Duality Theory: the Parts that Category Theory Can’t Reach?

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3 August, 2011
Rival approaches?

**Aim of talk**: to compare two duality notions:
Suppose we have two concrete categories $\mathcal{A}$ and $\mathcal{X}_T$, where the subscript $T$ signals presence of topological structure. Concreteness (in the weak sense that there are faithful grounding functors to SET) enables us to talk about **finite objects**.
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- **Stone-type duality** (as the term is used by Johnstone in *Stone Spaces*): a dual equivalence between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ arising from a dual equivalence between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}\mathcal{T}}$ (the ‘finite level’) by taking, respectively, Ind-completion and Pro-completion.
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Suppose we have two concrete categories $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$, where the subscript $\mathcal{T}$ signals presence of topological structure. Concreteness (in the weak sense that there are faithful grounding functors to $\text{SET}$) enables us to talk about **finite objects**.

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- **Natural duality (full)**: dual equivalence between suitable categories $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ set up by hom-functors into $\mathcal{M}$ and $\mathcal{M}_\mathcal{T}$ with the same finite underlying set, with the unit and counit maps of the adjunction being isomorphisms, given by evaluations. Here $(\mathcal{M}, \mathcal{M}_\mathcal{T})$ acts as a **dualising object**.
Two classic examples, each both Stone-type and natural

(1) Stone duality

\[ \mathcal{A} = \text{BA} — \text{Boolean algebras} \]
\[ \mathcal{X}_J = \text{STONE} — \text{Stone (alias Boolean) spaces} \]

(2) A related dual equivalence:

\[ \mathcal{A}_J = \text{STONE} – \text{BA} — \text{Boolean-topological Boolean algebras} \]
\[ \mathcal{X} = \text{SET} \]

A category isomorphic to \( \mathcal{A}_J \) is

CABA—complete and atomic Boolean algebras

[Obvious choice of morphisms in all cases.]
Features of the preceding examples

- BA is a finitary algebraic category
- SET is (degenerately) a finitary algebraic category
- STONE is a category of topological spaces
- We have isomorphic categories:
  \[
  \begin{cases}
    \text{CABA is an infinitary algebraic category,} \\
    \text{STONE-BA is a topological algebraic category}
  \end{cases}
  \]

NOTE: operating in ZFC throughout—spaces, not frames; no glimpse of an Elephant here.
Generation properties

BA is expressible as

\[ \text{HSP}(2) \] —a variety, definable by equations;
\[ \text{ISP}(2) \] —a quasivariety, definable by implications

Each of the other categories too is generated by a 2-element member, assumed to carry the discrete topology in case of STONE and STONE-BA. [More detail later.]
Putting the pieces together

\[ CABA \cong STONE - BA \leftarrow \overset{D}{\rightarrow} \overset{E}{\leftarrow} \overset{F}{\rightarrow} \overset{G}{\leftarrow} \overset{b}{\rightarrow} \overset{b}{\leftarrow} \overset{n_{BA}}{\downarrow} \overset{b}{\downarrow} \]

- \( b \) is the obvious forgetful functor.
- The two dual equivalences (horizontal arrows) are related by ‘topology-swapping’.
- Topology-swapping originates at the level of the generating objects.
- The left-hand vertical arrow is the canonical extension functor, when regarded as being into \( CABA \).
- Coalgebra buffs would now want to add endofunctors. \( \ldots \)
The distributive lattice case, likewise

* = completely distributive linked bialgebraic lattices (≡ up-set lattices)

This time, one of the categories without topology is

- a category of algebras (DL)

and the other is

- a category of structures (POS).

[This feature doesn’t show up with Stone duality.]
Duality theory: parts that category theory does reach
[informal discussion]

Assume we have a category $\mathcal{C}$ which is concrete in the strong sense that

- there is a faithful grounding functor $U: \mathcal{C} \to \text{SET}$;
- $U$ is \textbf{representable} in the sense that there exists $C_0 \in \mathcal{C}$ such that $U \cong \mathcal{C}(C_0, -)$.

Then $C_0$ acts as the ‘free object on one generator’.
Dualising objects

Now assume $\mathcal{C}$ and $\mathcal{Y}$ are concrete, with grounding functors $U$ and $V$, and that there are contravariant functors $S : \mathcal{Y} \to \mathcal{C}$ and $T : \mathcal{C} \to \mathcal{Y}$ yielding a dual adjunction $\langle S, T, e, \varepsilon \rangle$:

$$\mathcal{C}(C, SY) \cong \mathcal{Y}(Y, TC).$$

This set-up guarantees the existence of a dualising object, given by $(SY_0, TC_0)$ which (modulo a bijection) have the same underlying set. Refer to $SY_0$ and $TC_0$ as alter egos of each other.

