# PROFINITE COMPLETIONS AND CANONICAL EXTENSIONS OF SEMILATTICE REDUCTS OF DISTRIBUTIVE LATTICES

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ABSTRACT. A bounded distributive lattice $\mathbf{L}$ has two unital semilattice reducts, denoted $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$. These ordered structures have a common canonical extension $\mathbf{L}^\delta$. As algebras, they also possess profinite completions, $\widehat{\mathbf{L}}$, $\widehat{\mathbf{L}_\wedge}$ and $\widehat{\mathbf{L}_\vee}$; the first of these is well known to coincide with $\mathbf{L}^\delta$. Depending on the structure of $\mathbf{L}$, these three completions may coincide or may be different. Necessary and sufficient conditions are obtained for the canonical extension of $\mathbf{L}$ to coincide with the profinite completion of one, or of each, of its semilattice reducts. The techniques employed here draw heavily on duality theory and on results from the theory of continuous lattices.

## 1. INTRODUCTION

Canonical extensions are particular completions of ordered algebraic structures. We recall the definition in Section 2, noting only here that the canonical extension $\mathbf{L}^\delta$ of a lattice $\mathbf{L}$ depends only on the underlying order, so is unchanged if we pass to either of the semilattice reducts of $\mathbf{L}$. For contextual background on canonical extensions we refer to the papers of Gehrke and Vosmaer [10] and also to our paper [13]. The former paper outlines the theory of canonical extensions for lattice-based algebras and the applications to the study of associated logics which, historically, initiated the theory and which has influenced the way it has evolved.

In our paper [13] we focused on canonical extensions of (unital) semilattices in relation to their profinite completions. Given a residually finite variety $\mathcal{V}$ and an algebra $\mathbf{A} \in \mathcal{V}$, we denote by $\mathrm{Pro}_{\mathcal{V}}(\mathbf{A})$ the profinite completion of $\mathbf{A}$. The assumption of residual finiteness is satisfied for any $\mathcal{V}$ which is generated as a quasivariety by a finite algebra; it ensures that $\mathbf{A}$ embeds in $\mathrm{Pro}_{\mathcal{V}}(\mathbf{A})$ for each $\mathbf{A} \in \mathcal{V}$. Profinite completions are available in any of the following varieties: $\mathbf{S}_\wedge$ (meet semilattices with 1), $\mathbf{S}_\vee$ (join semilattices with 0) and $\mathcal{D}$ (distributive lattices with $0, 1$). The profinite completions of semilattices have a very rich theory. They can be described in a variety of ways: concretely, via a set-based representation; abstractly, via iterated free meet- and join-completions; via the duality theory for semilattices; or, categorically, directly as projective limits of finite semilattices. Our study in [13] revealed clearly the benefits for the treatment of canonical extensions of semilattices of viewing these as sitting inside the associated semilattice profinite completions. These benefits extend to the study of canonical extensions of bounded lattices, by consideration of one or both of the semilattice reducts. The richness of the theory increases further when we consider (semilattice reducts of) bounded distributive lattices.

It is well known that the profinite completion $\mathrm{Pro}_{\mathcal{D}}(\mathbf{L})$ for $\mathbf{L} \in \mathcal{D}$ is isomorphic to the canonical extension $\mathbf{L}^\delta$; details are recalled in Proposition 3.1. However

the situation changes radically when we consider semilattice reducts. Unlike the canonical extension, the profinite completion of a member $\mathbf{L}$ of $\mathcal{D}$ is not in general invariant under passage from $\mathbf{L}$ to its reducts $\mathbf{L}_\wedge \in \mathcal{S}_\wedge$ and $\mathbf{L}_\vee \in \mathcal{S}_\vee$. We showed already in [13] that a range of behaviours can arise: the semilattice profinite completion and canonical extension may coincide or be different, structurally or as regards cardinality.

Our main results in this paper, Theorems 3.8 and 4.3 below, give a complete answer to the following questions concerning a lattice $\mathbf{L} \in \mathcal{D}$: when is it the case that

(1) the canonical extension $\mathbf{L}^\delta$, as it naturally sits inside $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{L}_\wedge)$, coincides with $\mathrm{Pro}_{\mathcal{S}_\wedge}(\mathbf{L}_\wedge)$, and

(2) both (1) and the order dual assertion are true.

These problems are tractable thanks to the descriptions of profinite completions in $\mathcal{D}$ and in $\mathcal{S}_\wedge$ made available through duality theory. Besides involving both Priestley duality for $\mathcal{D}$ and Hofmann–Mislove–Stralka duality for $\mathcal{S}_\wedge$, the techniques we employ also draw heavily on the theory of continuous lattices. Our key tools here are results on the presence or absence of infinite antichains in semilattices which were obtained more than 20 years ago by Mislove [19, 20, 21] and Lawson, Mislove and Priestley [16, 17, 18] and which have lain largely dormant ever since.

## 2. Profinite and canonical extensions of semilattices: resumé

As the default, we shall work with $\mathcal{S}_\wedge$ rather than $\mathcal{S}_\vee$, noting that the difference is more one of notation than of substance. When we consider these classes as categories the morphisms are the homomorphisms.

We begin by recalling some order-theoretic notions. The underlying order of a semilattice $\mathbf{S} \in \mathcal{S}_\wedge$ is given in the expected way by the relation $\leqslant$ defined by $a \leqslant b$ if and only if $a \wedge b = a$. A *completion* of a semilattice $\mathbf{S} \in \mathcal{S}_\wedge$ is a pair $(e, C)$ where $C$ is a complete lattice and $e \colon \mathbf{S} \to C$ is an order-embedding. In the cases in which we are interested, $e$ will in fact be an $\mathcal{S}_\wedge$-morphism. Two completions $(e, C)$ and $(e', C')$ of $\mathbf{S}$ are *isomorphic* if there exists an order-isomorphism $\phi \colon C \to C'$ such that $e' = \phi \circ e$. When working with a given completion $(e, C)$ of a semilattice $\mathbf{S}$ we shall often leave tacit the embedding $e$ and refer to the completion simply as $C$.

Given $\mathbf{S} \in \mathcal{S}_\wedge$, a *filter* is a non-empty up-set in $\mathbf{S}$ which is closed under $\wedge$ and an *ideal* is a down-set in $\mathbf{S}$ which is (up-)directed; by convention, a directed set is required to be non-empty. We denote by $\mathrm{Filt}(\mathbf{S})$ the set of filters of $\mathbf{S}$ and by $\mathrm{Idl}(\mathbf{S})$, both ordered by set inclusion. The family $\mathrm{Filt}(\mathbf{S})$ is a complete lattice. A completion $(e, C)$ of $\mathbf{S}$ is said to be *dense* if every element of $C$ is both a join of meets of filters of $e(S)$ and a meet of joins of ideals of $e(S)$. The completion $(e, C)$ is *compact* if, whenever $F$ and $J$ are, respectively, a filter and an ideal in $\mathbf{S}$, then $\bigwedge e(F) \leqslant \bigvee e(J)$ in $C$ implies $F \cap J \neq \emptyset$. Finally, $(e, C)$ is a *canonical extension* of $\mathbf{S}$ if it is both dense and compact. It is uniquely determined, up to an isomorphism fixing $S$ (see [8, Section 2], noting that our definition of the canonical extension $\mathbf{S}^\delta$ of $\mathbf{S}$ agrees with that of the canonical extension of the underlying poset of $\mathbf{S}$).

