Duality Theory: the Parts that Category Theory Can't Reach?

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Rival approaches?

Aim of talk: to compare two duality notions:
Suppose we have two concrete categories $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$, where the subscript $\mathcal{T}$ signals presence of topological structure. Concreteness (in the weak sense that there are faithful grounding functors to SET) enables us to talk about **finite objects**.

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- **Stone-type duality** (as the term is used by Johnstone in *Stone Spaces*): a dual equivalence between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ arising from a dual equivalence between $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}\,\mathcal{T}}$ (the 'finite level') by taking, respectively, Ind-completion and Pro-completion.

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- **Natural duality (full)**: dual equivalence between suitable categories $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ set up by hom-functors into $\mathbf{M}$ and $\utilde{\mathbf{M}}_{\mathcal{T}}$ with the same finite underlying set, with the unit and counit maps of the adjunction being isomorphisms, given by evaluations. Here $(\mathbf{M}, \utilde{\mathbf{M}}_{\mathcal{T}})$ acts as a **dualising object**.

Two classic examples, each both Stone-type and natural

(1) Stone duality

$$\mathcal{A} = \text{BA—Boolean algebras}$$
$$\mathfrak{X}_{\mathfrak{T}} = \text{STONE—Stone (\textit{alias} Boolean) spaces}$$

(2) A related dual equivalence:

$$\mathcal{A}_{\mathfrak{T}} = \text{STONE–BA—Boolean-topological Boolean algebras}$$
$$\mathfrak{X} = \text{SET}$$

A category isomorphic to $\mathcal{A}_{\mathfrak{T}}$ is

CABA—complete and atomic Boolean algebras

[Obvious choice of morphisms in all cases.]

Features of the preceding examples

- BA is a finitary algebraic category
- SET is (degenerately) a finitary algebraic category
- STONE is a category of topological spaces
- We have isomorphic categories:

$$\begin{cases} \text{CABA is an infinitary algebraic category,} \\ \text{STONE-BA is a topological algebraic category} \end{cases}$$

NOTE: operating in ZFC throughout—spaces, not frames; no glimpse of an Elephant here.

Generation properties

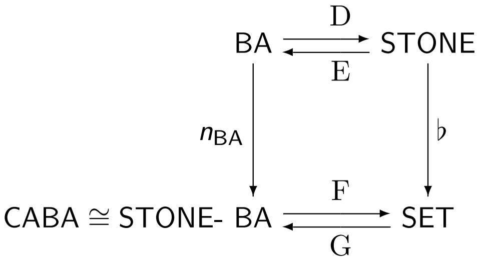
BA is expressible as

$\mathbb{HSP}(\mathbf{2})$—a variety, definable by equations;

$\mathbb{ISP}(\mathbf{2})$ —a quasivariety, definable by implications

Each of the other categories too is generated by a 2-element member, assumed to carry the discrete topology in case of STONE and STONE-BA. [More detail later.]

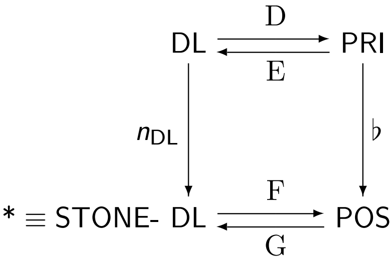
Putting the pieces together



- $^\flat$ is the obvious forgetful functor.
- The two dual equivalences (horizontal arrows) are related by ‘topology-swapping’.
- Topology-swapping originates at the level of the generating objects.
- The left-hand vertical arrow is the canonical extension functor, when regarded as being into CABA.

- Coalgebra buffs would now want to add endofunctors. . . .

The distributive lattice case, likewise



$*$ $=$ completely distributive linked bialgebraic lattices ($\equiv$ up-set lattices)

This time, one of the categories without topology is

- a category of algebras (DL)

and the other is

- a category of structures (POS).

[This feature doesn't show up with Stone duality.]

Duality theory: parts that category theory does reach [informal discussion]

Assume we have a category $\mathfrak{C}$ which is concrete in the strong sense that

- there is a faithful grounding functor $U \colon \mathfrak{C} \to \mathrm{SET}$;
- U is **representable** in the sense that there exists $C_0 \in \mathfrak{C}$ such that $U \cong \mathfrak{C}(C_0, -)$.

