

The coalgebraic many-valued cover modality

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Moss' logic

- Moss defined a logic \mathcal{L} for coalgebras for $T : \text{Set} \rightarrow \text{Set}$ which is parametric in T .
- \mathcal{L} is defined inductively by closing under infinite conjunctions and by closing under the functor T itself, so that the logic can be seen as an algebra

$$(\mathcal{P} + T)(\mathcal{L}) \rightarrow \mathcal{L}$$

- This means that \mathcal{L} is closed under ∇ , for if $\gamma \in T(\mathcal{L})$ then $\nabla\gamma \in \mathcal{L}$.

The semantics of ∇

The semantics of the logic w.r.t. a coalgebra $\xi : X \rightarrow TX$ and a state x in X is described via a relation

$$\Vdash_{\xi} \subseteq X \times \mathcal{L}$$

The semantics of the operator ∇ is then given using the relation lifting \bar{T} via the inductive clause

$$\Vdash (x, \nabla \gamma) \Leftrightarrow \bar{T}(\Vdash)(\xi(x), \gamma).$$

Thus the next diagram commutes in the category Rel of sets and relations

$$\begin{array}{ccc} T\mathcal{L} & \xrightarrow{\nabla} & \mathcal{L} \\ \bar{T}(\Vdash) \downarrow & & \downarrow \Vdash \\ TX & \xrightarrow{\xi} & X \end{array}$$

We need T to preserve weak pullbacks. Under this assumption Moss also showed that ∇ is invariant under bisimilarity.

Classical result on relation lifting

Theorem

For a functor $T : \text{Set} \rightarrow \text{Set}$ the following are equivalent:

- There is a functor $\bar{T} : \text{Rel}(\text{Set}) \rightarrow \text{Rel}(\text{Set})$ such that the square

$$\begin{array}{ccc} \text{Rel}(\text{Set}) & \xrightarrow{\quad \bar{T} \quad} & \text{Rel}(\text{Set}) \\ \uparrow (-)_\diamond & & \uparrow (-)_\diamond \\ \text{Set} & \xrightarrow{\quad T \quad} & \text{Set} \end{array}$$

commutes.

- T preserves weak pullbacks.

Example: ∇ for the powerset functor P

If X is a Kripke frame with the accessibility relation R and γ is a set of formulas then

$\Vdash (x, \nabla\gamma)$ when

- for all $\alpha \in \gamma$ there is $y \in R(x)$ s.t. $\Vdash (y, \alpha)$ and
- for all $y \in R(x)$ there exists $\alpha \in \gamma$ s.t. $\Vdash (y, \alpha)$.

In the standard modal language we have

$$\nabla\gamma = \Box \bigvee \gamma \wedge \bigwedge \Diamond \gamma$$

but also \Box and \Diamond can be defined in terms of ∇ .

Moving to the many-valued setting

The satisfaction relation $\Vdash: X \times \mathcal{L} \rightarrow 2$ could take more values, say in $[0, 1]$.

The idea is to use ‘relations’ in an enriched setting, so we will consider

$$\Vdash: X^{op} \otimes \mathcal{L} \rightarrow \mathcal{V}$$

where \mathcal{V} is a commutative quantale.

We will enrich all our categories, functors, etc. in a (co)complete symmetric monoidal closed base category \mathcal{V} .

Commutative Quantales

A commutative quantale

$$\mathcal{V} = (\mathcal{V}_0, \otimes, I, [-, -])$$

is a symmetric monoidal structure such that

- \mathcal{V}_0 is a complete lattice with the lattice order written as \leq , the symbol \perp denotes the least element and \top the greatest element of \mathcal{V}_0 .
- \otimes is a symmetric monoidal structure on \mathcal{V}_0 with a unit element I .
- The closed structure (the internal hom) of \mathcal{V}_0 is denoted by $[x, y]$. Hence we have adjunction relations

$$x \otimes y \leq z \quad \text{iff} \quad y \leq [x, z]$$

for every x, y, z in \mathcal{V}_0 .

