Mundici's Γ-functor theorem for star-shaped sets via Minkowski's duality with gauge functions.

Andrea Pedrini
andrea.pedrini@unimi.it

Università degli Studi di Milano
Dipartimento di Informatica e Comunicazione

Duality Theory in Algebra, Logic and Computer Science
Workshop II – 15th - 18th August 2012

ℓ-groups

A lattice-ordered abelian group is an algebra $\mathbf{G} = (G, +, -, \leq, 0)$ such that

- $(G, +, -, 0)$ is an abelian group,
- $(G, \leq)$ defines a lattice structure,
- for all $t, x, y \in V$, if $x \leq y$ then $t + x \leq t + y$.

ℓ-groups

A lattice-ordered abelian group is an algebra $\mathbf{G} = (G, +, -, \leq, 0)$ such that

- $(G, +, -, 0)$ is an abelian group,
- $(G, \leq)$ defines a lattice structure,
- for all $t, x, y \in V$, if $x \leq y$ then $t + x \leq t + y$.

A strong (order) unit is an element $u \in G$ such that for all $0 \leq x \in G$ there exists an integer $0 \leq n$ such that $x \leq nu$.

ℓ-groups

A lattice-ordered abelian group is an algebra $\mathbf{G} = (G, +, -, \leq, 0)$ such that

- $(G, +, -, 0)$ is an abelian group,
- $(G, \leq)$ defines a lattice structure,
- for all $t, x, y \in V$, if $x \leq y$ then $t + x \leq t + y$.

A strong (order) unit is an element $u \in G$ such that for all $0 \leq x \in G$ there exists an integer $0 \leq n$ such that $x \leq nu$.

Given two ℓ-groups G and H with units respectively u and v, a unital ℓ-homomorphism is a map $h : G \to H$ which is both a group-homomorphism and a lattice-homomorphism and that preserves the units ($h(u) = v$).

The functor Γ

Given an ℓ-group G with a unit u, the unital interval is the set

$$[0, u] = \{x \in G : 0 \leq x \leq u\}.$$

Theorem

The structure $\Gamma(G, u) = \langle [0, u], \oplus, \neg, 0 \rangle$*, where*

$$x \oplus y = u \wedge (x + y) \quad \textit{and} \quad \neg x = u - x,$$

is an MV-algebra.

The functor Γ

Given an ℓ-group G with a unit u, the unital interval is the set

$$[0, u] = \{x \in G : 0 \leq x \leq u\}.$$

Theorem

$$\Gamma : (G, u) \mapsto \langle [0, u], \oplus, \neg, 0 \rangle$$
$$h \mapsto h|_{[0,u]}$$

is a functor from the category of ℓ-groups with distinguished strong units and the category of MV-algebras.

Good sequences

Given an MV-algebra A, a good sequence is a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of A such that

1) there exists an index $j \in \mathbb{N}$ such that, for all $i \geq j$, $a_i = 0$;
2) $a_i \oplus a_{i+1} = a_i$, for all $i \in \mathbb{N}$.

Good sequences

Given an MV-algebra A, a good sequence is a sequence $(a_i)_{i \in \mathbb{N}}$ of elements of A such that

1) there exists an index $j \in \mathbb{N}$ such that, for all $i \geq j$, $a_i = 0$;
2) $a_i \oplus a_{i+1} = a_i$, for all $i \in \mathbb{N}$.

Lemma

Let G be an ℓ-group with unit u and let $A = \Gamma(G, u)$. Then for each $0 \leq a \in G$ there exists a unique good sequence $(a_i)_{i \in \mathbb{N}}$ of elements of A such that $a = a_1 + a_2 + \ldots$

Mundici's Γ-functor Theorem

Theorem

The functor Γ defines a natural equivalence between the category of ℓ-groups with strong unit, and the category of MV-algebras.

Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
- for all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,
  if $\lambda \geq 0$ then $\lambda(v \wedge w) = \lambda v \wedge \lambda w$.

Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
- for all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,
  if $\lambda \geq 0$ then $\lambda(v \wedge w) = \lambda v \wedge \lambda w$.