We now briefly recall the duality theory for semilattices due to Hofmann, Mislove and Stralka [15]. It is well known that $\mathcal{S}_\wedge$ is generated, as a variety and as a quasivariety, by the two-element semilattice $\mathbf{2} = \langle \{0, 1\}; \wedge, 1 \rangle$ in which the underlying strict order is given by requiring $0 < 1$. We may form a topological structure $\mathbf{2}_\mathcal{T}$ by equipping $\mathbf{2}$ with the discrete topology $\mathcal{T}$. Furthermore, we may topologise the hom-set $\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ by treating $\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ as a subspace of $2^S$ with the product topology induced by $\mathcal{T}$. In addition, $\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2})$ can be given the structure of a meet semilattice with 1 by lifting the operations of $\mathbf{2}$ pointwise. The Hofmann–Mislove–Stralka duality sets up a dual equivalence between the category $\mathcal{S}_\wedge$ and

the category $\mathcal{Z}$ of topological semilattices belonging to the topological quasivariety $\mathbb{IS}_c\mathbb{P}^+(\mathbf{2}_{\mathcal{T}})$, that is, those which are isomorphic to closed subsemilattices of powers of $\mathbf{2}_{\mathcal{T}}$. Morphisms in $\mathbf{S}_{\wedge}$ are the maps preserving $\wedge$ and $1$ and those in $\mathcal{Z}$ are the maps which are continuous $\mathbf{S}_{\wedge}$ morphisms. The topology carried by any member $\mathbf{Z}$ of $\mathcal{Z}$ is compact and totally disconnected, that is, $\mathbf{Z}$ is topologically a Boolean space. The dual equivalence between $\mathbf{S}_{\wedge}$ and $\mathcal{Z}$ is given by the hom-functors

$$\mathrm{D} = \mathbf{S}_{\wedge}(-, \mathbf{2})\colon \mathbf{S}_{\wedge} \to \mathcal{Z} \quad \text{and} \quad \mathrm{E} = \mathcal{Z}(-, \mathbf{2}_{\mathcal{T}})\colon \mathbf{Z} \to \mathbf{S}_{\wedge}.$$

In particular, the duality for $\mathbf{S}_{\wedge}$ asserts that $\mathbf{S} \in \mathbf{S}_{\wedge}$ can be recovered from its dual $\mathrm{D}(\mathbf{S})$ as the $\mathcal{Z}$-morphisms from $\mathbf{S}_{\wedge}(\mathbf{S}, \mathbf{2})$ into $\mathbf{2}_{\mathcal{T}}$, with semilattice structure inherited from the power $\mathbf{2}^{\mathbf{S}_{\wedge}(\mathbf{S},\mathbf{2})}$. The evaluation map $e_S\colon \mathbf{S} \to \mathbf{2}^{\mathbf{S}_{\wedge}(\mathbf{S},\mathbf{2})}$ defined by

$$\forall a \in S \ \forall h \in \mathbf{S}_{\wedge}(\mathbf{S}, \mathbf{2}) \ e_S(a)(h) = h(a)$$

is an $\mathbf{S}_{\wedge}$-embedding. To see the duality subsumed within the general framework of natural duality theory, see Clark and Davey [4, 4.4.6 and 4.4.7]. We note that this duality is relatively unusual amongst natural dualities in that the generating algebra $\mathbf{2}$ in $\mathbf{S}$ and the alter ego $\mathbf{2}_{\mathcal{T}}$ which generates $\mathcal{Z}$ as a topological quasivariety differ only as regards the absence or presence of the topology. We shall make direct use of this feature shortly.

We have described the category $\mathcal{Z}$ dual to $\mathbf{S}_{\wedge}$ in terms of topological algebras. However the duality for semilattices is often made much more powerful thanks to the Fundamental Theorem for Compact Zero-dimensional Semilattices. This theorem, which originates in [15, Chapter II] (or see [11, VI-3.13]) allows the dual category $\mathcal{Z}$ to be alternatively described in algebraic and order-theoretic terms. Specifically, at the object level, the members of $\mathcal{Z}$ are precisely the algebraic lattices equipped with the Lawson topology. The Lawson topology of an algebraic lattice $\mathbf{M}$ has as a closed subbasis the sets $\uparrow k$ and $\downarrow x$, where $k$ ranges over the semilattice $K(\mathbf{M})$ of compact elements of $\mathbf{M}$ and $x$ ranges over $\mathbf{M}$. The $\mathcal{Z}$-morphisms are, by definition, the continuous maps which preserve $\wedge$ and $1$. Since the topology is the Lawson topology, these are precisely the maps preserving arbitrary meets and directed joins. Therefore we may, when it expedient to do so, work with $\mathbf{AL}$ rather than with $\mathcal{Z}$, where $\mathbf{AL}$ is the category of algebraic lattices with morphisms the maps preserving all meets and directed joins. We may also take advantage of the well-known duality between $\mathbf{AL}$ and the category $\mathbf{S}_{\vee}$. On objects, the functors setting up this duality send $\mathbf{S} \in \mathbf{S}_{\vee}$ to $\mathrm{Idl}(\mathbf{S})$ and $\mathbf{M} \in \mathbf{AL}$ to the set $K(\mathbf{M})$, the compact elements of $\mathbf{M}$. For details see for example [11, Chapter V].

We now fit profinite completions into the picture. Take $\mathbf{A} \in \mathcal{V}$, where $\mathcal{V}$ is a residually finite variety. Under the reverse inclusion order, the family of congruences of $\mathbf{A}$ of finite index is directed and there are natural bonding homomorphisms $\varphi_{\alpha\beta}\colon \mathbf{A}/\alpha \to \mathbf{A}/\beta$, for $\alpha \subseteq \beta$, given by $\varphi_{\alpha\beta}(a/\alpha) = a/\beta$. The categorical inverse limit $\widehat{\mathbf{A}}$ of the resulting inverse system can be realised concretely as a subalgebra $\mathrm{Pro}_{\mathcal{V}}(\mathbf{A})$ of the product of the finite algebras $\mathbf{A}/\alpha$. Equipping each finite algebra with the discrete topology and a product of finite spaces with the product topology, $\mathrm{Pro}_{\mathcal{V}}(\mathbf{A})$ becomes a topological algebra. The natural homomorphism $\mu_{\mathbf{A}}\colon \mathbf{A} \to \mathrm{Pro}_{\mathcal{V}}(\mathbf{A})$ given by $\mu_{\mathbf{A}}(a)(\alpha) := a/\alpha$, for all $a \in A$ and $\alpha \in \mathcal{S}_{\mathbf{A}}$ is an embedding. We shall henceforth refer to $\widehat{\mathbf{A}}$ as the profinite completion of $\mathbf{A}$.

We now specialise to $\mathbf{S}_{\wedge}$. The profinite limit $\mathrm{Pro}_{\mathbf{S}_{\wedge}}(\mathbf{S})$ has a number of noteworthy properties. Viewed as a member of $\mathbf{S}_{\wedge}$, it is an algebraic lattice, and $(\mu_{\mathbf{S}}, \mathrm{Pro}_{\mathbf{S}_{\wedge}}(\mathbf{S}))$ is a completion of $\mathbf{S}$, according to our definition above. Viewed as a topological algebra, it is a compact totally disconnected topological semilattice and, by the Fundamental Theorem cited above, its topology is necessarily the

Lawson topology. (In fact such topological semilattices are exactly the profinite ones, as was proved by Numakura [22], but we shall not need this fact.)

There is a more amenable description of the topological semilattice $\mathrm{Pro}_{\mathbf{S}_\wedge}(\mathbf{S})$ than that given above. As an algebra, it is, up to isomorphism, $\mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2})$, or, in terms of functors, $(^\flat \circ \mathrm{D} \circ {}^\flat \circ \mathrm{D})(\mathbf{S})$, where D is the hom-functor $\mathbf{S}_\wedge(-, \mathbf{2})$ and $^\flat \colon \mathfrak{Z} \to \mathbf{S}_\wedge$ is the functor forgetting topology. For a comment on the categorical roots of this claim see [15, I-3.11]. For any semilattice $\mathbf{S} \in \mathbf{S}$ the map $x \mapsto x^{-1}$ sets up an isomorphism from $\mathbf{S}_\wedge(\mathbf{S}, \mathbf{2})$ onto the filter lattice $\mathrm{Filt}(\mathbf{S})$ of $\mathbf{S}$, where $\mathrm{Filt}(\mathbf{S})$ is ordered by inclusion. Hence, as we observed in [13, Section 3] , $\mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2})$ can be identified with $\mathrm{Filt}^2(\mathbf{S})$, the filter lattice of the filter lattice of $\mathbf{S}$. Adopting this perspective, we can realise the profinite completion concretely as an algebraic closure system, with meets given by intersection and directed joins by union. We may summarise the preceding discussion in the following theorem. This is an abbreviated form of [13, Theorem 3.2], sufficient for our present needs.