Examples of quantales

- 1 The unit interval $[0; 1]$ with the usual order and the Łukasiewicz tensor $x \otimes y = \max\{x + y - 1, 0\}$.
The internal hom is given by $[x, y] =$ if $x \leq y$ then 1 else $1 - x + y$.
- 2 The unit interval $[0; 1]$ with the usual order and the Gödel tensor $x \otimes y = \min\{x, y\}$.
The internal hom is given by $[x, y] =$ if $x \leq y$ then 1 else y .
- 3 The unit interval $[0; 1]$ with the usual order and the product tensor $x \otimes y = x \cdot y$.
The internal hom is given by $[x, y] =$ if $x \leq y$ then 1 else $\frac{y}{x}$.

More examples

- 1 \mathcal{V}_0 is the two-element chain 2, i.e., there are two objects 0 and 1 with $0 \leq 1$.
The tensor in 2 is the meet and the internal hom is implication.
- 2 \mathcal{V}_0 is the unit interval $[0; 1]$ with \leq being the reversed order $\geq_{\mathbb{R}}$ of the real numbers.
The unit I is 0 and $x \otimes y = \max\{x, y\}$ where the maximum is taken w.r.t. the usual order $\leq_{\mathbb{R}}$.
The internal hom is given by $[0, 1](x, y) =$ if $x \geq_{\mathbb{R}} y$ then 0 else y .
- 3 \mathcal{V}_0 is the interval $[0; \infty]$ with \leq being the reversed order $\geq_{\mathbb{R}}$ of the reals. Extend the usual addition of nonnegative reals by putting $x + \infty = \infty + x = \infty$, for every $x \in [0; \infty]$ and let $x \otimes y = x + y$, the unit I being 0.
The internal hom is given by truncated subtraction $[0, 1](x, y) = y \dot{-} x =$ if $x \geq_{\mathbb{R}} y$ then 0 else $y - x$.

\mathcal{V} -categories

Definition

A category \mathcal{A} enriched in \mathcal{V} (or, a \mathcal{V} -category) consists of:

- 1 A class of *objects* denoted by a, b, \dots
- 2 For every pair a, b of objects a *hom-object* $\mathcal{A}(a, b)$ in \mathcal{V}_0 .

such that

- 1 For every object a there is an inequality

$$I \leq \mathcal{A}(a, a)$$

witnessing the “choice of the identity morphism on a ”.

- 2 For every triple a, b, c of objects there is an inequality

$$\mathcal{A}(b, c) \otimes \mathcal{A}(a, b) \leq \mathcal{A}(a, c)$$

witnessing “the composition of morphisms”.

Examples

- 1 When $\mathcal{V} = 2$, a \mathcal{V} -category is a (possibly large) preorder.
- 2 When \mathcal{V} is $[0, 1]$ with the reversed order, a \mathcal{V} -category \mathcal{A} is a (possibly large) *generalised ultrametric space*: the hom-object $\mathcal{A}(a, b)$ is the “distance” of a and b (notice that distance need not be symmetric), the axiom for identities becomes the requirement that $\mathcal{A}(a, a) = 0$ holds for all a , and the axiom for composition is the ultrametric triangle inequality $\mathcal{A}(a, c) \leq_{\mathbb{R}} \max\{\mathcal{A}(a, b), \mathcal{A}(b, c)\}$ for all a, b, c .
- 3 When \mathcal{V} is $[0, \infty]$ with the reversed order, a \mathcal{V} -category \mathcal{A} is a (possibly large) *generalised metric space*: the hom-object $\mathcal{A}(a, b)$ is the “distance” of a and b (notice that distance need not be symmetric and it may be infinite), the axiom for identities becomes the requirement that $\mathcal{A}(a, a) = 0$ holds for all a , and the axiom for composition is the triangle inequality $\mathcal{A}(a, c) \leq_{\mathbb{R}} \mathcal{A}(a, b) + \mathcal{A}(b, c)$ for all a, b, c .

\mathcal{V} -functors

Definition

Given \mathcal{V} -categories \mathcal{A} , \mathcal{B} , a \mathcal{V} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ is given by the following data:

- 1 An *object assignment*: for every object a in \mathcal{A} , there is a unique object fa in \mathcal{B} .
- 2 An *action on hom-objects*: for every pair a, a' of objects of \mathcal{A} there is an inequality

$$\mathcal{A}(a, a') \leq \mathcal{B}(fa, fa')$$

in \mathcal{V}_0 .