With the grounding functors now suppressed, the unit and counit maps of the adjunction are necessarily given by evaluation:

$$e_C : C \to STC, \quad e_C(c) : y \mapsto y(c),$$

$$\varepsilon_Y : Y \to TSY, \quad \varepsilon_Y(y) : c \mapsto c(y).$$
Levels of good behaviour of a dual adjunction with associated dualising object

(1) just a dual adjunction;
(2) a dual adjunction, with unit and counit maps being ‘embeddings’;
(3) as in (2) and with each $e_C$ an isomorphism;
(4) a dual equivalence, with each $e_C$ and each $\epsilon_Y$ an isomorphism.

How far one can get will depend on how well the alter egos making up the dualising object operate in tandem.

Given that (1) holds, (2) relies on a compatibility requirement.

(3) and (4) are much harder to address in full generality. Categorical methodology exists—but not pursued here.
Narrowing the focus: a more algebraic perspective

A motivation for studying lattice-based or poset-based algebras often comes from logic—modelling syntactic specifications algebraically.

Motivation for studying dualities for these comes from
- the power of relational semantics (Kripke frames), eg for modal logics;
- coalgebraic modelling, of transition systems, eg.

We shall restrict to finite $M$—because we want to talk about profiniteness.

So what sort of categories should we encompass? For sure, certain categories whose models are
- algebras, often lattice-based.

But more generally we shall want to
- admit partial(ly defined) algebraic operations.
Narrowing the focus, continued

Within this framework, we’d want to focus on

- **varieties**—specified by equations, or
- **quasivarieties**—specified by implications.

These are, respectively, classes closed under $H$, $S$, $P$, or $I$, $S$, $P$.

Which should we go for: $H$ or $I$?

To be specific, consider a single algebra $M$ and compare $HSP(M)$ and $ISP(M)$.

- $A \in A := ISP(M)$ iff morphisms from $A$ into $M$ separate points.

In general, $ISP(M) \subsetneq HSP(M)$.
The lattice-based algebra case, with $\mathcal{M}$ finite

- $\text{HSP}(\mathcal{M}) = \text{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras (actually can take $\mathcal{M} = \text{HS}(\mathcal{M})$)—by Jónsson’s Lemma.

So $\text{HSP}$ case reduces to $\text{ISP}$ case, at the small cost of working in a multisorted set-up.

Key duality results extend, or can be expected to extend, to the multisorted case.
From algebras to structures

We want to be even-handed in our use of categories, and to allow for structures—not just algebras.

Let $M$ be a finite non-empty set. We shall call $M$ a **structure** if $M = (M; G, H, R)$, where

- $G$ is a set of (total) operations;
- $H$ is a set of partial operations;
- $R$ is a set of relations,

all of finite arity.

We call $M$ a **total structure** if $H = \emptyset$.

The characterisations of varieties and quasivarieties extend, mutatis mutandis, to the structures we consider.

The corresponding topologised structure, $M_T$, will always carry the discrete topology.
A pair of structures, four categories

Given any two structures, $\mathbf{M}$ and $\mathbf{M} \sim$ (on the same underlying set, but no compatibility yet) we have four categories:

\[
\begin{align*}
\mathcal{A} & := \text{ISP}(\mathbf{M}), & \mathcal{X} & := \text{ISP}^0\mathcal{P}^+(\mathbf{M}), \\
\mathcal{A}_\mathcal{T} & := \text{ISP}_c\mathcal{P}(\mathbf{M}_\mathcal{T}), & \mathcal{X}_\mathcal{T} & := \text{ISP}_c^0\mathcal{P}^+(\mathbf{M}_\mathcal{T}).
\end{align*}
\]

The first pair are categories of structures, the second pair are categories of Stone-structures of type $\mathbf{M}$.