**Theorem 2.1.** *Let $\mathbf{S} \in \mathbf{S}_\wedge$. Then the following completions of $\mathbf{S}$ are isomorphic:*

  (i) $(\mu_{\mathbf{S}}, \mathrm{Pro}_{\mathbf{S}_\wedge}(\mathbf{S}))$*;*
 (ii) $(e_S, \mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2}))$*, where each hom-set is ordered pointwise and $e_S$ is the natural evaluation map given by $e_S(a)(x) = x(a)$, for $a \in \mathbf{S}$ and $x \in \mathbf{S}_\wedge(\mathbf{S}, \mathbf{2})$;*
(iii) $(e, \mathrm{Filt}^2(\mathbf{S}))$*, where at each stage the filter lattice is ordered by inclusion and $e$ sends $a \in S$ to the principal filter in $\mathrm{Filt}^2(\mathbf{S})$ generated by the principal filter $\uparrow a$ in $\mathrm{Filt}(\mathbf{S})$.*

The various incarnations of the profinite completion $\widehat{\mathbf{S}}$ of a semilattice $\mathbf{S}$ can be used interchangeably, according to need or taste. We shall henceforth use the generic notation $\widehat{\mathbf{S}}$ and we denote by $e \colon \mathbf{S} \to \widehat{\mathbf{S}}$ the associated natural embedding map.

We do not recapitulate here in any detail on the theory of canonical extensions as it applies to semilattices, referring the reader to [8, 10, 13] for background and details. For our present purposes the facts summarised in the following theorem will suffice (cf. [13], Section 2 and in particular Theorems 2.6 and 2.8).

**Theorem 2.2.** *Let $\mathbf{S} \in \mathbf{S}_\wedge$.*

  (i) *The profinite completion $(e, \widehat{\mathbf{S}})$ is 2/3 canonical, in that it satisfies*
    (a) *the compactness property: for $F \in \mathrm{Filt}(\mathbf{S})$ and $J \in \mathrm{Idl}(\mathbf{S})$,*

$$\bigwedge e(F) \leqslant \bigvee e(J) \Longrightarrow F \cap J \neq \emptyset;$$

    (b) $\bigvee \bigwedge$*-density: every element of $\widehat{\mathbf{S}}$ is a join of down-directed meets of elements drawn from $e(S)$.*
 (ii) *Let $\mathbf{S}^\delta$ be the subset of $\widehat{\mathbf{S}}$ consisting of those elements of $\widehat{\mathbf{S}}$ which are meets of directed joins of elements drawn from $e(S)$. Then $\mathbf{S}^\delta$ is a complete meet-subsemilattice of $\widehat{\mathbf{S}}$, $e$ maps $\mathbf{S}$ into $\mathbf{S}^\delta$ and the completion $(e, \mathbf{S}^\delta)$ satisfies*
    (a) *the compactness property: for $F \in \mathrm{Filt}(\mathbf{S})$ and $J \in \mathrm{Idl}(\mathbf{S})$,*

$$\bigwedge e(F) \leqslant \bigvee e(J) \Longrightarrow F \cap J \neq \emptyset;$$

    (b) $\bigvee \bigwedge$*-density: every element of $\mathbf{S}^\delta$ is a join of down-directed meets of elements drawn from $e(S)$;*
    (c) $\bigwedge \bigvee$*-density: every element of $\mathbf{S}^\delta$ is a meet of directed joins of elements drawn from $e(S)$.*
  *Consequently $(e, \mathbf{S}^\delta)$ is, up to isomorphism, the canonical extension of $\mathbf{S}$.*

A few comments are called for here. The derivation of (ii) from (i) is slightly less obvious than it might appear at first glance, since in (i), meets and joins are calculated in $\widehat{\mathbf{S}}$ and those in (ii) in $\mathbf{S}^\delta$. So it is necessary to show that the joins and

meets in $\mathbf{S}^\delta$ arising in (ii) coincide with those calculated in $\widehat{\mathbf{S}}$; see [13, Theorem 2.7]. We note also that in (i)(b) and (ii)(b) the joins are in fact directed, but the meets in (ii)(c) are in general not down-directed unless $\mathbf{S}$ is the reduct of a bounded lattice.

Subsequently in this paper we elect to work principally with $\widehat{\mathbf{S}}$ in its incarnation as $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2})$, as in Theorem 2.1(ii). It will be helpful to spell out how Theorem 2.2 operates in this setting. Proofs of the corresponding statements phrased in terms of the set-based representation given in Theorem 2.1(iii) are given in [13, Section 2]. In $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{S}, \mathbf{2}), \mathbf{2})$, all meets and all directed joins are computed pointwise. In particular joins of sets of the form $e(J)$, where $J$ is an ideal in $\mathbf{S}$, are given pointwise.

Analogues of Theorems 2.1 and 2.2 are available for $\mathcal{S}_\vee$. The lattice $\mathrm{Filt}^2(\mathbf{S})$ is replaced by $(\mathrm{Idl}((\mathrm{Idl}(\mathbf{S}))^\partial)^\partial$, or equivalently by $(\mathrm{Filt}(\mathrm{Idl}(\mathbf{S})))^\partial$. The embedding $e$ then sends $a \in S$ to the principal filter in $\mathrm{Filt}(\mathrm{Idl}(\mathbf{S}))$ generated by the principal ideal $\downarrow a$ in $\mathrm{Idl}(\mathbf{S})$, and the $\bigvee\bigwedge$-density in Theorem 2.2(i)(b) is replaced by $\bigwedge\bigvee$-density. (Here $P^\partial$ denotes the order dual of a poset $P$.)

## 3. THE COINCIDENCE PROBLEM: THE ROLE PLAYED BY COUNTABLY GENERATED FREE SEMILATTICES

We now specialise to the variety $\mathcal{D}$ and the classes $\mathcal{D}_\wedge$ and $\mathcal{D}_\vee$ with which this paper is principally concerned. Here $\mathcal{D}_\wedge$ and $\mathcal{D}_\vee$ denote, respectively, the classes of semilattices of the form $\mathbf{L}_\wedge$ and of the form $\mathbf{L}_\vee$, for $\mathbf{L} \in \mathcal{D}$. In the remainder of the paper we shall assume familiarity with Priestley duality for $\mathcal{D}$; an elementary treatment can be found in [7].

We first record classic facts about canonical extensions of members of $\mathcal{D}$, exploiting Priestley duality for $\mathcal{D}$. The following proposition originates in [9]; see also [6]. In it, and subsequently, we use $\mathbf{2}$ to denote the two-element algebra in $\mathcal{D}$, in $\mathcal{S}_\wedge$ or in $\mathcal{S}_\vee$. Which is intended will be clear from the context.

**Proposition 3.1.** *Let $\mathbf{L} \in \mathcal{D}$ and let $\mathbf{X} = \mathcal{D}(\mathbf{L}, \mathbf{2})$ be its Priestley dual space and identify $\mathbf{L}$ with the lattice of clopen up-sets of $\mathbf{X}$. Then, up to isomorphism, the canonical extension $\mathbf{L}^\delta$ of $\mathbf{L}$ is the lattice of all up-sets of $\mathbf{X}$, where the embedding of $\mathbf{L}$ into $\mathbf{L}^\delta$ is the inclusion map. Moreover, $\mathbf{L}^\delta$ is an algebraic and dually algebraic lattice.*

We now begin to study, for a bounded distributive lattice $\mathbf{L}$, the relationship between the canonical extension $\mathbf{L}^\delta$ and the profinite completions $\widehat{\mathbf{L}_\wedge}$ and $\widehat{\mathbf{L}_\vee}$ formed relative to $\mathcal{S}_\wedge$ and $\mathcal{S}_\vee$, respectively. The following preliminary result will be used later.