Relations in enriched setting

For a general base category \mathcal{V} , a “relation”

$$R : \mathcal{A} \multimap \mathcal{B}$$

from a \mathcal{V} -category \mathcal{A} to a \mathcal{V} -category \mathcal{B} is a \mathcal{V} -functor of the form

$$R : \mathcal{B}^{\text{op}} \otimes \mathcal{A} \rightarrow \mathcal{V}$$

called a *module* and

Given modules

$$R : \mathcal{A} \multimap \mathcal{B} \quad S : \mathcal{B} \multimap \mathcal{C}$$

we define their *composite*

$$S \cdot R : \mathcal{A} \multimap \mathcal{C}$$

to be the functor with values

$$S \cdot R(c, a) = \bigvee_b S(c, b) \otimes R(b, a)$$

for all c and a .

By $\mathcal{V}\text{-mod}$ we denote the 2-category of \mathcal{V} -modules (= “relations”)

A 2-functor $(-)_\diamond : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-mod}$

Definition

Given $f : \mathcal{A} \rightarrow \mathcal{B}$ in $\mathcal{V}\text{-cat}$, the module $f_\diamond : \mathcal{A} \dashv\vdash \mathcal{B}$ given by

$$f_\diamond(b, a) = \mathcal{B}(b, fa)$$

is called the *graph of f* .

every module f_\diamond is a left adjoint in $\mathcal{V}\text{-mod}$, having the module

$$f^\diamond(a, b) = \mathcal{B}(fa, b)$$

as a right adjoint.

When can we lift a \mathcal{V} -cat functor to \mathcal{V} -mod?

A *relation lifting* of a 2-functor $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ is a 2-functor $\bar{T} : \mathcal{V}\text{-mod} \rightarrow \mathcal{V}\text{-mod}$, making the square

$$\begin{array}{ccc} \mathcal{V}\text{-mod} & \xrightarrow{\bar{T}} & \mathcal{V}\text{-mod} \\ (-)_{\diamond} \uparrow & & \uparrow (-)_{\diamond} \\ \mathcal{V}\text{-cat} & \xrightarrow{T} & \mathcal{V}\text{-cat} \end{array}$$

commutative.

Exact squares

Definition

Call a lax square

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{p_1} & \mathcal{B} \\ p_0 \downarrow & \nearrow & \downarrow g \\ \mathcal{A} & \xrightarrow{f} & \mathcal{C} \end{array}$$

in \mathcal{V} -cat exact, if the equality

$$\mathcal{C}(fa, gb) = \bigvee_w \mathcal{A}(a, p_0 w) \otimes \mathcal{B}(p_1 w, b)$$

holds, naturally in a and b .

The main result

Theorem

For $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$, the following are equivalent:

- 1 There exists $\bar{T} : \mathcal{V}\text{-mod} \rightarrow \mathcal{V}\text{-mod}$ such that the following square

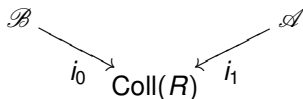
$$\begin{array}{ccc} \mathcal{V}\text{-mod} & \xrightarrow{\bar{T}} & \mathcal{V}\text{-mod} \\ (-)_{\diamond} \uparrow & & \uparrow (-)_{\diamond} \\ \mathcal{V}\text{-cat} & \xrightarrow{T} & \mathcal{V}\text{-cat} \end{array}$$

commutes.

- 2 T preserves exact squares.

Relations in enriched setting

A module can be represented by a *cospan*



called the *collage* of R that becomes a two-sided discrete fibration in $(\mathcal{V}\text{-cat})^{op}$.
the category $\text{Coll}(R)$ is defined as follows:

- 1 Objects of $\text{Coll}(R)$ are the disjoint union of objects of \mathcal{A} and \mathcal{B} .
- 2 $\text{Coll}(R)(a, a') = \mathcal{A}(a, a')$ in case both a and a' are in \mathcal{A} .
- 3 $\text{Coll}(R)(b, b') = \mathcal{B}(b, b')$ in case both b and b' are in \mathcal{B} .
- 4 $\text{Coll}(R)(b, a) = R(b, a)$ in case b is in \mathcal{B} and a is in \mathcal{A} .
- 5 $\text{Coll}(R)(a, b) = \perp$ in case a is in \mathcal{A} and b is in \mathcal{B} .