$FVL(n)$ is the free vector lattice on n generators.

Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
- for all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,
  if $\lambda \geq 0$ then $\lambda(v \wedge w) = \lambda v \wedge \lambda w$.

$FVL(n)$ is the free vector lattice on n generators.

The lattice structure induces a partial order (defined as usual):

$$v \leq w \text{ if and only if } v \wedge w = v.$$

Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda\in\mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
- for all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,
  if $\lambda \geq 0$ then $\lambda(v \wedge w) = \lambda v \wedge \lambda w$.

$FVL(n)$ is the free vector lattice on n generators.

A strong unit is an element $u \in V$ such that for all $0 \leq v \in V$ there exists a $0 \leq \lambda \in \mathbb{R}$ such that $v \leq \lambda u$.

Vector lattices

A (real) vector lattice is an algebra $\mathbf{V} = (V, +, \wedge, \vee, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ such that

- $(V, +, \{\lambda\}_{\lambda \in \mathbb{R}}, 0)$ is a vector space,
- $(V, \wedge, \vee)$ is a lattice,
- for all $t, v, w \in V$, $\quad t + (v \wedge w) = (t + v) \wedge (t + w)$,
- for all $v, w \in V$ and for all $\lambda \in \mathbb{R}$,
  if $\lambda \geq 0$ then $\lambda(v \wedge w) = \lambda v \wedge \lambda w$.

$FVL(n)$ is the free vector lattice on n generators.

A strong unit is an element $u \in V$ such that for all $0 \leq v \in V$ there exists a $0 \leq \lambda \in \mathbb{R}$ such that $v \leq \lambda u$.
A unital vector lattice is a pair $(\mathbf{V}, u)$, where $\mathbf{V}$ is a vector lattice and u is a strong unit of $\mathbf{V}$.

Representation of $FVL(n)$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is piecewise linear if there are finitely many linear polynomials $w_1, \ldots, w_s$ such that

$$\forall x \in \mathbb{R}^n \, \exists i \in \{1, \ldots, s\} \; : \; f(x) = w_i(x).$$

Representation of $FVL(n)$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is piecewise linear if there are finitely many linear polynomials $w_1, \ldots, w_s$ such that

$$\forall x \in \mathbb{R}^n \, \exists i \in \{1, \ldots, s\} \; : \; f(x) = w_i(x).$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous if, for each $x \in \mathbb{R}^n$ and for all $0 \leq \lambda$,

$$f(\lambda x) = \lambda f(x).$$

Representation of $FVL(n)$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is piecewise linear if there are finitely many linear polynomials $w_1, \ldots, w_s$ such that

$$\forall x \in \mathbb{R}^n \, \exists i \in \{1, \ldots, s\} \; : \; f(x) = w_i(x).$$

A function $f : \mathbb{R}^n \to \mathbb{R}$ is positively homogeneous if, for each $x \in \mathbb{R}^n$ and for all $0 \leq \lambda$,

$$f(\lambda x) = \lambda f(x).$$

Baker-Beynon duality: $FVL(n)$ is isomorphic to the set of all the continuous, piecewise linear and positively homogeneous functions $f : \mathbb{R}^n \to \mathbb{R}$, equipped with min, max, $+$ and products by real scalars.

Gauge functions

The 1-cut of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$C_f = \{x \in \mathbb{R}^n : f(x) \leq 1\}.$$

Gauge functions

The 1-cut of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$C_f = \{x \in \mathbb{R}^n : f(x) \leq 1\}.$$

The gauge function of a subset A of $\mathbb{R}^n$ is the function $g_A : \mathbb{R}^n \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$

$$g_A(x) = \inf\{\lambda \geq 0 : x \in \lambda A\},$$

where $\lambda A = \{\lambda y : y \in A\}$.