**Proposition 3.2.** *Let $\mathbf{L}$ be a bounded distributive lattice. Then the canonical extension $\mathbf{L}^\delta$ is an $\mathcal{S}_\wedge$-retract of the $\mathcal{S}_\wedge$-profinite completion $\widehat{\mathbf{L}_\wedge}$ via a retraction that preserves arbitrary meets.*

*Proof.* By Proposition 3.1, the canonical extension of the bounded distributive lattice $\mathbf{L}$ is, up to isomorphism, the complete lattice of all order-preserving maps from $\mathcal{D}(\mathbf{L}, \mathbf{2})$ into $\{0, 1\}$. Since the members of $\mathcal{S}_\wedge(\mathcal{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ are order-preserving maps and $\mathcal{D}(\mathbf{L}, \mathbf{2}) \subseteq \mathcal{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$, there is a natural restriction map $\psi \colon \widehat{\mathbf{L}_\wedge} \to \mathbf{L}^\delta$ given by $\psi(\alpha) = \alpha{\restriction}_{\mathcal{D}(\mathbf{L}, \mathbf{2})}$. Since $\wedge$ and $1$ are defined pointwise in both its domain and codomain, $\psi$ is an $\mathcal{S}_\wedge$-morphism that in fact preserves arbitrary meets. Now take $\gamma \in \mathbf{L}^\delta$ and define the map $\phi(\gamma)$ from $\mathcal{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ into $\{0, 1\}$ by

$$(\phi(\gamma))(h) = \bigwedge \{ \gamma(f) \mid f \in \mathcal{D}(\mathbf{L}, \mathbf{2}), \ f \geqslant h \}.$$

Every $f \in \mathcal{D}(\mathbf{L}, \mathbf{2})$ satisfies $f(0_{\mathbf{L}}) = 0$ and consequently $(\phi(\gamma))(1) = \bigwedge \emptyset = 1$. Also, for every $h_1, h_2 \in \mathcal{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ and $f \in \mathcal{D}(\mathbf{L}, \mathbf{2})$, we have $f \geqslant h_1 \wedge h_2$ if and only

if $f \geqslant h_1$ or $f \geqslant h_2$. Therefore it follows immediately from the definition of $\phi(\gamma)$ that $(\phi(\gamma))(h_1 \wedge h_2) = (\phi(\gamma))(h_1) \wedge (\phi(\gamma))(h_2)$, and so $\phi(\gamma) \in \widehat{\mathbf{L}_\wedge}$. Hence the map $\phi\colon L^\delta \to \widehat{\mathbf{L}_\vee}$ is well defined and satisfies $\psi(\phi(\gamma)) = \gamma$. Finally we claim that $\phi$ is an $\mathbf{S}_\wedge$-morphism. In fact, for every $h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ and every $\gamma_1, \gamma_2 \in \widehat{\mathbf{L}_\wedge}$ we have

$$\phi(\gamma_1 \wedge \gamma_2)(h) = \bigwedge \{\, (\gamma_1 \wedge \gamma_2)(f) \mid f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}),\ f \geqslant h \,\}$$

$$= \bigwedge \{\, \gamma_1(f) \wedge \gamma_2(f) \mid f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}),\ f \geqslant h \,\}$$

$$= \bigwedge \{\, \gamma_1(f) \mid f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}),\ f \geqslant h \,\} \wedge$$

$$\bigwedge \{\, \gamma_2(f) \mid f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}),\ f \geqslant h \,\}$$

$$= \phi(\gamma_1)(h) \wedge \phi(\gamma_2)(h)$$

$$= (\phi(\gamma_1) \wedge \phi(\gamma_2))(h).$$

Also

$$\phi(1_{\mathbf{L}^\delta})(h) = \bigwedge \{\, 1(f) \mid f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}),\ f \geqslant h \,\} = 1,$$

for every $h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$, so that $\phi(1_{\mathbf{L}^\delta})$ is the top element in $\widehat{\mathbf{L}_\wedge}$.                    $\square$

As noted earlier, a canonical extension of an ordered structure is uniquely determined by the underlying poset, so that the canonical extension $\mathbf{L}^\delta$ of a bounded lattice $\mathbf{L}$ (not necessarily distributive) can be obtained by considering either of its semilattice reducts. Specifically, we have models, denoted respectively by $\mathbf{L}_\wedge^\delta$ and $\mathbf{L}_\vee^\delta$, of the canonical extension of $\mathbf{L}$ constructed, as in Theorem 2.2 and its order dual version, within $\widehat{\mathbf{L}_\wedge}$ and within $\widehat{\mathbf{L}_\vee}$. There is a natural isomorphism between these models; see [13, Section 2]. When we refer to $\mathbf{L}^\delta$ coinciding with $\widehat{\mathbf{L}_\wedge}$, we mean that the subset $\mathbf{L}_\wedge^\delta$ of $\widehat{\mathbf{L}_\wedge}$ is equal to $\widehat{\mathbf{L}_\wedge}$, and likewise for the $\vee$-reduct. As we recalled in Section 1, we already showed in [13] that, even in the restricted setting of semilattices in $\boldsymbol{\mathcal{D}}_\vee$ or $\boldsymbol{\mathcal{D}}_\wedge$, coincidence, as defined above, may or may not occur. Here, in this same restricted setting, we shall obtain necessary and sufficient conditions for coincidence, thereby contributing a deeper understanding of the relationship between canonical extensions and profinite completions. Our arguments will rely heavily on duality theory.

What distinguishes the analysis of the distributive case from that of bounded lattices in general is that we have a fully-fledged topological duality available for $\boldsymbol{\mathcal{D}}$. Furthermore, this connects well with the Hofmann–Mislove–Stralka duality for the semilattice reducts. Fix $\mathbf{L} \in \boldsymbol{\mathcal{D}}$. The hom-set $\boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2})$ can be viewed as a subset of $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$, that is, the subset consisting of those maps into $\mathbf{2}$ which preserve $\vee$ and $0$ as well as $\wedge$ and $1$. Under the correspondence $h \mapsto h^{-1}(1)$ ($h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$), a non-constant map $h$ is sent to a prime filter if and only if it is a prime element of the complete lattice $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$, that is, if and only if $h \geqslant h_1 \wedge h_2$ implies $h \geqslant h_1$ or $h \geqslant h_2$. (By convention here the top element of a complete lattice does not qualify as a prime.) Furthermore, for each $h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$,

$$h = \bigwedge \{\, f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2}) \mid f \geqslant h \,\}.$$

This last assertion—that every filter in a bounded distributive lattice $\mathbf{L}$ is an intersection of prime filters—is a classic result, following easily from the Prime Filter Theorem. The non-proper filter $\mathbf{L}$ may be subsumed here by regarding it as an empty meet of prime filters. In what follows we shall switch backwards and forwards as expedient between (prime) elements of $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ and (prime) elements in the filter lattice of $\mathbf{L}$. Because $\mathbf{L}$ is distributive, for any $f \in \boldsymbol{\mathcal{D}}(\mathbf{L}, \mathbf{2})$, the complementary sets $f^{-1}(1)$ and $f^{-1}(0)$ are respectively a prime filter and a prime ideal. Note that coincidence of the $\mathbf{S}_\wedge$-profinite completion and the canonical completion

of $\mathbf{L}$ can be stated by the assertion that, up to isomorphism, the lattice of all $\mathbf{S}_\wedge$-morphisms from $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ to $\mathbf{2}$ is the lattice of all order-preserving maps from $\mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$ to $\{0, 1\}$.

The following theorem is a first step towards describing the members $\mathbf{L}$ of $\mathbf{\mathcal{D}}$ for which the canonical extension $\mathbf{L}^\delta$ coincides with $\widehat{\mathbf{L}_\wedge}$ (and likewise for the order dual version). It is the equivalence of (1) and (2) in Theorem 3.3 that we shall exploit. We prove that (2) implies (1) by going via condition (3) but note that this implication can also be proved directly.