Then C_0 acts as the 'free object on one generator'.

Dualising objects

Now assume $\mathfrak{C}$ and $\mathcal{Y}$ are concrete, with grounding functors U and V, and that there are contravariant functors $S\colon \mathcal{Y} \to \mathfrak{C}$ and $T\colon \mathfrak{C} \to \mathcal{Y}$ yielding a dual adjunction $\angle S, T, e, \epsilon\rangle$:

$$\mathfrak{C}(C, SY) \cong \mathcal{Y}(Y, TC).$$

This set-up **guarantees** the existence of a **dualising object**, given by (SY_0, TC_0) which (modulo a bijection) have the same underlying set. Refer to SY_0 and TC_0 as **alter egos** of each other.
With the grounding functors now suppressed
The unit and counit maps of the adjunction are necessarily given by evaluation:

$$\begin{aligned} e_C &\colon C \to STC, & e_C(c)&\colon y \mapsto y(c),\\ \epsilon_Y &\colon Y \to TSY, & \epsilon_Y(y)&\colon c \mapsto c(y). \end{aligned}$$

Levels of good behaviour of a dual adjunction with associated dualising object

(1) just a dual adjunction;
(2) a dual adjunction, with unit and counit maps being 'embeddings';
(3) as in (2) and with each e_C an isomorphism;
(4) a dual equivalence, with each e_C and each ϵ_Y an isomorphism.

How far one can get will depend on how well the alter egos making up the dualising object operate in tandem.

Given that (1) holds, (2) relies on a compatibility requirement.

(3) and (4) are much harder to address in full generality. Categorical methodology exists—but not pursued here.

Narrowing the focus: a more algebraic perspective

A motivation for studying lattice-based or poset-based algebras often comes from **logic**—modelling syntactic specifications algebraically.

Motivation for studying dualities for these comes from

- the power of relational semantics (Kripke frames), eg for modal logics;
- coalgebraic modelling, of transition systems, eg.

We shall restrict to **finite** M—because we want to talk about profiniteness.

So what sort of categories should we encompass? For sure, certain categories whose models are

- algebras, often lattice-based.

But more generally we shall want to

- admit **partial**(ly defined) algebraic operations.

Narrowing the focus, continued

Within this framework, we'd want to focus on

- **varieties**—specified by equations, or
- **quasivarieties**—specified by implications.

These are, respectively, classes closed under $\mathbb{H}$, $\mathbb{S}$, $\mathbb{P}$, or $\mathbb{I}$, $\mathbb{S}$, $\mathbb{P}$.

Which should we go for: $\mathbb{H}$ or $\mathbb{I}$?

To be specific, consider a single algebra $\mathbf{M}$ and compare $\mathbb{HSP}(\mathbf{M})$ and $\mathbb{ISP}(\mathbf{M})$.

- $\mathbf{A} \in \mathcal{A} := \mathbb{ISP}(\mathbf{M})$ iff morphisms from $\mathbf{A}$ into $\mathbf{M}$ separate points.

In general, $\mathbb{ISP}(\mathbf{M}) \subsetneq \mathbb{HSP}(\mathbf{M})$.

The lattice-based algebra case, with $\mathbf{M}$ finite

- $\mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathcal{M})$, where $\mathcal{M}$ is a finite set of finite algebras (actually can take $\mathcal{M} = \mathbb{HS}(\mathbf{M})$)—by Jónsson's Lemma.

So $\mathbb{HSP}$ case reduces to $\mathbb{ISP}$ case, at the small cost of working in a multisorted set-up.

Key duality results extend, or can be expected to extend, to the multisorted case.

From algebras to structures

We want to be even-handed in our use of categories, and to allow for structures—not just algebras.

Let M be a finite non-empty set We shall call $\mathbf{M}$ a **structure** if $\mathbf{M} = (M; G, H, R)$, where

- G is a set of (total) operations;
- H is a set of partial operations;
- R is a set of relations,

all of finite arity.

We call $\mathbf{M}$ a **total structure** if $H = \varnothing$.