Gauge functions

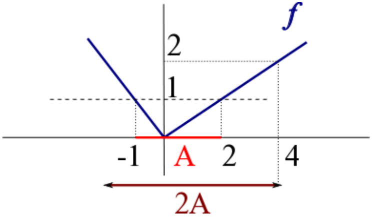
The 1-cut of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$C_f = \{x \in \mathbb{R}^n : f(x) \leq 1\}.$$

The gauge function of a subset A of $\mathbb{R}^n$ is the function $g_A : \mathbb{R}^n \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$

$$g_A(x) = \inf\{\lambda \geq 0 : x \in \lambda A\},$$

where $\lambda A = \{\lambda y : y \in A\}$.



Gauge functions

The 1-cut of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set

$$C_f = \{x \in \mathbb{R}^n : f(x) \leq 1\}.$$

The gauge function of a subset A of $\mathbb{R}^n$ is the function $g_A : \mathbb{R}^n \to \mathbb{R}$ such that, for all $x \in \mathbb{R}^n$

$$g_A(x) = \inf\{\lambda \geq 0 : x \in \lambda A\},$$

where $\lambda A = \{\lambda y : y \in A\}$.

What kind of sets are the 1-cuts of the elements of $FVL(n)^+$? Is $FVL(n)^+$ the set of the gauge functions of some reasonable subsets of $\mathbb{R}^n$?

Gauge functions: a simplification

We define the set of the gauge functions as

$$\mathcal{G}^n = \{f : \mathbb{R}^n \to \mathbb{R}^+, \text{ continuous and positively homogeneous}\}$$

and we equip it with pointwise defined operations of min, max, $+$ and products by real scalars.

Gauge functions: a simplification

We define the set of the gauge functions as

$$\mathcal{G}^n = \{f : \mathbb{R}^n \to \mathbb{R}^+, \text{ continuous and positively homogeneous}\}$$

and we equip it with pointwise defined operations of min, max, $+$ and products by real scalars.

What kind of sets are the 1*-cuts of the elements of* $\mathcal{G}^n$*?*
Is $\mathcal{G}^n$ *the set of the gauge functions of some reasonable subsets of* $\mathbb{R}^n$*?*

Star-shaped sets

The ray departing from the origin 0 and through the point $x \neq 0$ is the set

$$\sigma_x = \{\lambda x : 0 \leq \lambda \in \mathbb{R}\}.$$

$A \subseteq \mathbb{R}^n$ is a star-shaped set if and only if

1. 0 is in its interior
2. for each $x \neq 0$, $\sigma_x \cap A = [0, w]$ or $\sigma_x \cap A = \sigma_x$
3. A is closed
4. its formal boundary
   $\mathrm{bd}(A) = \{w : \exists x \neq 0 \text{ such that } \sigma_x \cap A = [0, w]\}$ is closed.

Star-shaped sets

The ray departing from the origin 0 and through the point $x \neq 0$ is the set

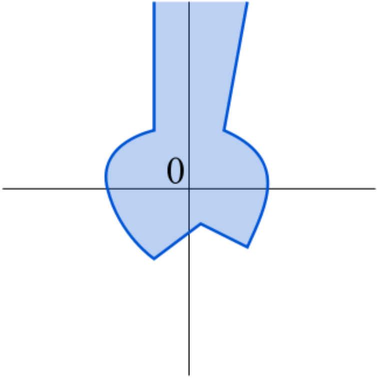
$$\sigma_x = \{\lambda x : 0 \leq \lambda \in \mathbb{R}\}.$$

$A \subseteq \mathbb{R}^n$ is a star-shaped set if and only if

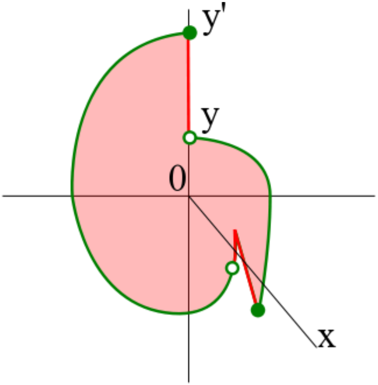
1. 0 is in its interior
2. for each $x \neq 0$, $\sigma_x \cap A = [0, w]$ or $\sigma_x \cap A = \sigma_x$
3. A is closed
4. its formal boundary
   $\mathrm{bd}(A) = \{w : \exists x \neq 0 \text{ such that } \sigma_x \cap A = [0, w]\}$ is closed.