**Theorem 3.3.** *Let* $\mathbf{L}$ *be a bounded distributive lattice. Then the following statements are equivalent:*

(1) *the canonical extension* $\mathbf{L}^\delta$ *coincides with the* $\mathbf{S}_\wedge$-*profinite completion* $\widehat{\mathbf{L}_\wedge}$;
(2) *every element of* $\mathrm{Filt}(\mathbf{L}) \setminus \{\mathbf{L}\}$ *is a non-empty meet of finitely many primes;*
(3) *every member of* $\mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ *preserves arbitrary meets.*

*Proof.* For (1) implies (2) we establish the contrapositive. So assume that $1 \neq h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ but that $h$ is not a finite meet in $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ of elements from $\mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$. Take the filter $\mathcal{F}$ of $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ defined as follows:

$$\mathcal{F} = \{\, x \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}) \mid \exists y_1, \cdots, y_n \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2}) \; x \geqslant y_1 \wedge \cdots \wedge y_n \geqslant h \,\}.$$

If it were the case that $h \in \mathcal{F}$, then we would have elements $y_1, \ldots, y_n \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$ such that $h \geqslant y_1 \wedge \cdots \wedge y_n \geqslant h$. This would imply that $h = y_1 \wedge \cdots \wedge y_n$, contrary to hypothesis. Take $\alpha \in \mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ such that $\alpha^{-1}(1) = \mathcal{F}$. This implies that $\alpha(h) = 0$. We claim that $\alpha$ does not belong to the canonical extension and therefore the two completions do not coincide. Suppose for a contradiction that $\alpha \in \mathbf{L}_\wedge^\delta$. Then there exists a family $\mathcal{J}$ of ideals of $\mathbf{L}$ such that

$$\alpha = \bigwedge \{\, \bigvee e(J) \mid J \in \mathcal{J} \,\}.$$

Since $\alpha(h) = 0$, there exists $J \in \mathcal{J}$ such that $J \subseteq h^{-1}(0)$, or equivalently $J \cap h^{-1}(1) = \emptyset$. By the Prime Ideal Theorem for $\mathbf{\mathcal{D}}$, there exists a prime ideal $I$ such that $J \subseteq I$ and $I \cap h^{-1}(1) = \emptyset$. Take $g \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$ to satisfy $g^{-1}(0) = I$. Since $I \subseteq h^{-1}(0)$ we have $g \geqslant h$ and hence $g \in \mathcal{F}$, that is, $g \in \alpha^{-1}(1)$. However $g(I) = \{0\}$, which implies that $\alpha(g) = 0$. Therefore we have reached the required contradiction.

Now assume that (2) holds. To establish that (3) holds, we shall first prove that the retraction $\psi$ defined in the proof of Proposition 3.2 is bijective. Take $\alpha_1, \alpha_2 \in \mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ and suppose that $\alpha_1 \neq \alpha_2$. Then we may assume there exists $h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ such that $\alpha_1(h) = 1$ and $\alpha_2(h) = 0$. By (2), there exist $f_1, \cdots, f_n \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$ such that $h = f_1 \wedge \cdots \wedge f_n$. Since $\alpha_1$ and $\alpha_2$ preserve finite meets, $\alpha_1(f_i) = 1$ and $\alpha_2(f_i) = 0$ for some $i \in \{1, \cdots, n\}$. Consequently $\psi(\alpha_1) \neq \psi(\alpha_2)$. Hence every $\alpha$ in $\mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ is of the form $\phi(\psi(\alpha))$. But then the preservation of arbitrary meets by $\alpha$ follows from the definition of $\phi$ and from the fact that every element of $\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ is a finite meet of elements of $\mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2})$.

Finally assume that (3) holds. To establish that (1) holds, it is enough to prove that every $\alpha$ in $\mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$ is a meet of directed joins of elements of $e(L)$. Take $\alpha \in \mathbf{S}_\wedge(\mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2}), \mathbf{2})$. Since $\alpha$ preserves arbitrary meets, for every $h \in \mathbf{S}_\wedge(\mathbf{L}_\wedge, \mathbf{2})$ the following holds:

$$\alpha(h) = \alpha(\bigwedge \{\, f \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2}) \mid f \geqslant h \,\}) = \bigwedge \{\, \alpha(f) \mid f \in \mathbf{\mathcal{D}}(\mathbf{L}, \mathbf{2}), f \geqslant h \,\}$$

and

$$\alpha(h) = 0 \Longleftrightarrow \exists f \in \mathbf{D}(\mathbf{L}, \mathbf{2}) \ (f \geqslant h \wedge \alpha(f) = 0)$$
$$\Longleftrightarrow \exists f \in \mathbf{D}(\mathbf{L}, \mathbf{2}) \ (\alpha(f) = 0 \wedge f^{-1}(0) \subseteq h^{-1}(0))$$
$$\Longleftrightarrow \exists f \in \mathbf{D}(\mathbf{L}, \mathbf{2}) \ (\alpha(f) = 0 \wedge \bigvee h(f^{-1}(0)) = 0)$$
$$\Longleftrightarrow \exists f \in \mathbf{D}(\mathbf{L}, \mathbf{2}) \ (\alpha(f) = 0 \wedge (\bigvee e(f^{-1}(0)))(h) = 0).$$

Thus $\alpha = \bigwedge \{\, \bigvee e(f^{-1}(0)) \mid f \in \mathbf{D}(\mathbf{L}, \mathbf{2}), \alpha(f) = 0 \,\}$.                    $\square$

We have the following corollary of the contrapositive of (1) implies (2) in Theorem 3.3.

**Corollary 3.4.** *Let $\mathbf{L} \in \mathbf{D}$. Then every element $F$ of* $\mathrm{Filt}(\mathbf{L})$ *which is not a finite non-empty meet of primes gives rise to an element $\alpha_F$ of* $\widehat{\mathbf{L}_\wedge} \setminus \mathbf{L}_\wedge^\delta$. *Moreover the map $F \mapsto \alpha_F$ is injective.*

We now give an example in which we exploit the above methodology to witness non-coincidence.

**Example 3.5.** Let $C$ be an uncountable order-dense chain (for example, $\mathbb{R}$ with the usual order). Give $C^C$ the pointwise order and let $\mathbf{L}$ be $C^C$ with top and bottom elements, $\top$ and $\bot$, adjoined. For every $a, b \in C$, let $F_a^b$ be

$$F_a^b = \left(\uparrow a \times C^{C \setminus \{b\}}\right) \cup \{\top\} := \prod_{x \in C} G_x \cup \{\top\}$$

where $G_x = \uparrow a$ for $x = b$ and $G_x = C$ otherwise. Fix $a \in C$. The family $\left\{F_a^b\right\}_{b \in C}$ is an antichain of prime filters of $\mathbf{L}$. Note that $\bigwedge_{b \in C} F_a^b = (\uparrow a)^C \cup \{\top\}$. Also observe that if $F \cup \{\top\}$ is a prime filter of $\mathbf{L}$ such that $\pi_b(F) = \uparrow a$, for some projection map $\pi_b \colon C^C \to C$, with $b \in C$, then $F \cup \{\top\} = F_a^b$. Clearly a finite intersection of prime filters of $C^C$ must have image $C$ under some projection map $\pi_b$. Consequently $\bigwedge_{b \in C} F_a^b$ is not a finite meet of prime filters of $\mathbf{L}$. Hence we have exhibited a set of elements of $\widehat{\mathbf{L}_\wedge} \setminus \mathbf{L}_\wedge^\delta$ of cardinality $|C|$.

We now enquire exactly when it is the case that condition (2) in Theorem 3.3 holds. Here we are fortunate: much of the work has essentially already been done for us since the literature contains a wealth of relevant information. The sources on which we draw work in part with join semilattices (and their ideal lattices) and in part with meet semilattices (and their filter lattices). At each stage of our exposition we shall accordingly work with whichever formulation aligns best with our source or works more smoothly. We shall show that a necessary and sufficient condition for $\mathbf{L}_\wedge^\delta$ to coincide with $\widehat{\mathbf{L}_\wedge}$ is that $\mathrm{Filt}(\mathbf{L})$ should have locally finite meet breadth (the definition is given below). We then draw on results from the theory of continuous lattices to derive a number of conditions equivalent to this one, the aim being to find conditions expressed directly in terms of $\mathbf{L}$. One of these conditions is that $\mathbf{L}_\wedge$ should fail to contain the free meet semilattice $^\mathbb{N}\mathbf{2}$ on $\aleph_0$ generators as an $\mathbf{S}_\wedge$-subobject (see Theorem I.1.5 of [15]). The full results are presented in Theorems 3.6 and 3.8 below.