Examples of functors with BCC

The *Kripke-polynomial* 2-functors $T : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$, given by the grammar

$$T ::= Id \mid \text{const}_{\mathcal{X}} \mid T + T \mid T \times T \mid T \otimes T \mid T^\partial \mid \mathbb{L}T$$

The 2-functor T^∂ (the *dual* of the 2-functor T) is defined as the following composite

$$\begin{array}{ccccccc} \mathcal{V}\text{-cat} & \xrightarrow{(-)^{op}} & \mathcal{V}\text{-cat}^{co} & \xrightarrow{T^{co}} & \mathcal{V}\text{-cat}^{co} & \xrightarrow{(-)^{op}} & \mathcal{V}\text{-cat} \\ \mathcal{A} & \longmapsto & \mathcal{A}^{op} & \longmapsto & T(\mathcal{A}^{op}) & \longmapsto & (T(\mathcal{A}^{op}))^{op} \end{array}$$

The 2-functor \mathbb{L} sends \mathcal{A} to $[\mathcal{A}^{op}, \mathcal{V}]$ and $f : \mathcal{A} \rightarrow \mathcal{B}$ is sent to the left Kan extension along f^{op} .

The 2-functor \mathbb{U} is defined as \mathbb{L}^∂ . It sends \mathcal{A} to $[\mathcal{A}, \mathcal{V}]^{op}$.

The powerset functor on preorders

A preorder \mathcal{A} is mapped to the set of *all subsets of the carrier* of \mathcal{A} , ordered by the so-called Egli-Milner order

$$B \leq A \Leftrightarrow \left\{ \begin{array}{l} \forall b \in B . \exists a \in A . b \leq_{\mathcal{A}} a \\ \wedge \\ \forall a \in A . \exists b \in B . b \leq_{\mathcal{A}} a \end{array} \right.$$

The Pos-collapse of \mathbb{P} is the convex powerspace functor, which provides the Kripke semantics for negation-free modal logic in the same way as the usual powerset provides the Kripke semantics for classical modal logic.

The powerset functor on \mathcal{V} -cat

The objects of $\mathbb{P}\mathcal{A}$ are arbitrary \mathcal{V} -subsets $\varphi : |\mathcal{A}| \rightarrow \mathcal{V}$ of \mathcal{A} . For any $\varphi, \psi : |\mathcal{A}| \rightarrow \mathcal{V}$ put

$$\mathbb{P}\mathcal{A}(\varphi, \psi) = [|\mathcal{A}|, \mathcal{V}](\varphi, \psi^\downarrow) \otimes [|\mathcal{A}|, \mathcal{V}](\psi, \varphi^\uparrow)$$

or, in a detailed formula, by

$$\mathbb{P}\mathcal{A}(\varphi, \psi) = \bigwedge_a [\varphi(a), \bigvee_{a'} \psi(a') \otimes \mathcal{A}(a, a')] \otimes \bigwedge_{a'} [\psi(a'), \bigvee_a \varphi(a) \otimes \mathcal{A}(a, a')]$$

that can be perceived as the “Egli-Milner condition in the \mathcal{V} -setting”.

The powerset functor on \mathcal{V} -cat

Given a \mathcal{V} -functor $f : \mathcal{A} \rightarrow \mathcal{B}$ and \mathcal{V} -subset $\varphi : |\mathcal{A}| \rightarrow \mathcal{V}$, define $\mathbb{P}f(\varphi) : |\mathcal{B}| \rightarrow \mathcal{V}$ by

$$b \mapsto \bigvee_a |\mathcal{B}|(fa, b) \otimes \varphi a.$$

In other words, $\mathbb{P}f(\varphi)$ is the value of a left Kan extension of φ along $|f| : |\mathcal{A}| \rightarrow |\mathcal{B}|$. In particular, the equality

$$[|\mathcal{B}|, \mathcal{V}](\mathbb{P}f(\varphi), \psi) = [|\mathcal{A}|, \mathcal{V}](\varphi, \psi \cdot |f|)$$

holds for all $\varphi : |\mathcal{A}| \rightarrow \mathcal{V}$, $\psi : |\mathcal{B}| \rightarrow \mathcal{V}$.