The set of all the star-shaped sets of R^n is denoted by $\mathcal{C}^n$.
It is closed by $\cap$ and $\cup$.

Star-shaped sets



star-shaped set



set not star-shaped

A first correspondence

Lemma

If $A \in \mathcal{C}^n$, then

i) $g_A(0) = 0$;
ii) $g_A(x) = 0$ *for each* x *such that the ray* σ_x *is completely contained in* A;
iii) *for all* $x \in \mathbb{R}^n$, $g_A(x) = 1$ *if and only if* $x \in \mathrm{bd}(A)$.

A first correspondence

Theorem

The functionals $\omega : \mathcal{G}^n \to \mathcal{C}^n$ *defined as* $\omega(f) = C_f$ *and* $\gamma : \mathcal{C}^n \to \mathcal{G}$ *defined as* $\gamma(A) = g_A$ *are one the inverse of the other and define a bijection between* $\mathcal{G}^n$ *and* $\mathcal{C}^n$*. They are also order reversing, and they give the correspondences:*

$$\omega(f \wedge g) = \omega(f) \cup \omega(g), \qquad \gamma(A \cup B) = \gamma(A) \wedge \gamma(B),$$
$$\omega(f \vee g) = \omega(f) \cap \omega(g), \qquad \gamma(A \cap B) = \gamma(A) \vee \gamma(B),$$

and

$$\omega(0) = \mathbb{R}^n, \qquad \gamma(\mathbb{R}^n) = 0.$$

Gauge sum

Given $A, B \in \mathcal{C}^n$, their gauge sum $A +_{\mathrm{g}} B$ is the element $C \in \mathcal{C}^n$ such that, for all $x \in \mathbb{R}^n$,

$$g_C(x) = g_A(x) + g_B(x).$$

Thus, $\omega(f + g) = \omega(f) +_{\mathrm{g}} \omega(g)$ and $\gamma(A +_{\mathrm{g}} B) = \gamma(A) + \gamma(B)$.

Gauge sum

Given $A, B \in \mathcal{C}^n$, their gauge sum $A +_{\mathrm{g}} B$ is the element $C \in \mathcal{C}^n$ such that, for all $x \in \mathbb{R}^n$,

$$g_C(x) = g_A(x) + g_B(x).$$

Thus, $\omega(f+g) = \omega(f) +_{\mathrm{g}} \omega(g)$ and $\gamma(A +_{\mathrm{g}} B) = \gamma(A) + \gamma(B)$.

Lemma

Each $A \in \mathcal{C}^n$ is completely described by its intersections with the rays in $\mathbb{R}^n$ departing from the origin:

$$A = \bigsqcup_{|x|=1} \sigma_x \cap A.$$

Gauge sum

Given $A, B \in \mathcal{C}^n$, their gauge sum $A +_{\mathrm{g}} B$ is the element $C \in \mathcal{C}^n$ such that, for all $x \in \mathbb{R}^n$,

$$g_C(x) = g_A(x) + g_B(x).$$

Thus, $\omega(f+g) = \omega(f) +_{\mathrm{g}} \omega(g)$ and $\gamma(A +_{\mathrm{g}} B) = \gamma(A) + \gamma(B)$.

For each $x \in \mathbb{R}^n$ with $|x| = 1$,

$$\sigma_x \cap (A +_{\mathrm{g}} B) = \begin{cases} \sigma_x \cap A & \text{if } \sigma_x \cap B = \sigma_x, \\ \sigma_x \cap B & \text{if } \sigma_x \cap A = \sigma_x, \\ \left[0, \dfrac{ab}{a+b} x\right] & \text{if } \sigma_x \cap A = [0, ax],\ \sigma_x \cap B = [0, bx]. \end{cases}$$

The multiplication by a natural scalar is not an iterated gauge sum

$$nA = \{nx : x \in A\}$$

Thus, if $\sigma_x \cap A = [0, a]$, then $\sigma_x \cap nA = [0, na]$.