We recall that a lattice $\mathbf{M}$ is said to have *locally finite meet breadth* if, for any given $x \in M$, the set

$$\{\, |A| \mid A \text{ is a finite and meet irredundant subset of } M \text{ such that } \bigwedge A = x \,\}$$

has a finite upper bound. A subset $A$ (not necessarily finite) of $M$ is *meet irredundant* if for every finite subset $B$ of $A$ we have $\bigwedge B < \bigwedge C$ whenever $\emptyset \neq C \subsetneq B$. We stress that the definition of locally finite meet breadth requires the sets $A$ to be finite. This is in line with the definition as originally given by Gierz, Lawson and

Stralka [12] and as employed by Mislove in [20]. We shall shortly draw on results from [20]; there and in some other later sources the restriction to finite $A$ is not made explicit. Note that the presence of an infinite meet irredundant subset in $\mathbf{M}$ does not necessarily imply that $\mathbf{M}$ fails to have locally finite meet breadth.

Theorem 3.6 below originates with Mislove ([19], in particular Proposition 6.3, and [20, Section 2]). For further context and discussion see also [21]. We shall require the theorem in the special case that $\mathbf{M}$ is the filter lattice of some $\mathbf{L} \in \mathfrak{D}$. Then the requirement that each element is the meet of primes is simply the assertion that every filter is the intersection of prime filters, a fact we have recalled earlier; note also [11, Proposition IV-1.21].

None of the implications in Theorem 3.6 is new but the result as we state it is not given explicitly in this form in the literature. We therefore indicate how the various constituent results combine to yield the result as presented here.

**Theorem 3.6.** *Let $\mathbf{M}$ be a distributive continuous lattice in which every element is the meet of primes. Then the following statements are equivalent:*

(1) *some element of $\mathbf{M}$ fails to be the meet of finitely many primes;*
(2) *there exists an element $x$ of $\mathbf{M}$ such that the set of minimal primes above $x$ forms an infinite antichain, which is necessarily meet irredundant;*
(3) *$\mathbf{2}^{\mathbb{N}'}$ embeds in $\mathbf{M}$ by a map preserving all non-empty meets and taking primes to primes, where $\mathbf{2}^{\mathbb{N}'}$ denotes the meet semilattice formed by taking the non-empty subsets of $\mathbb{N}$ with union as the semilattice operation;*
(4) *$\mathbf{2}^{\mathbb{N}}$ embeds in $\mathbf{M}$ by a map preserving all meets and directed joins;*
(5) *$\mathbf{M}$ fails to have locally finite meet breadth.*

*Proof.* (Outline) The equivalence of (1), (2) and (3) is valid under the less restrictive assumption that $\mathbf{M}$ is a complete (meet) semilattice (morphisms in this context being maps which preserve *non-empty* meets; see [16, Section 1]). The equivalence was established by Mislove in [20, Proposition 2.1] (or see [19, Proposition 6.3]).

Under the assumption that $\mathbf{M}$ is a distributive continuous lattice, the proof of Corollary 2.2 in [20] establishes that (3) implies (4). We emphasise that this proof draws heavily on the theory of continuous lattices, and in particular relies on the Lemma on Primes [11, V-1.1]. (We cannot simply assert that the map supplied by (3) extends to yield the map demanded in (4).)

The implication (4) implies (5) is elementary. To prove it, consider the meet $x$ of the copy of $\mathbf{2}^{\mathbb{N}}$ in $\mathbf{M}$. Then it can easily be shown that $x$ is the meet of $B$, where $B$ can be taken to be a meet irredundant set of arbitrarily large finite cardinality; see the proof of [20, Corollary 2.3] for the details. Finally, for any element $x$ which is the meet of $n$ primes, the cardinality of a finite meet irredundant subset with meet $x$ must have cardinality at most $n$ (as shown in the proof of [20, Corollary 2.3]). Therefore, via the contrapositive, (5) implies (1). □

We may deploy Theorem 3.6 with $\mathbf{M}$ as the filter lattice of a member $\mathbf{L}$ of $\mathfrak{D}$ or as the ideal lattice of $\mathbf{L}$; the latter may when required be identified with the lattice of filters of the order dual $\mathbf{L}^{\partial}$. The equivalence of primary interest to us in Theorem 3.6 is that between (4) and (5). But we wish to go further and to present conditions expressed in terms of $\mathbf{L}$ rather than in terms of $\mathrm{Filt}(\mathbf{L})$ and/or $\mathrm{Idl}(\mathbf{L})$. We shall make use of Hofmann–Mislove–Stralka duality to achieve this.

We note that it is elementary to see that a countably infinite meet irredundant subset generates a free semilattice on countably many generators, denoted $^{\mathbb{N}}2$, within a (meet) semilattice, so that a meet semilattice contains such a set if and only if it contains $^{\mathbb{N}}\mathbf{2}$ as a subsemilattice, where $\wedge$ is taken to be union. (This observation appears as Remark 1.6 in [16].)

The following lemma amplifies a statement made in the proof of [17, Theorem 3.1]. It mirrors [16, Proposition 1.13] but is slightly simpler to formulate thanks to the additional assumption of distributivity, which gives us access to Theorem 3.6. Here it will be convenient to work with the duality between $\mathbf{S}_\vee$ and $\mathbf{AL}$, in which the first dual of $\mathbf{S} \in \mathbf{S}_\vee$ is taken to be $\mathrm{Idl}(\mathbf{S})$; see the discussion in Section 2. We stress that, in the lemma, while the first pair of equivalences and the second pair come from elementary duality theory, the link between the two pairs is established with the aid of Theorem 3.6.

**Lemma 3.7.** *Let* $\mathbf{L} \in \mathfrak{D}$. *Then the following statements are equivalent:*

(1) *there is an embedding of* $\mathbf{2}^\mathbb{N}$ *into* $\mathrm{Idl}(\mathbf{L})$ *which preserves arbitrary meets and directed joins;*
(2) *there is a surjective* $\mathbf{S}_\vee$-*morphism of* $\mathbf{L}$ *onto* $^\mathbb{N}\mathbf{2}$;
(3) *there is an* $\mathbf{S}_\vee$-*embedding of* $^\mathbb{N}\mathbf{2}$ *into* $\mathbf{L}$;
(4) *there is a surjective map from* $\mathrm{Idl}(\mathbf{L})$ *onto* $\mathbf{2}^\mathbb{N}$ *which preserves arbitrary meets and directed joins.*

*Proof.* In (2) and (4) we are in fact dealing with epimorphisms. In addition, $\mathbf{2}^\mathbb{N}$ and $^\mathbb{N}\mathbf{2}$ are mutually dual. Hence (1) and (2) are mutually dual, and hence equivalent statements, and likewise for (3) and (4). See [15] for details.

Now assume that $f \colon \mathbf{L} \to {}^\mathbb{N}\mathbf{2}$ is a surjective $\mathbf{S}_\vee$-morphism. Then any set of points $a_n \in \mathbf{L}$ such that $f(a_n) = \{n\}$ for $n \in \mathbb{N}$ generates a copy of $^\mathbb{N}\mathbf{2}$ in $\mathbf{L}$. Therefore (2) implies (3). Finally assume (4) holds. This implies that $\mathrm{Idl}(\mathbf{L})$ cannot have locally finite meet breadth, since $\mathbf{2}^\mathbb{N}$ does not. Then the implication (5) implies (4) in Theorem 3.6 tells us that (1) holds.                                    $\square$

Combining all the preceding results, applied to $\mathbf{L}$ and, order dually, to $\mathbf{L}^\partial$, we obtain the following theorem, which includes criteria in terms of $\mathbf{L}$ itself for the canonical extension $\mathbf{L}^\delta$ to coincide with the profinite completion of either of its unital semilattice reducts.

**Theorem 3.8.** *Let* $\mathbf{L} \in \mathfrak{D}$. *Then the following statements are equivalent:*

(M1)  $\mathbf{L}$ *fails to contain a meet subsemilattice isomorphic to the countably generated free semilattice;*
(M2)  $\mathrm{Filt}(\mathbf{L})$ *has locally finite meet breadth;*
(M3)  $\mathbf{L}^\delta_\wedge$ *coincides with* $\widehat{\mathbf{L}_\wedge}$.