The characterisations of of varieties and quasivarieties extend, mutatis mutandis, to the structures we consider.

The corresponding topologised structure, $\mathbf{M}_{\mathcal{T}}$, will always carry the discrete topology.

A pair of structures, four categories

Given any two structures, $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ (on the same underlying set, but no compatibility yet) we have four categories:

$$\mathcal{A} := \mathbb{ISP}(\mathbf{M}), \qquad \mathfrak{X} := \mathbb{IS}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}}),$$
$$\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_{\mathrm{c}}\mathbb{P}(\mathbf{M}_{\mathcal{T}}), \qquad \mathfrak{X}_{\mathcal{T}} := \mathbb{IS}^0_{\mathrm{c}}\mathbb{P}^+(\underset{\sim}{\mathbf{M}}_{\mathcal{T}}).$$

The first pair are categories of structures, the second pair are categories of Stone-structures of type $\underset{\sim}{\mathbf{M}}$.

Technical note

$\mathbb{P}$ *allows empty indexed products, yielding the total one-element structure;* $\mathbb{P}^+$ *doesn't. Operator* $\mathbb{S}$ *excludes the empty structure while* $\mathbb{S}^0$ *includes it, when the type does not include nullary operations.*

A dual equivalence on the cheap: Hofmann–Mislove–Stralka duality for semilattices

$\mathcal{S} = \mathbb{ISP}(\mathbf{2})$	$\mathsf{SL} = \wedge, 1 -$ semilattices
$\mathcal{Z} = \mathcal{S}_{\mathcal{T}} = \mathbb{IS}_c\mathbb{P}(\mathbf{2}_{\mathcal{T}})$	STONE-SL

(Here we have two categories rather than four.)

On finite, discretely topologised, objects the topology does no work, so

$$\mathcal{Z}_{\mathsf{fin}} \text{ ``is'' } \mathcal{S}_{\mathsf{fin}}.$$

With this identification the evaluation maps are just identities. SO we have a dual equivalence at the finite level.

HMS duality, continued

Easy:

$\mathcal{S}$ built from $\mathcal{S}_{\mathsf{fin}}$ by taking directed (cofiltered) limits,

$\mathcal{Z}$ built from $\mathcal{Z}_{\mathsf{fin}}$ by taking projective limits (filtered colimits).

and the limits/colimits are preserved by the functors.

$$\begin{array}{ccc}
\mathcal{S} & \underset{\mathrm{E}}{\overset{\mathrm{D}}{\rightleftarrows}} & \mathcal{Z} \\
\mathsf{Ind}\uparrow & & \uparrow\mathsf{Pro} \\
\mathcal{S}_{\mathsf{fin}} & \rightleftarrows & \mathcal{Z}_{\mathsf{fin}}
\end{array}$$

Back a step . . .

Profiniteness, concretely

Profinite objects, are, loosely, those which are built from finite ones by means of filtered colimits.

Let $\mathfrak{C}$ be a concrete category and $\mathfrak{C}_{\mathsf{fin}}$ the full subcategory of finite objects. We may ask whether

$$\mathfrak{C} = \text{Pro-}\mathfrak{C}_{\mathsf{fin}},$$

the categorical pro-completion of $\mathfrak{C}_{\mathsf{fin}}$. ($\mathfrak{C}$ a finitely accessible category.)

There is a mismatch between concretely-realised profinite limits and abstract ones (viewed as diagrams). This is why profinite widgets (groups, BAs, ...) are built from $\mathfrak{C}_{\mathsf{fin}\mathfrak{T}}$ rather than $\mathfrak{C}_{\mathsf{fin}}$ and so are treated as topological algebras. Henceforth keep the $_{\mathfrak{T}}$ tacit since we always use the discrete topology on finite objects.

The dual notion: Ind-completion

This behaves well, for many classes of algebras.
In any variety $\mathcal{V}$ of algebras which is locally finite (ie, finitely generated algebras are finite), then every every $\mathbf{A} \in \mathcal{V}$ is the directed union of its finite subalgebras. Hence

$$\mathcal{V} = \mathsf{Ind}\text{-}\mathcal{V}_{\mathsf{fin}}.$$

This extends, easily, to a corresponding local finiteness result for any $\mathbb{ISP}(\underset{\sim}{\mathbf{M}})$ where $\underset{\sim}{\mathbf{M}}$ is a finite structure.