$$n.A = \underbrace{A +_{\mathrm{g}} A +_{\mathrm{g}} \ldots +_{\mathrm{g}} A}_{n \text{ times}}$$

Thus, if $\sigma_x \cap A = [0, a]$, then $\sigma_x \cap n.A = [0, \frac{1}{n}a]$

$$n.A = \frac{1}{n}A$$

Units

A unit of $\mathcal{C}^n$ is any element $U \in \mathcal{C}^n$ such that for any element $A \in \mathcal{C}^n$ there exists a positive integer n such that $n.U \subseteq A$.

A unit is any element which gauge function is a unit of $\mathcal{G}^n$.

The units of $\mathcal{C}^n$ are exactly the compact elements of $\mathcal{C}^n$.

Fixed a unit $U \in \mathcal{C}^n$, the unital interval is the set

$$[U, \mathbb{R}^n] = \{A \in \mathcal{C}^n : U \subseteq A\}.$$

Truncated gauge sum

Given $A, B \in \mathcal{C}^n$ and the unit U, the truncated gauge sum $A \oplus_{\mathrm{g}} B$ is the element $C = (A +_{\mathrm{g}} B) \cup U \in \mathcal{C}^n$.

Thus, $\gamma(A \oplus_{\mathrm{g}} B) = \gamma(A) \oplus \gamma(B)$.

$$\sigma_x \cap (A \oplus_{\mathrm{g}} B) =$$
$$\begin{cases} \sigma_x \cap A & \text{if } \sigma_x \cap B = \sigma_x \text{ and } \sigma_x \cap U \subseteq \sigma_x \cap A, \\ \sigma_x \cap B & \text{if } \sigma_x \cap A = \sigma_x \text{ and } \sigma_x \cap U \subseteq \sigma_x \cap B, \\ \sigma_x \cap U & \text{if } \sigma_x \cap B = \sigma_x \text{ and } \sigma_x \cap A \subseteq \sigma_x \cap U, \\ \sigma_x \cap U & \text{if } \sigma_x \cap A = \sigma_x \text{ and } \sigma_x \cap B \subseteq \sigma_x \cap U, \\ (\sigma_x \cap U) \cup \left[0, \dfrac{ab}{a+b}\right] & \text{otherwise.} \end{cases}$$

Good sequences of star-shaped sets

Given a fixed unit $U \in \mathcal{C}^n$, a good sequence of star-shaped sets is a sequence $(A_i)_{i \in \mathbb{N}}$ of elements $A_i \in \mathcal{C}^n$ such that

1) there exists an index $j \in \mathbb{N}$ such that, for all $i \geq j$, $A_i = \mathbb{R}^n$;
2) $U \subseteq A_i$, for all $i \in \mathbb{N}$;
3) $A_i \oplus_{\mathrm{g}} A_{i+1} = A_i$, for all $i \in \mathbb{N}$.

Good sequences of star-shaped sets

Given a fixed unit $U \in \mathcal{C}^n$, a good sequence of star-shaped sets is a sequence $(A_i)_{i \in \mathbb{N}}$ of elements $A_i \in \mathcal{C}^n$ such that

1) there exists an index $j \in \mathbb{N}$ such that, for all $i \geq j$, $A_i = \mathbb{R}^n$;
2) $U \subseteq A_i$, for all $i \in \mathbb{N}$;
3) $A_i \oplus_{\mathrm{g}} A_{i+1} = A_i$, for all $i \in \mathbb{N}$.

Lemma

Fixed a unit $U \in \mathcal{C}^n$, for each $A \in \mathcal{C}^n$ there exists a unique good sequence of star-shaped sets $(A_i)_{i \in \mathbb{N}}$ such that $A = A_1 +_{\mathrm{g}} A_2 +_{\mathrm{g}} \cdots$.