*Order dually, the following statements are equivalent:*

(J1)  $\mathbf{L}$ *fails to contain a join subsemilattice isomorphic to the countably generated free semilattice;*
(J2)  $\mathrm{Idl}(\mathbf{L})$ *has locally finite meet breadth;*
(J3)  $\mathbf{L}^\delta_\vee$ *coincides with* $\widehat{\mathbf{L}_\vee}$.

We note that we showed directly in [13, Example 6.7] that a free semilattice on an infinite number of generators has the property that the canonical extension and $\mathbf{S}_\wedge$-profinite completion fail to coincide; indeed they have different cardinalities.

We have concentrated in this paper on distributive lattices, and distributivity of the ideal and/or filter lattice was central in the derivations of the theorems in this section. It is however of interest to know the extent to which the presence or absence of a suitably embedded free semilattice influences the relationship between the profinite completion and canonical extension of a unital semilattice.

To align with the literature on which we principally draw we shall work below with $\mathbf{S}_\vee$ rather than $\mathbf{S}_\wedge$. We follow the notation of Chakir and Pouzet [2, 3] and denote by $[\kappa]^{<\omega}$ the free $\mathbf{S}_\vee$-algebra on $\kappa$ generators, where $\kappa$ is an infinite cardinal; it can be identified with the join semilattice of finite subsets of $\kappa$, with $\vee$ as $\cup$ and

0 as the empty set. We now elucidate when the free $\mathbf{S}_\vee$-algebra on $\kappa$ generators. embeds into a given $\mathbf{S}$ in $\mathbf{S}_\vee$ (or a given $\mathbf{S} \in \mathbf{S}_\wedge$).

Proposition 3.9, in an order dual form, is given by Chakir and Pouzet [3, Section 2], in part drawing on earlier studies on the presence or absence of infinite antichains given in [16, 17, 18] and elsewhere. Chakir and Pouzet refer to a [join] independent set in a join semilattice: order dually, a subset $B$ of a meet semilattice is [meet] independent if for all $x \in B$ and every non-empty finite subset $F$ of $B \setminus \{x\}$ we have $x \not\geqslant \bigwedge F$. It is elementary to check that this condition is satisfied if and only if $B$ is meet irredundant, as we defined this term earlier. In the statement of the proposition, preservation of the element 1 is not important: if an embedding exists in (2) without 1 being preserved, then an embedding can be found which does preserve 1 (see [3, Lemma 4(i)]).

**Proposition 3.9.** *Let* $\mathbf{S} \in \mathbf{S}_\wedge$*. Then the following statements are equivalent:*

(1) $\mathbf{S}$ *contains a meet irredundant subset of cardinality* $\kappa$*;*
(2) $\mathbf{S}$ *contains a meet-subsemilattice isomorphic to the free semilattice in* $\mathbf{S}_\wedge$ *on* $\kappa$ *generators.*
(3) $\mathbf{S}$ *contains a subposet isomorphic to the free algebra on* $\kappa$ *generators in* $\mathbf{S}_\vee$*;*
(4) $\mathrm{Filt}(\mathbf{S})$ *contains a subposet isomorphic to* $\mathbf{2}^\kappa$*;*
(5) $\mathbf{2}^\kappa$ *embeds into* $\mathrm{Filt}(\mathbf{S})$ *by a map preserving arbitrary meets.*

This result contributes to our study in several ways. When $\mathbf{S} \in \mathbf{D}_\wedge$, condition (3) is, by Theorem 3.8, necessary and sufficient for $\mathbf{S}^\delta$ to fail to coincide with $\widehat{\mathbf{S}}$. The fact that in (5) the embedding map preserves arbitrary meets, rather than arbitrary meets and directed joins, signals that the proposition does not subsume our results for the distributive lattice case and, given the tools we needed to employ there, we would not expect it to do so. However the fact that (3) implies (2) is new information, even when $\mathbf{S} \in \mathbf{D}_\wedge$. We might also apply the proposition to $\mathbf{S} = \mathrm{Filt}(\mathbf{L})$ where $\mathbf{L} \in \mathbf{D}$. Here, when $\kappa = \aleph_0$, (1) is a necessary condition for $\mathrm{Filt}(\mathbf{L})$ to fail to have locally finite meet breadth. Hence all of conditions (1)–(5) must hold for $\mathbf{S} = \mathrm{Filt}(\mathbf{L})$ whenever $\mathbf{L}_\wedge^\delta$ and $\widehat{\mathbf{L}_\wedge}$ do not coincide. For a semilattice $\mathbf{S}$ which is not the meet reduct of some $\mathbf{L} \in \mathbf{D}$, we cannot expect condition (3) to be necessary and sufficient for $\mathbf{S}^\delta$ and $\widehat{\mathbf{S}}$ to fail to coincide. However the proposition does indicate that, if its equivalent conditions are satisfied, then $\widehat{\mathbf{S}}$ is 'big'. In such a case coincidence is probably unlikely. (As noted earlier, we know that coincidence fails if $\mathbf{S} = [\kappa]^{<\omega}$.)

## 4. CHARACTERISATION OF COINCIDENCE IN TERMS OF PRIESTLEY DUALITY

Thus far in our comparison of $\mathbf{L}_\wedge^\delta$ and $\widehat{\mathbf{L}_\wedge}$ for $\mathbf{L} \in \mathbf{D}$, we have not made direct use of Priestley duality. We do so now.

We call on some ideas originating with Lawson, Mislove and Priestley as part of Theorem 3.1 in [17]. We note that the proofs of some implications in the full theorem given there are abbreviated and cryptic. We shall take a somewhat different route. On the way we obtain, in Proposition 4.1 and Corollary 4.2, a slight refinement of the result given as Theorem 3.1 in [17]. Finally, in Theorem 4.3 we give necessary and sufficient conditions on a distributive lattice to have coincidence of its canonical extension with the profinite completions of its semilattice reducts.

**Proposition 4.1.** *Let* $\mathbf{L} \in \mathbf{D}$ *and let* $X = \mathbf{D}(\mathbf{L}, \mathbf{2})$ *be its Priestley dual space. Then the following statements are equivalent:*

(1) $\mathbf{L}$ *fails one, and hence all, of conditions* (M1)–(M3) *in Theorem* 3.8*;*
(2) *there is a countably infinite antichain of points* $x_n$ *in* $X$ *converging to a point* $x$ *and such that there exists* $n$ *for which* $x \geqslant x_n$*.*

*Order dually, the following statements are equivalent:*

$(1)^\partial$ **L** *fails one, and hence all, of conditions* (J1)–(J3) *in Theorem* 3.8;

$(2)^\partial$ *there is a countably infinite antichain of points $x'_n$ in $X$ converging to a point $x'$ and such that there exists $n$ for which $x' \leqslant x'_n$.*

*Proof.* We recall that under Priestley duality, and with **L** identified with the clopen up-sets of $X$, the filter lattice of **L** can be identified with the lattice of open down-sets of $X$, via the map $F \mapsto \bigcup\{\, (X \setminus a) \mid a \in F \,\}$. Under this correspondence the prime filters are exactly the open sets of the form $X \setminus {\uparrow}u$ for $u \in X$. Given an open down-set $U$, the sets $X \setminus {\uparrow}z$ for $z \in \mathrm{Min}(X \setminus U)$ are precisely the prime filters minimal with respect to lying above $U$ in $\mathrm{Filt}(\mathbf{L})$. Here $\mathrm{Min}\,S$ denotes the minimal points of a subset $S$ of $X$. Note that because $X$ is a Priestley space,, $S = {\uparrow}\mathrm{Min}\,S$ when $S$ is closed in $X$; this is easily proved by Zorn's Lemma, using elementary properties of Priestley spaces. Further details concerning duality for ideals and, order dually, for filters, can be found in [7, Chapter 11].

Assume (1) holds and apply condition (2) in Theorem 3.6 to find an open down-set $U$ such that there exists an infinite antichain of points $y_k$ minimal in $X \setminus U$. Since $X \setminus U$ is closed, this antichain has a limit point, $x$ say, and this belongs to $X \setminus U$. Then there is a subsequence $(y_{k_n})_{n \geqslant 2}$ of $(y_k)$ converging to $x$. The point $x$ need not lie above any of the elements $y_{k_n}$. However there is a minimal element of $X \setminus U$ below $x$. Define $y_{k_1}$ to be such an element. Then let $x_n = y_{k_n}$ $(n = 1, 2, \dots)$ to obtain a antichain sequence satisfying condition (2).

