Duality for archimedian l-groups with strong order unit

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- Joint work with Daniela Petrişan

Roughly what I will talk about

- Archimedean *l*-groups are well studied, particularly representation as *l*-groups of real continuous functions.
- Want a better handle on the finitistic aspects of representation.
- In particular, we aim for a finitary logic for integration.
- Duality theory gives us logics for certain categories of 0-dimensional spaces.
- Yet, *l*-group representation needs non-0-dimensional spaces.
- Strong proximity lattices with negation provide a duality for compact Hausdorff spaces.
- Hence, these *l*-groups have a (choice free) representation via strong proximity lattices.
- Conclusion, for representation, "*l*" is almost enough. Need only the additional group theoretic data:
  - Positive and negative parts of an element;
  - the lattice filter of strong order units.

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Outline

Introducing *l*-groups

Definitions

l-group: A group with a compatible lattice structure: $\wedge$, $\vee$, ...

archimedean: if every multiple of a is bounded by some b, then $a \leq 0$

strong order unit: an element u so that every element is bounded by a multiple of u

S: The category consisting of archimedean *l*-groups equipped with a strong order unit.

Examples

The usual suspects: *l*-groups of bounded real-valued functions

Basics of S

Common (but not trivial) facts about **S** objects

- The lattice reduct of any *l*-group is distributive (but not bounded).
- The group reduct of an archimedean *l*-group is abelian
  – justifies writing groups additively

Standard notation

- a is positive: $0 \leq a$
- $a^+ := a \vee 0$
- $a^- := (-a)^+ = -(a \wedge 0)$
- $|a| := a^+ + a^-$
- $na := a + a + \ldots + a$ (n will always be a positive integer)

Not so standard notation and facts

Definition

$0 \lll a$ a is a strong order unit

Warning: In l-group literature $a \ll b$ means a is infinitely smaller than b.

Facts about strong order units

- the strong order units form a lattice filter (never empty in $A \in \mathbf{S}$);
- for $0 \leq a, b, c$, any d:
  - $0 \lll (a \vee b) + c \quad \Leftrightarrow \quad 0 \lll a + b + c$;
  - $0 \lll a + d^+$ and $0 \lll a + d^- \Rightarrow 0 \lll a$;
  - $0 \lll a + b \quad \Leftrightarrow \quad 0 \lll ma + nb$ for some m, n
  - $a \wedge b = 0$ and $0 \lll a + c \Rightarrow b \leq nc$ for some n.

The idea behind strong proximity lattices

Idea: How to get non-0-dimensional duality

- **KHaus** consists of retracts of Stone spaces.
- A compact Hausdorff space is specified by a retraction (X, r) where $r\colon X \to X$ with $r^2 = r$.
- **BA** $\equiv$ **Stone**$^{\mathrm{op}}$.
- Hence, something like (co)retractions of Boolean algebras are dual to compact Hausdorff spaces.
- Stably compact spaces are also retracts of spectral spaces, but not all continuous retracts will be spectral morphisms.
- To get duality for stably compact spaces, more care is needed.

Either way, this is not especially convenient. We want categories that are simple to calculate with.

What strong proximity lattices are

Definition

$(L, \prec)$
- L, a bounded distributive lattice (or Boolean algebra) $(\sqcap, \sqcup, \top, \bot, \ldots)$
- $\prec$, a sublattice of $\sqsubseteq$
- $\prec ; \sqsubseteq \; = \; \prec \; = \; \sqsubseteq ; \prec$
- $\prec ; \prec \; \subseteq \; \prec$
- $a \sqcap b \prec c$ implies $\exists a'$ so that $a \prec a'$ and $a' \sqcap b \prec c$
- similarly for $\sqcup$ on the right.
- Boolean case: $a \sqcap b \prec c$ iff $b \prec c \sqcup \neg a$.

$\vdash : L \to M$
- A binary relation compatible with the structures:

$$\prec_L ; \vdash \; = \; \vdash \; = \; \vdash ; \prec_M$$

Prox* Category of proximity lattices and compatible relations

Prox$^*_\neg$ Category of Boolean strong proximity lattices and compatible relations

Locating the functional morphisms in Prox*

Adjoint pairs

For $\mathcal{A}$ any order-enriched category

Map$(\mathcal{A})$ The subcategory of $\mathcal{A}$ with morphisms $R_* : L \to M$ having an upper adjoint R^*:

$$R^*; R_* \leq \mathrm{id}_L \qquad \text{and} \qquad \mathrm{id}_M \leq R_*; R^*$$

Will track pairs (R_*, R^*), even though R^* is determined by R_*

Theorem

With morphisms in **Prox** *ordered by* $\supseteq$*,*

- **Map**(**Prox***) $\equiv$ **StK**, *the category of stably compact spaces and perfect maps.*
- **Map**(**Prox**$^*_\neg$) $\equiv$ **KHaus**, *the category of compact Hausdorff spaces (where all continuous maps are perfect anyway).*

Toward continuous sequent calculus

Observation

- $\prec$ determines a congruence on L: $a \equiv_L b$ meaning

$$a \prec c \;\Leftrightarrow\; b \prec c \quad \& \quad c \prec a \;\Leftrightarrow\; c \prec b.$$

- Proximity lattice morphisms are invariant with respect to this: $a \equiv_L b \vdash c$ implies $a \vdash c$, etc.