HMS duality: lifting from the finite level

- At the finite level, the duality works trivially.
- $\mathcal{S}$ is the Ind-completion of $\mathcal{S}_{\mathsf{fin}}$.
- $\mathfrak{Z}$ is the Pro-completion of $\mathfrak{Z}_{\mathsf{fin}}$.
- The finite-level dual equivalence lifts to that between $\mathcal{S}$ and $\mathfrak{Z}$, by categorically routine arguments.

This was how the HMS duality was first proved.

What is NOT trivial is the recasting of $\mathfrak{Z}$ in (infinitary) algebraic terms—this is the Fundamental Theorem on Compact Zero-Semilattices..

Compatibility of a pair of structures

- $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ are compatible structures on the same finite set M (operations, relations and partial operations allowed); No presumption that $\mathbf{M}$ is "algebraic" and $\underset{\sim}{\mathbf{M}}$ "relational".
- Compatibility: the structure of $\underset{\sim}{\mathbf{M}}$ is preserved by the operations and partial operations of $\mathbf{M}$ and the relations are substructures.

This notion is symmetric.

Paired adjunctions.

Let

- $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ and $\mathcal{X}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_{\mathcal{T}})$.
  $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}})$ and $\mathcal{X} := \mathbb{ISP}(\underset{\sim}{\mathbf{M}})$.

We have a dual adjunction between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$ set up by the hom-functors

$$\mathrm{D}\colon \mathcal{A} \to \mathcal{X}_{\mathcal{T}}, \qquad \begin{cases} \mathrm{D}(\mathbf{A}) = \mathcal{A}(\mathbf{A}, \mathbf{M}) \\ \mathrm{D}(f) = - \circ f \end{cases}$$

$$\mathrm{E}\colon \mathcal{X}_{\mathcal{T}} \to \mathcal{A}, \qquad \begin{cases} \mathrm{E}(\mathbf{X}) = \mathcal{X} + \mathcal{T}(\mathbf{X}, \underset{\sim}{\mathbf{M}}_{\mathcal{T}}) \\ \mathrm{E}(\phi) = - \circ \phi \end{cases}$$

and likewise for F and G.
Compatibility ensures that all the unit and counit maps are given by evaluations and are embeddings.

Levels of good behaviour

Focus first on $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ and the dual adjunction $\langle D, E, e, \epsilon \rangle$.
Treat $\mathbf{M}$ as given, $\underset{\sim}{\mathbf{M}}$ a candidate alter ego. Can $\mathbf{M}, \underset{\sim}{\mathbf{M}}$ be chosen so that

- $\underset{\sim}{\mathbf{M}}$ yields a **duality** (e_A is an isomorphism for all $\mathbf{A} \in \mathcal{A}$):
- $\underset{\sim}{\mathbf{M}}$ yields a **full duality** (ie $\langle D, E, e, \epsilon \rangle$ sets up a dual equivalence)
- $\underset{\sim}{\mathbf{M}}$ yields a **[full] duality** between $\mathcal{A}_{\text{fin}}$ and $\mathfrak{X}_{\text{fin}\mathcal{T}}$ ($\underset{\sim}{\mathbf{M}}$ yields a [full] duality, at the finite level.

Independence results: quasivarieties of algebras

- Given $\mathbf{A}$, dualisability and full dualisability (and strong dualisability too) are independent of the choice of the generator $\mathbf{M}$.

Cautionary example (Hyndman/Willard)
In general $\mathbf{A}$ can be dualisable but not fully dualisable: example provided by an $\mathbf{M}$ which is a 3-element chain with 2 unary operations,

The finite level, in the natural duality set-up

We want, at the very least, a duality at the finite level.

- This may be trivial (as for SL) or easy (as for BA or DL).

- It may be possible only with an alter ego of infinite type. Example:

  $$\mathbf{M} = (\{0,1\}, \to)$$

  ($\mathcal{A} = \mathbb{HSP}(\mathbf{M}) = \mathbb{ISP}(\mathbf{M})$ is the variety of **implication algebras**).