Now assume that (2) holds. Let $Y = {\uparrow}\{\, x_n \mid n \in \mathbb{N} \,\}$. Since the sequence $(x_n)$ is assumed to converge to $x$ and $X$ is Hausdorff, $\{\, x_n \mid n \in \mathbb{N} \,\} \cup \{x\}$ is closed, and $Y$ is the up-set it generates, by the assumption on $x$ in (2). Therefore $Y$ is closed. Consider the filter $F$ corresponding to the open down-set $U := X \setminus Y$. The prime filters minimal with respect to lying below $F$ in $\mathrm{Filt}(\mathbf{L})$ correspond to the sets $X \setminus {\uparrow}u$, where $u$ is minimal in $Y$. These are exactly the sets $X \setminus {\uparrow}x_n$ for $n \in \mathbb{N}$. Therefore condition (2) in Theorem 3.6 holds.

Since $\mathrm{Idl}(\mathbf{L})$ is $\mathrm{Filt}(\mathbf{L}^\partial)$ and $X^\partial$ (with the same topology as on $X$) is the dual space of $\mathbf{L}^\partial$, the dual conditions $(1)^\partial$ and $(2)^\partial$ are equivalent, by the previous argument applied to $\mathbf{L}^\partial$.                                                $\square$

**Corollary 4.2.** *Let $\mathbf{L} \in \mathfrak{D}$ and let $X = \mathfrak{D}(\mathbf{L}, \mathbf{2})$ be its Priestley dual space. Then the following statements are equivalent:*

(1) $\mathrm{Filt}(\mathbf{L})$ *or* $\mathrm{Idl}(\mathbf{L})$ *fails to have locally finite meet breadth;*

(2) $X$ *has an infinite antichain.*

*Both* $\mathrm{Filt}(\mathbf{L})$ *and* $\mathrm{Idl}(\mathbf{L})$ *fail to have locally finite meet breadth if both conditions* (2) *and* $(2)^\partial$ *in Proposition* 4.1 *hold.*

*Proof.* Proposition 4.1 tells us that (1) implies (2).

Conversely, assume that $X$ contains a countably infinite antichain of points $x_n$, which, by passing to a subsequence if necessary, we may assume converges to some point $x$. Then we have the following cases:

(I) $x \geqslant x_n$ for some $n$;

(II) $x \leqslant x_n$ for some $n$;

(III) $x$ is incomparable to all the elements $x_n$.

In the first two cases, the results of Proposition 4.1 can be applied. In the third case, we may add $x$ to the original antichain to obtain one which satisfies both of conditions (2) and $(2)^\partial$ in Proposition 4.1.                                $\square$

The next result deserves to be recorded explicitly. It follows from Theorem 3.8.

**Theorem 4.3.** *Let* $\mathbf{L} \in \mathbf{D}$. *Then in order that each of* $\widehat{\mathbf{L}_\wedge}$ *and* $\widehat{\mathbf{L}_\vee}$ *serves as the canonical extension* $\mathbf{L}^\delta$ *of* $\mathbf{L}$ *it is necessary and sufficient that the Priestley dual space of* $\mathbf{L}$ *contain no infinite antichain. Equivalently,* $\mathbf{L}$ *must not contain a copy of the free countably generated semilattice, either as an* $\mathbf{S}_\wedge$-*subobject or as an* $\mathbf{S}_\vee$-*subobject.*

We recollect that $\mathbf{L}$ in $\mathbf{D}$ may contain an infinite antichain even when its Priestley dual space does not. In [17, Proposition 3.3] it is shown that $\mathbf{L}$ contains an infinite antichain but $X = \mathbf{D}(\mathbf{L}, \mathbf{2})$ fails to contain an infinite antichain precisely when $X$ contains a subposet isomorphic to the disjoint union, $Y$ say, of the chain $\omega$ and its dual $\omega^\partial$. By way of an example we may consider $\mathbf{L} = (\omega \oplus \mathbf{1}) \times (\mathbf{1} \oplus \omega^\partial)$, for which $\mathbf{D}(\mathbf{L}, \mathbf{2}) = Y$. In this case we can verify directly that $\mathbf{L}^\delta = (\omega \oplus \mathbf{2}) \times (\mathbf{2} \oplus \omega^\partial)$ and that this serves also as the profinite completion of each of $\mathbf{L}_\wedge$ and $\mathbf{L}_\vee$.

We now have criteria which enable us to recognise from the dual space of a bounded distributive lattice $\mathbf{L}$ when the various completions, $\mathbf{L}^\delta$, $\widehat{\mathbf{L}_\wedge}$ and $\widehat{\mathbf{L}_\vee}$ do and do not coincide. There is a very special class of Priestley spaces which yield examples in which we can recognise coincidence or non-coincidence of the completions very directly from the original lattice without the need to consider whether it contains an infinite free semilattice.

We start from the observation that any algebraic lattice, equipped with its Lawson topology, is a Priestley space and so the dual space of a member of $\mathbf{D}$. The lattices arising in this fashion have very special properties, as set out in [24]. Let $\mathbf{L}$ be such a lattice, with dual space $X$, and let $\mathcal{J}(\mathbf{L})$ denote the ordered set of join-irreducible elements of $\mathbf{L}$. Then every non-zero element of $\mathbf{L}$ is a join of a finite set of elements from $\mathcal{J}(\mathbf{L})$, and the ordered set $\mathcal{J}(\mathbf{L})$ is dually isomorphic to $K(X)$, the join semilattice of compact elements in the algebraic lattice $X$. The ideal lattice $\mathrm{Idl}(\mathbf{L})$ is isomorphic to the lattice of Scott-open subsets of $X$. As a consequence $\mathrm{Idl}(\mathbf{L})$ is completely distributive and is isomorphic to the lattice of down-sets of $\mathcal{J}(\mathbf{L})$. The lattice of down-sets of an ordered set $P$ fails to have locally finite meet breadth if and only if it contains $\mathbf{2}^\mathbb{N}$ as a complete sublattice (here the bounds are not required to be preserved) or, equivalently, if and only if $P$ has an infinite antichain (see [18], where additional references to the result can also be found).

**Proposition 4.4.** *Let* $X$ *be an algebraic lattice equipped with the Lawson topology and let* $\mathbf{L}$ *be the lattice of clopen up-sets of* $X$. *Then the following conditions are equivalent:*

(1) $\mathrm{Idl}(\mathbf{L})$ *has locally finite meet breadth;*
(2) $\mathcal{J}(\mathbf{L})$ *does not contain an infinite antichain.*

*Corresponding order-dual statements can be made in the case that* $X^\partial$ *is an algebraic lattice.*

We can now construct many examples in which $\mathbf{L}^\delta$ fails to coincide with $\widehat{\mathbf{L}_\vee}$: we simply take any join semilattice $\mathbf{S}$ having an infinite antichain, equip $\mathrm{Idl}(\mathbf{S})$ with the Lawson topology and let $\mathbf{L}$ be the lattice of clopen up-sets of $\mathrm{Idl}(\mathbf{S})$.

An even more special case arises from lattices $X$ which are linked bi-algebraic, that is, both algebraic and dually algebraic and such that the interval topology on $X$ is Hausdorff. Any such lattice $X$ carries the interval topology as its unique Priestley space topology, this coincides with both the Lawson topology and dual Lawson topology, and $X$ is a compact zero-dimensional topological lattice; see [23] and [11, Section VII-2]. In this situation Proposition 4.4 applies to both $X$ and $X^\partial$. Hence the lattice $\mathbf{L}$ of clopen up-sets of $X$ is such that $\mathbf{L}^\delta$ coincides with both $\widehat{\mathbf{L}_\vee}$ and $\widehat{\mathbf{L}_\wedge}$ if and only if $X$ has neither an infinite antichain of compact elements nor an infinite antichain of cocompact elements.

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