Then $\mathbb{P} : \mathcal{V}\text{-cat} \rightarrow \mathcal{V}\text{-cat}$ is a 2-functor.

Coalgebras and bisimilarity in enriched setting

Definition

A T -coalgebra is a \mathcal{V} -functor $\xi : \mathcal{X} \rightarrow T\mathcal{X}$. Elements of \mathcal{X} are called states and ξ is the transition structure. A coalgebra morphism from $\xi : \mathcal{X} \rightarrow T\mathcal{X}$ to $\xi' : \mathcal{X}' \rightarrow T\mathcal{X}'$ is \mathcal{V} -functor $f : \mathcal{X} \rightarrow \mathcal{X}'$ such that $\xi' \cdot f = Tf \cdot \xi$. The category of T -coalgebras is denoted by $\text{Coalg}(T)$ and we write $U : \text{Coalg}(T) \rightarrow \mathcal{V}\text{-cat}$ and $V : \mathcal{V}\text{-cat} \rightarrow \text{Set}$ for the respective forgetful functors.

Definition

Bisimilarity, or behavioural equivalence, is the smallest equivalence relation on elements of coalgebras generated by pairs

$$(x, VUf(x))$$

where x is an element of a coalgebra and f is a coalgebra morphism.

∇ over \mathcal{V} -cat

The satisfaction relation should be a \mathcal{V} -module

$$\Vdash: \mathcal{X} \otimes \mathcal{L} \rightarrow \mathcal{V},$$

or equivalently,

$$\Vdash: \mathcal{L} \dashrightarrow \mathcal{X}^{op}.$$

Next we want to assume that \mathcal{L} comes equipped with a ∇ -operator.

The op makes it necessary to take formulas of the kind $\nabla\gamma$ not from $T\mathcal{L}$ but from $T^\partial\mathcal{L}$.

Recall that $T^\partial(\mathcal{X}) = (T(\mathcal{X}^{op}))^{op}$, so that T and T^∂ agree on discrete \mathcal{X} . So we assume that we have an algebra

$$T^\partial\mathcal{L} \rightarrow \mathcal{L}$$

Semantics of ∇

Given a T -coalgebra ξ , we define the semantics of ∇ via the relation lifting of T^∂ :

$$\Vdash(x, \nabla\gamma) = \overline{T^\partial}(\Vdash)(\xi(x), \gamma)$$

Notice that T^∂ preserves exact squares whenever T does.
In a diagram

$$\begin{array}{ccc} T^\partial \mathcal{L} & \xrightarrow{\nabla^\diamond} & \mathcal{L} \\ \overline{T^\partial}(\Vdash) \downarrow & & \downarrow \Vdash \\ T^\partial(X^{op}) = (TX)^{op} & \xrightarrow{(\xi^{op})^\diamond} & X^{op} \end{array}$$

Invariance under bisimulations

Proposition

If T preserves exact squares, then ∇ is invariant under bisimilarity.

Idea of the proof: To $\Vdash: \mathcal{X} \otimes \mathcal{L} \rightarrow \mathcal{V}$ corresponds a \mathcal{V} -functor $[[\cdot]] : \mathcal{L} \rightarrow [\mathcal{X}, \mathcal{V}]$. The fact that all $\varphi \in \mathcal{L}$ are invariant under bisimilarity implies that

$$[[\cdot]]_{\xi} : \mathcal{L} \rightarrow [U\xi, \mathcal{V}]$$

is natural in ξ .