Finitely generated lattice-based (quasi)varieties

Let $\mathcal{A} = \mathbb{ISP}(\mathbf{M})$, where $\mathbf{M}$ is a finite lattice with (maybe) additional operations.

- Then there exists $\underset{\sim}{\mathbf{M}}$ yielding a duality, with $\underset{\sim}{\mathbf{M}}$ a total structure.
- There exists $\underset{\sim}{\mathbf{M}}$ yielding a full duality (in fact a strong duality), but $\underset{\sim}{\mathbf{M}}$ cannot always to chosen to be a total structure: in general partial homomorphisms of arity $\leqslant 2$ need to be included in $\underset{\sim}{\mathbf{M}}$.

There is a corresponding result, in terms of multisorted dualities, for a variety $\mathcal{A} = \mathbb{HSP}(\mathbf{M})$.

Lifting from the finite level, via Pro- and Ind-completion

Proposition Take M finite, and $\mathbf{M}, \underset{\sim}{\mathbf{M}}$ compatible structures,

$$\mathcal{A} := \mathbb{ISP}(\mathbf{M}) \text{ and } \mathfrak{X}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{\mathbf{M}}_{\mathcal{T}}).$$

Assume that $\underset{\sim}{\mathbf{M}}$ yields a duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$. Then $\underset{\sim}{\mathbf{M}}$ yields a full duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ if and only if

- $\underset{\sim}{\mathbf{M}}$ yields a full duality at the finite level, and
- $\mathfrak{X}_{\mathcal{T}} = \text{Pro} - \mathfrak{X}_{\text{fin}}$.

This is a purely categorical result.

SO we're all set–or are we?
Must $\mathfrak{X}_{\mathcal{T}} = \text{Pro} - \mathfrak{X}_{\text{fin}}$??

Lifting from the finite level: compactness

The Duality Compactness Theorem
Assume M finite and that $(\mathbf{M}, \underset{\sim}{\mathbf{M}})$ yields a duality at the finite level and $\underset{\sim}{\mathbf{M}}$ is of **finite type**. Then $\mathbf{M}$ yields a duality:

- for $\mathbf{M}$ an algebra (Zadori, Willard);
- for $\mathbf{M}$ a structure (D. Hofmann, 2002—established in context of finitary limit sketches).

We'd like to insert 'full' in assumption and conclusion. BUT WE CAN'T!

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Cautionary examples (Davey/Haviar/Willard et al.)

- for **M** the 3-element chain, there exists $\underset{\sim}{\mathbf{M}}$ of finite type which
  - ⋆ dualises DL,
  - ⋆ fully dualises DL at the finite level
  - ⋆ but which does NOT lift to a full duality for DL.
- Same conclusion with **M** replaced by **any** finite non-Boolean $\mathbf{M} \in$ DL.

Lifting from the finite level: better news

A Weak Full Duality Compactness Theorem (Davey, 2006) A full duality at the finite level based on $(\mathbf{M}, \underset{\sim}{\mathbf{M}})$ does lift to a full duality if

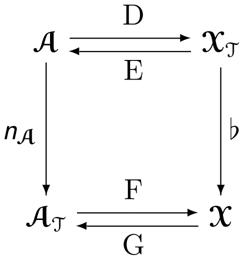
- $\underset{\sim}{\mathbf{M}}$ is of finite type, and
- $\underset{\sim}{\mathbf{M}}$ is a **total structure** (no partial operations).

and then the dual equivalence between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$ does arise from Ind- and Pro-completion.

Catch 22: We often get full dualisability only at the expense pf adding partial operations, and doing this may mean the duality is NOT given by Ind- and Pro-completion,

Dualities in partnership, stage 1

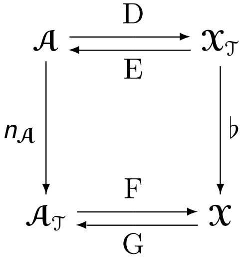
Back to the four category set-up.