Consequence

- No point in requiring L to be a bounded distributive lattice at all
- L can be any algebra for the signature of bounded lattices
- With unary $\neg$, in the case of $\mathbf{Prox}^*_\neg$
- $\prec$ can be suitably well-behaved ensuring that $(L/\equiv, \prec/\equiv)$ is a strong proximity lattice

Continuous sequent calculus: presentation of strong proximity lattices

The categories **MLS** and **MLS**$_\neg$

MLS the category of strong proximity lattices in the relaxed sense

MLS$_\neg$ Likewise for strong proximity lattices with negation

Some details need to be watched

- Convenient to write in the sequent calculus style: $\Gamma \prec \Delta$ for finite sets
- Idempotency, commutativity are "for free"
- Associativity, absorptivity and distributivity coded easily in inference rules
- Details more important, e.g., for coalgebraic logic, another topic.

Equivalences of categories

Theorem

MLS *is equivalent as an order-enriched category to* $\mathbf{Prox}^*$
$\mathbf{MLS}_\neg$ *is equivalent as an order-enriched category to* $\mathbf{Prox}^*_\neg$

Corollary

The corresponding categories of maps (adjoint pairs) are equivalent to **StK** *and* **KHaus**, *respectively.*

S objects are strong proximity lattices

Definition

For $(A, +, 0, -, \wedge, \vee, u) \in \mathbf{S}$,

$$U(A) := (A, \wedge, \vee, u, -u, -)$$
$$a \prec_A b \;:\Leftrightarrow\; 0 \lll a^- + b^+$$

Observations

- $U(A)$ has the right signature
- Certainly not a Boolean algebra:
  - u and $-u$ are not bounds
  - $a \wedge -a = -u$ fails
  - So does $a \vee -a = u$
  - Not even the interval $[-u, u]$ is Boolean

The structures are right

Lemma

For any $A \in \mathbf{S}$*,* $U(A) := (U(A), \prec_A)$ *is a strong proximity lattice with negation in the relaxed sense of* **MLS**$_\neg$*.*

Proof sketch

Mostly, applying the previous list of facts about strong order units.

Cut rule If $a \prec b \vee c$ and $c \wedge a \prec b$, then also $c^+ \wedge c^- = 0$
So $a \prec b$.

L$\wedge$ rule

$$
\begin{aligned}
a \prec b \,\&\, a \prec c &\Leftrightarrow 0 \lll a^- + b^+ \,\&\, 0 \lll a^- + c^+ \\
&\Leftrightarrow 0 \lll (a^- + b^+) \wedge (a^- + c^+) \\
&\Leftrightarrow 0 \lll a^- + (b \wedge c)^+ \\
&\Leftrightarrow a \prec b \wedge c
\end{aligned}
$$

Proof sketch continued

Negation rule

$$\begin{aligned} a \prec b \vee c &\Leftrightarrow 0 \lll a^- + b^+ + c^+ = a^- + b^+ + (-c)^- \\ &\Leftrightarrow -c \wedge a \prec b \end{aligned}$$

Interpolation Suppose $a \prec b$

- $0 \lll a^- + b^+$
- For some m,
  $0 \lll u \leq ma^- + mb^+ - u \leq ma^- + (mb - u)^+$
- So $ma \prec mb - u$ and $a \prec mb - u$
- Also, $0 \lll u \leq (mb - u)^- + mb^+$
- So $mb - u \prec mb$ and $mb - u \prec b$.

Strong interpolation follows from this and negation.

Extending U to morphisms

Definition

Let $h\colon A \to B$ be a morphism in **S**.

$\vdash_h\colon U(B) \to U(A)$ $b \vdash_h a \;:\Leftrightarrow\; b \prec_B h(a)$

$\vdash^h\colon U(A) \to U(B)$ $b \vdash a \;:\Leftrightarrow\; h(a) \prec_B b$

Lemma

The pair $(\vdash_h, \vdash^h)$ *constitutes an adjoint pair in* **Prox**$_\neg$. *Moreover,* $\vdash_{\mathrm{id}_A} = \prec_A$ *and* $\vdash_{hg} = \vdash_h ; \vdash_g$.