Technical note

$\mathcal{P}$ allows empty indexed products, yielding the total one-element structure; $\mathcal{P}^+$ doesn’t. Operator $\mathcal{S}$ excludes the empty structure while $\mathcal{S}^0$ includes it, when the type does not include nullary operations.
A dual equivalence on the cheap: 
Hofmann–Mislove–Stralka duality for semilattices

\[ S = \text{ISP}(2) \quad \quad \quad \text{SL} = \land, 1 - \text{semilattices} \]
\[ Z = S_\mathcal{J} = \text{ISP}_c(2_\mathcal{J}) \quad \text{STONE-SL} \]

(Here we have two categories rather than four.)

On finite, discretely topologised, objects the topology does no work, so

\[ Z_{\text{fin}} \text{ “is” } S_{\text{fin}}. \]

With this identification the evaluation maps are just identities. SO we have a dual equivalence at the finite level.
HMS duality, continued

Easy:

\[ S \text{ built from } S_{\text{fin}} \text{ by taking directed (cofiltered) limits,} \]
\[ Z \text{ built from } Z_{\text{fin}} \text{ by taking projective limits (filtered colimits).} \]

and the limits/colimits are preserved by the functors.

\[
\begin{array}{ccc}
S & \xleftarrow{D} & Z \\
\uparrow{E} & & \downarrow{\text{Pro}} \\
S_{\text{fin}} & \xleftarrow{\text{Ind}} & Z_{\text{fin}}
\end{array}
\]

Back a step ...
Profiniteness, concretely

Profinite objects, are, loosely, those which are built from finite ones by means of filtered colimits.

Let $\mathcal{C}$ be a concrete category and $\mathcal{C}_{\text{fin}}$ the full subcategory of finite objects. We may ask whether

$$\mathcal{C} = \text{Pro-}\mathcal{C}_{\text{fin}},$$

the categorical pro-completion of $\mathcal{C}_{\text{fin}}$. ($\mathcal{C}$ a finitely accessible category.)

There is a mismatch between concretely-realised profinite limits and abstract ones (viewed as diagrams). This is why profinite widgets (groups, BAs, ...) are built from $\mathcal{C}_{\text{fin}}$ rather than $\mathcal{C}_{\text{fin}}$ and so are treated as topological algebras. Henceforth keep the $\mathcal{T}$ tacit since we always use the discrete topology on finite objects.
The dual notion: Ind-completion

This behaves well, for many classes of algebras. In any variety $\mathcal{V}$ of algebras which is locally finite (ie, finitely generated algebras are finite), then every $A \in \mathcal{V}$ is the directed union of its finite subalgebras. Hence

$$\mathcal{V} = \text{Ind-}\mathcal{V}_{\text{fin}}.$$  

This extends, easily, to a corresponding local finiteness result for any $\text{ISP}(\mathcal{M})$ where $\mathcal{M}$ is a finite structure.
HMS duality: lifting from the finite level

- At the finite level, the duality works trivially.
- $S$ is the Ind-completion of $S_{\text{fin}}$.
- $\mathcal{Z}$ is the Pro-completion of $\mathcal{Z}_{\text{fin}}$.
- The finite-level dual equivalence lifts to that between $S$ and $\mathcal{Z}$, by categorically routine arguments.

This was how the HMS duality was first proved.

What is NOT trivial is the recasting of $\mathcal{Z}$ in (infinitary) algebraic terms—this is the Fundamental Theorem on Compact Zero-Semilattices.
Compatibility of a pair of structures

- $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are compatible structures on the same finite set $\mathcal{M}$ (operations, relations and partial operations allowed); No presumption that $\mathcal{M}$ is “algebraic” and $\tilde{\mathcal{M}}$ “relational”.
- Compatibility: the structure of $\tilde{\mathcal{M}}$ is preserved by the operations and partial operations of $\mathcal{M}$ and the relations are substructures.

This notion is symmetric.
Paired adjunctions.

Let

\[ \mathcal{A} := \text{ISP}(M) \text{ and } \mathcal{X}_\mathcal{T} := \text{ISP}(\mathcal{M}_\mathcal{T}). \]
\[ \mathcal{A}_\mathcal{T} := \text{ISP}(\mathcal{M}_\mathcal{T}) \text{ and } \mathcal{X} := \text{ISP}(\mathcal{M}). \]

We have a dual adjunction between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ set up by the hom-functors

\[ D: \mathcal{A} \to \mathcal{X}_\mathcal{T}, \quad \begin{cases} 
D(A) = \mathcal{A}(A, M) \\
D(f) = - \circ f 
\end{cases} \]

\[ E: \mathcal{X}_\mathcal{T} \to \mathcal{A}, \quad \begin{cases} 
E(X) = X + \mathcal{I}(X, \mathcal{M}_\mathcal{T}) \\
E(\phi) = - \circ \phi 
\end{cases} \]

and likewise for $F$ and $G$.
Compatibility ensures that all the unit and counit maps are given by evaluations and are embeddings.
Levels of good behaviour

Focus first on $\mathcal{A}$ and $\mathcal{X}_T$ and the dual adjunction $\langle D, E, e, \epsilon \rangle$.