with

- $\mathcal{A} := \mathbb{ISP}(\mathbf{M})$ and $\mathcal{X} := \mathbb{IS}^0\mathbb{P}^+(\underset{\sim}{\mathbf{M}})$;
  $\mathcal{A}_{\mathcal{T}} := \mathbb{IS}_c\mathbb{P}(\mathbf{M}_{\mathcal{T}})$ and $\mathcal{X} := \mathbb{ISP}(\underset{\sim}{\mathbf{M}})$.
- $\mathbf{M}$ and $\underset{\sim}{\mathbf{M}}$ compatible structures on the same underlying finite set,

Dualities in partnership, stage 2



Paired Adjunctions Theorem The following are equivalent:

(i) the outer square commutes, ie, $n_{\mathcal{A}}(\mathbf{A}) = \mathrm{G}(\mathrm{D}(\mathbf{A})^\flat)$, for all $\mathbf{A} \in \mathcal{A}$;

(ii) $n_{\mathcal{A}}(\mathbf{A})$ consists of all maps $\alpha\colon \mathcal{A}(\mathbf{A}, \mathbf{M}) \to M$ that preserve the structure on $\underset{\sim}{\mathbf{M}}$, for all $\mathbf{A} \in \mathcal{A}$;

(iii) $\underset{\sim}{\mathbf{M}}$ yields a duality between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$ at the finite level.

Moreover, if $\underset{\sim}{\mathbf{M}}$ is of finite type, then (i)–(iii) are equivalent to

(iv) $\underset{\sim}{\mathbf{M}}_{\mathcal{T}}$ yields a duality between $\mathcal{A}$ and $\mathcal{X}_{\mathcal{T}}$.

Dualities in partnership, stage 3

The TopSwap Theorem (Davey/Haviar/Priestley, 2011)
Assume **M** a total structure of finite type.

(1) If $\underset{\sim}{\mathbf{M}}$ yields a finite-level duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$, then **M** yields a duality between $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}$.
(2) If $\underset{\sim}{\mathbf{M}}$ yields a finite-level full duality between $\mathcal{A}$ and $\mathfrak{X}_{\mathcal{T}}$, then F and G set up a dual equivalence between $\mathcal{A}_{\mathcal{T}}$ and $\mathfrak{X}$.

The substance here is in (1): the dualisability claim. It doesn't come from Ind- and Pro-completion arguments.

Describing pro-completions: a theorem with a long genealogy

Result is part algebraic, part categorical:

- $\mathsf{Pro}\text{-}\mathsf{BA}_{\mathsf{fin}} = \mathsf{STONE}\text{-}\mathsf{BA}$;
- $\mathsf{Pro}\text{-}\mathsf{DL}_{\mathsf{fin}} = \mathsf{STONE}\text{-}\mathsf{DL}$;
- $\mathsf{Pro}\text{-}\mathsf{SL}_{\mathsf{fin}} = \mathsf{STONE}\text{-}\mathsf{SL}$ ($\mathsf{SL} =$ meet-semilattices with $\top$);
- $\mathsf{Pro}\text{-}\mathcal{V}_{\mathsf{fin}} = \mathsf{STONE}\text{-}\mathcal{V}$, for $\mathcal{V}$ any **finitely generated** variety of lattices;
- ....

But there are limits to how far this goes.

And, beyond algebras,

- $\mathsf{Pro}\text{-}\mathsf{POS}_{\mathsf{fin}} \neq \mathsf{STONE}\text{-}\mathsf{POS}$—LHS is the strictly smaller category PRI of Priestley spaces.

Describing the category $\mathcal{A}_{\mathfrak{T}}$

Assume that $\mathcal{A} = \mathbb{ISP}(\mathfrak{M})$, where $\mathfrak{M}$ is a finite set of finite lattice-based algebras, and assume that $\mathcal{A}$ is a variety (or equivalently, assume that every homomorphic image of every subalgebra of each algebra in $\mathfrak{M}$ is in $\mathcal{A}$). Then $\mathcal{A}$ is such that

$$\mathcal{A}_{\mathfrak{T}} = STONE - \mathcal{A}.$$

This is not the most general theorem possible, but it gives a widely applicable result avoiding hard-to-state algebraic conditions. These same conditions imply that $\mathcal{A}_{Tp} = \text{Pro-}\mathcal{A}_{\text{fin}}$.

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