Polyhedral star-shaped sets

A closed half-space in $\mathbb{R}^n$ is a subset $H \subset \mathbb{R}^n$ of the form

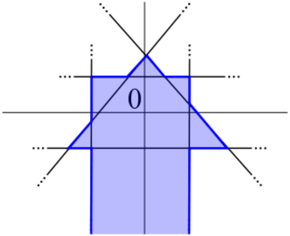
$$H = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : a \cdot x + b = a_1 x_1 + \cdots + a_n x_n + b \geq 0\},$$

where $0 \neq a = (a_1, \ldots, a_n) \in \mathbb{R}^n$ and b is a fixed real number.

Polyhedral star-shaped sets

A star-shaped set $A \in \mathcal{C}^n$ is polyhedral if it is a finite union of finite intersections of closed half-spaces, that is if there exists a finite number of closed half-spaces H_{ij}, such that $A = \bigcup_i \bigcap_j H_{ij}$.

$\mathcal{SP}^n$ is the set of polyhedral elements in $\mathcal{C}^n$.



Polyhedral star-shaped sets

A star-shaped set $A \in \mathcal{C}^n$ is polyhedral if it is a finite union of finite intersections of closed half-spaces, that is if there exists a finite number of closed half-spaces H_{ij}, such that $A = \bigcup_i \bigcap_j H_{ij}$.

$\mathcal{SP}^n$ is the set of polyhedral elements in $\mathcal{C}^n$.

Theorem (Characterization of $\mathcal{SP}^n$)

The elements of $\mathcal{SP}^n$ are exactly those subsets of $\mathbb{R}^n$ that can be written as finite unions of finite intersections of closed half-spaces whose interiors contain the point 0.

Two Lemmas for a polyhedral translation

Lemma

$FVL(n)^+$ is the subset of $FVL(n)$ of those elements that can be written as finitely many meets of finitely many joins of linear words joined with 0:

$$FVL(n)^+ = \left\{ f \in FVL(n) : f = \bigwedge_{k \in K} \bigvee_{j \in J} \left(\sum_{i=1}^{n} \lambda_{ijk} \pi_i \vee 0 \right) \right\}.$$

Lemma

If H is a closed half-space of $\mathbb{R}^n$ which contains 0 in its interior, then g_H is an element of $FVL(n)^+$ of the form $\sum_{i=1}^{n} \lambda_i \pi_i \vee 0$, and vice versa.

Theorem

ω and γ provide an isomorphism between $\langle FVL(n)^+, \wedge, \vee, +, \{\lambda\}_{\lambda\in\mathbb{R}}, 0\rangle$ and $\left\langle \mathcal{SP}^n, \cup, \cap, +_{\mathrm{g}}, \{\frac{1}{\lambda}\}_{\lambda\in\mathbb{R}}, \mathbb{R}^n\right\rangle$ which preserves the units.

Theorem (Representation)

Fixed a unit $U \in \mathcal{SP}^n$, for each $A \in \mathcal{SP}^n$ there exists a unique good sequence of polyhedral star-shaped sets $(A_i)_{i\in\mathbb{N}}$ such that $A = A_1 +_{\mathrm{g}} A_2 +_{\mathrm{g}} \cdots$.

Thank you for your attention.

Bibliography

- K. A. Baker, Free Vector Vattices, Canadian Journal of Mathematics. **20** (1968), 58-66.
- W. M. Beynon, Duality Theorems for Finitely Generated Vector Lattices, Proc. London Math. Soc. (3) **31** (1975 part 1), 114-128.
- G. Birkchoff, *Lattice Theory*, third ed., American Mathematical Society Colloquium Publications vol. 25, American Mathematical Society, Providence, R.I., 1979.

Bibliography

- R. L. O. Cignoli, I. M. D'Ottaviano and D. Mundici, *Algebraic Foundations of Many-Valued Reasoning*, Kluwer Academic Publishers, Dordrecht, 2000.
- R. J.Gardner *Geometric Tomography*, second ed., Encyclopedia of Mathematics and Its Applications, vol. 58, Cambridge University Press, Cambridge, 2006.
- D. Mundici, *Interpretation of AF C*-algebras in Łukasiewicz sentential calculus*. J. Funct. Anal. 65 (1986), no. 1, 1563.