Treat $M$ as given, $\tilde{M}$ a candidate alter ego. Can $M, \tilde{M}$ be chosen so that

- $\tilde{M}$ yields a **duality** ($e_A$ is an isomorphism for all $A \in \mathcal{A}$):
- $\tilde{M}$ yields a **full duality** (ie $\langle D, E, e, \epsilon \rangle$ sets up a dual equivalence)
- $\tilde{M}$ yields a **[full] duality** between $\mathcal{A}_{\text{fin}}$ and $\mathcal{X}_{\text{fin}T}$ ($M$ yields a [full] duality, at the finite level.

Independence results: quasivarieties of algebras

- Given $A$, dualisability and full dualisability (and strong dualisability too) are independent of the choice of the generator $M$.

**Cautionary example** (Hyndman/Willard)

In general $A$ can be dualisable but not fully dualisable: example provided by an $M$ which is a 3-element chain with 2 unary operations,
The finite level, in the natural duality set-up

We want, at the very least, a duality at the finite level.

- This may be trivial (as for SL) or easy (as for BA or DL).

- It may be possible only with an alter ego of infinite type.
  Example:
  \[ M = (\{0, 1\}, \rightarrow) \]
  \[ (\mathcal{A} = \text{HSP}(M) = \text{ISP}(M) \text{ is the variety of implication algebras}). \]
Finitely generated lattice-based (quasi)varieties

Let $\mathcal{A} = \text{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite lattice with (maybe) additional operations.

- Then there exists $\mathbf{M}$ yielding a duality, with $\mathbf{M}$ a total structure.

- There exists $\mathbf{M}$ yielding a full duality (in fact a strong duality), but $\mathbf{M}$ cannot always to chosen to be a total structure: in general partial homomorphisms of arity $\leq 2$ need to be included in $\mathbf{M}$.

There is a corresponding result, in terms of multisorted dualities, for a variety $\mathcal{A} = \text{HSP}(\mathbf{M})$. 
Lifting from the finite level, via Pro- and Ind-completion

**Proposition** Take $M$ finite, and $M, \bar{M}$ compatible structures,

$\mathcal{A} := \text{ISP}(M)$ and $\mathcal{X}_\mathcal{T} := \text{IScP}(M_\mathcal{T})$.

Assume that $\bar{M}$ yields a duality between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$. Then $\bar{M}$ yields a full duality between $\mathcal{A}$ and $\mathcal{X}_\mathcal{T}$ if and only if

- $\bar{M}$ yields a full duality at the finite level, and
- $\mathcal{X}_\mathcal{T} = \text{Pro} - \mathcal{X}_{\text{fin}}$.

This is a purely categorical result.

SO we’re all set—or are we? Must $\mathcal{X}_\mathcal{T} = \text{Pro} - \mathcal{X}_{\text{fin}}$??
Lifting from the finite level: compactness

The Duality Compactness Theorem
Assume $M$ finite and that $(M, \tilde{M})$ yields a duality at the finite level and $\tilde{M}$ is of finite type. Then $M$ yields a duality:

- for $M$ an algebra (Zadori, Willard);
- for $M$ a structure (D. Hofmann, 2002—established in context of finitary limit sketches).

We’d like to insert ‘full’ in assumption and conclusion. BUT WE CAN’T!
Lifting from the finite level: compactness

The Duality Compactness Theorem
Assume $M$ finite and that $(M, \sim)$ yields a duality at the finite level and $\sim$ is of finite type. Then $M$ yields a duality:

- for $M$ an algebra (Zadori, Willard);
- for $M$ a structure (D. Hofmann, 2002—established in context of finitary limit sketches).

We’d like to insert ‘full’ in assumption and conclusion. BUT WE CAN’T!

Cautionary examples (Davey/Haviar/Willard et al.)

