{!y_j})$;
- ϵ^{∂}-*formulas* γ (negative occurrences)
- PIA-formulas

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