We have to show that

$$[[\nabla \cdot]]_{\xi} : T^{\partial} \mathcal{L} \rightarrow [U\xi, \mathcal{V}]$$

is also natural in ξ . We use the commutativity of:

$$\begin{array}{ccc} T^{\partial} \mathcal{L} & \xrightarrow{\quad \nabla \quad} & \mathcal{L} \\ T^{\partial} [[\cdot]] \downarrow & & \downarrow [[\cdot]] \\ T^{\partial} [\mathcal{X}, \mathcal{V}] & \xrightarrow{\delta_{\mathcal{X}^{op}}} [T\mathcal{X}, \mathcal{V}] & \xrightarrow{[\xi, \mathcal{V}]} [\mathcal{X}, \mathcal{V}] \end{array}$$

Example: ∇ for \mathbb{U} -coalgebras

Recall that ∇ is an algebra for the functor $\mathbb{U}^\partial = \mathbb{L}$, hence

$$\nabla : \mathbb{L}(\mathcal{L}) \rightarrow \mathcal{L}.$$

We have

Given a \mathbb{U} -coalgebra $\xi : \mathcal{X} \rightarrow \mathbb{U}\mathcal{X}$ and $\gamma \in \mathbb{U}^\partial(\mathcal{L}) = \mathbb{L}(\mathcal{L})$ we have

$$\begin{aligned} \Vdash(x, \nabla\gamma) &= \overline{\mathbb{U}^\partial}(\Vdash)(\xi(x), \gamma) \\ &= \bigwedge_{y \in \mathcal{X}} [\xi(x)(y), \bigvee_{\varphi \in \mathcal{L}} \Vdash(y, \varphi) \otimes \gamma(\varphi)] \end{aligned}$$

For $\gamma = \mathcal{L}(-, \varphi)$ we obtain the semantics of \square from Bou et al.

$$\Vdash(x, \nabla\mathcal{L}(-, \varphi)) = \bigwedge_{y \in \mathcal{X}} [\xi(x)(y), \Vdash(y, \varphi)]$$

Example: ∇ for \mathbb{L} -coalgebras

Given a \mathbb{L} -coalgebra $\xi : \mathcal{X} \rightarrow \mathbb{L}\mathcal{X}$ and $\gamma \in \mathbb{L}^\partial(\mathcal{L}) = \mathbb{U}(\mathcal{L})$ we have

$$\begin{aligned}\Vdash(x, \nabla\gamma) &= \overline{\mathbb{L}^\partial}(\Vdash)(\xi(x), \gamma) \\ &= \bigwedge_{\varphi \in \mathcal{L}} [\gamma(\varphi), \bigvee_{y \in \mathcal{X}} \Vdash(y, \varphi) \otimes \xi(x)(y)]\end{aligned}$$

If $\gamma = \mathcal{L}(\varphi, -)$, we obtain the semantics of the \diamond -operator from Bou et al

$$\Vdash(x, \nabla\mathcal{L}(\varphi, -)) = \bigvee_{y \in \mathcal{X}} \Vdash(y, \varphi) \otimes \xi(x)(y).$$

Example: ∇ for \mathbb{P} -coalgebras

Consider a quantale \mathcal{V} such that $\otimes = \wedge$. Given a \mathbb{P} -coalgebra $\xi : \mathcal{X} \rightarrow \mathbb{P}\mathcal{X}$, the ∇ -semantics wrt ξ is given as follows. Observe that $\mathbb{P} = \mathbb{P}^\partial$, thus \mathcal{L} is a \mathbb{P} -algebra. For every $x \in \mathcal{X}$ and $\gamma \in \mathbb{P}^\partial(\mathcal{L}) = \mathbb{P}(\mathcal{L})$ we have

$$\begin{aligned} \Vdash(x, \nabla\gamma) &= \overline{\mathbb{P}^\partial}(\Vdash)(\xi(x), \gamma) \\ &= \bigwedge_{y \in \mathcal{X}} [\xi(x)(y), \bigvee_{\varphi \in \mathcal{L}} \Vdash(y, \varphi) \otimes \gamma(\varphi)] \\ &\otimes \bigwedge_{\varphi \in \mathcal{L}} [\gamma(\varphi), \bigvee_{y \in \mathcal{X}} \Vdash(y, \varphi) \otimes \xi(x)(y)] \end{aligned}$$