- for $M$ the 3-element chain, there exists $\sim$ of finite type which
  * dualises DL,
  * fully dualises DL at the finite level
  * but which does NOT lift to a full duality for DL.
- Same conclusion with $M$ replaced by any finite non-Boolean $M \in DL$. 
A Weak Full Duality Compactness Theorem (Davey, 2006) A full duality at the finite level based on \((M, \cong)\) does lift to a full duality if

- \(M\) is of finite type, and
- \(\cong\) is a total structure (no partial operations).

and then the dual equivalence between \(A\) and \(X_T\) does arise from Ind- and Pro-completion.

Catch 22: We often get full dualisability only at the expense of adding partial operations, and doing this may mean the duality is NOT given by Ind- and Pro-completion,
Dualities in partnership, stage 1

Back to the four category set-up.

\[
\begin{array}{c}
\mathcal{A} \quad \xrightarrow{D} \quad \mathcal{X}_\mathcal{J} \\
\downarrow n_A \quad \quad \quad \downarrow b \\
\mathcal{A}_\mathcal{J} \quad \xleftarrow{F} \quad \mathcal{X}
\end{array}
\]

with

\begin{itemize}
  \item \( \mathcal{A} := \text{ISP}(M) \) and \( \mathcal{X} := \text{ISP}^0(M) \);
  \item \( \mathcal{A}_\mathcal{J} := \text{ISP}_c(M) \) and \( \mathcal{X} := \text{ISP}(\sim) \).
  \item \( M \) and \( \sim \) compatible structures on the same underlying finite set,
\end{itemize}
Paired Adjunctions Theorem The following are equivalent:

(i) the outer square commutes, i.e., \( n_\mathcal{A}(A) = G(D(A)^\flat) \), for all \( A \in \mathcal{A} \);
(ii) \( n_\mathcal{A}(A) \) consists of all maps \( \alpha : \mathcal{A}(A, M) \to M \) that preserve the structure on \( M \), for all \( A \in \mathcal{A} \);
(iii) \( M \) yields a duality between \( \mathcal{A} \) and \( \mathcal{X}_\mathcal{J} \) at the finite level.

Moreover, if \( M \) is of finite type, then (i)–(iii) are equivalent to
(iv) \( M_\mathcal{J} \) yields a duality between \( \mathcal{A} \) and \( \mathcal{X}_\mathcal{J} \).
The TopSwap Theorem (Davey/Haviar/Priestley, 2011)
Assume $\mathbf{M}$ a total structure of finite type.

(1) If $\mathbf{M}$ yields a finite-level duality between $\mathcal{A}$ and $\mathcal{X}_J$, then $\mathbf{M}$ yields a duality between $\mathcal{A}_J$ and $\mathcal{X}$.

(2) If $\mathbf{M}$ yields a finite-level full duality between $\mathcal{A}$ and $\mathcal{X}_J$, then $F$ and $G$ set up a dual equivalence between $\mathcal{A}_J$ and $\mathcal{X}$.

The substance here is in (1): the dualisability claim. It doesn’t come from Ind- and Pro-completion arguments.
Describing pro-completions: a theorem with a long genealogy

Result is part algebraic, part categorical:

- $\text{Pro-BA}_{\text{fin}} = \text{STONE-BA}$;
- $\text{Pro-DL}_{\text{fin}} = \text{STONE-DL}$;
- $\text{Pro-SL}_{\text{fin}} = \text{STONE-SL}$ (SL = meet-semilattices with $\top$);
- $\text{Pro-}$\text{V}_{\text{fin}} = \text{STONE-}$\text{V}$, for $\text{V}$ any finitely generated variety of lattices;
- ...

But there are limits to how far this goes.

And, beyond algebras,

- $\text{Pro-POS}_{\text{fin}} \neq \text{STONE-POS}$—LHS is the strictly smaller category PRI of Priestley spaces.
Assume that $\mathcal{A} = \text{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite lattice-based algebras, and assume that $\mathcal{A}$ is a variety (or equivalently, assume that every homomorphic image of every subalgebra of each algebra in $\mathcal{M}$ is in $\mathcal{A}$). Then $\mathcal{A}$ is such that

$$\mathcal{A}_{\mathcal{J}} = \text{STONE} - \mathcal{A}.$$ 

This is not the most general theorem possible, but it gives a widely applicable result avoiding hard-to-state algebraic conditions. These same conditions imply that $\mathcal{A}_{\mathcal{T}p} = \text{Pro-}\mathcal{A}_{\text{fin}}$. 
Selected recent references concerning natural dualities I

Selected recent references concerning natural dualities II