Proof sketch

Check that $\vdash^h$ and $\vdash_h$ are indeed morphisms
Adjointness follows easily from the definition
Also easy:

$$\vdash_h ; \vdash_g \subseteq \vdash_{hg} \text{ and } \vdash^g ; \vdash^h \subseteq \vdash^{hg}$$

The proximity lattice of the extended reals

Definition

Extended reals $L\mathbb{R} := (\mathbb{Q} \cup \{-\infty, \infty\}, \wedge, \vee, \infty, -\infty, <)$ where

- $-\infty < -\infty < p < \infty < \infty$ for all p
- $p < q$ for finite rationals as usual

Observations

- $L\mathbb{R}$ in **MLS**, not in **MLS**$_\neg$
- Dual space is $\mathbb{R}^*_w$ ($\mathbb{R}^*$ with weak lower topology)
- For compact Hausdorff X, the perfect maps $X \to \mathbb{R}^*_w$ are exactly the continuous maps from $X \to \mathbb{R}^*_E$ (with Euclidean topology)

Bounded real morphisms in Map(MLS$_\neg$)

Definition

Adjoint pair $(\Rightarrow_f, \Rightarrow^f) = f\colon L \to L\mathbb{R}$ is

f bounded iff $\top \Rightarrow_f q$ and $p \Rightarrow^f \bot$ for some finite rationals p and q.

$B(L)$ the set of all bounded adjoint pairs from L.

[Note: "*f*" is merely a label here.]

$B(L)$ is a "proof-theoretic" l-group

Main idea

- Algebraic structure of $B(L)$ must involve constructing adjoint pairs
- E.g, $f + g = (\Rightarrow_{f+g}, \Rightarrow^{f+g})$ constructed from data

$$f = (\Rightarrow_f, \Rightarrow^f)$$
$$g = (\Rightarrow_g, \Rightarrow^g)$$

- Perhaps can formulate these as proof rules:

$$\frac{\text{Conditions on } \Rightarrow_f \text{ and } \Rightarrow_g}{a \Rightarrow_{f+g} r} \qquad \frac{\text{Conditions on } \Rightarrow^f \text{ and } \Rightarrow^g}{r \Rightarrow^{f+g} a}$$

The structure of $B(L)$ via proof rules

Constants For each rational p,

$$\frac{q < r}{a \Rightarrow_q r} \qquad \frac{p < q}{p \Rightarrow^q a}$$

Addition For bounded adjoint pairs f and g,

$$\frac{a \Rightarrow_f p \quad a \Rightarrow_g q \quad p + q < r}{a \Rightarrow_{f+g} r} \qquad \frac{p < q + r \quad q \Rightarrow^f a \quad r \Rightarrow^g a}{p \Rightarrow^{f+g} a}$$

Meet For bounded adjoint pairs f and g,

$$\frac{a \Rightarrow_f p \quad a \Rightarrow_g p}{a \Rightarrow_{f \sqcap g} p} \qquad \frac{p \Rightarrow^f a}{p \Rightarrow^{f \sqcap g} a} \qquad \frac{p \Rightarrow^g a}{p \Rightarrow^{f \sqcap g} a}$$

Negation

$$\frac{\neg a \Rightarrow_f -p}{p \Rightarrow^{-f} a} \qquad \frac{-p \Rightarrow^f \neg a}{a \Rightarrow_f p}$$

Main Lemma

Lemma

- *$B(L)$ is a divisible archimedean l-group with strong order unit $(\Rightarrow_1, \Rightarrow^1)$.*
- *$B(-)$ extends to a functor via composition of adjoint pairs:*
  - *For $h = (\vdash_h, \vdash^h)\colon L \to M$,*
    define $B(h)\colon B(M) \to B(L)$ by

$$B(h)(f) := (\vdash_h; \Rightarrow_f,\ \Rightarrow^f; \vdash^h)$$

Main Theorem

Theorem

- *B is dually right adjoint to U.*
- *The adjunction is a reflection: $L \simeq UB(L)$.*

Moreover, we get Gelfand-Naimark-Stone Duality

- Can extend the structure of $B(L)$ to an ordered ring.
- Result is a self-adjoint C^* algebra (completeness uses AC)
- U cut down to reducts of self-adjoint C^* algebras yields a dual equivalence.

Where I think we are going

Though Daniela may not agree entirely

1. Integration
   - Want to have a proof theory for integration:

$$\frac{\text{Conditions on } \Rightarrow^{f} \text{ and } \Rightarrow_{f}}{\varphi \Rightarrow^{\int f} p}$$

     where φ makes some assertion about (regular Borel) measures on a space.
   - Have sound rules for this, but do not have a completeness proof
2. Yosida representation
   - Replace strong order units with weak ones: **W**
   - Yosida: **W** represented by l-groups of continuous maps $f\colon X \to \mathbb{R}^*$ where $f^{-1}(\mathbb{R})$ is dense
   - Will need to replace boundedness with “dense domain of reality”
   - Frame theorists: Ball, Banaschewski, Pultr have some ideas that may work

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More

Even less sure whether Daniela is on board

3. Assuming (1) and (2), integration should extend allowing for $\int f \, d\mu = \infty$
4. Assuming (3), this should tie up with Daniell's theory of integration.