The Chromatic Number of Dense Random Graphs

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**Definitions**

**Chromatic number** $\chi(G)$

Minimum number of colours required to colour the vertices of a graph $G$ so that no two adjacent vertices are coloured the same.
$n = 7$

$\chi(G) = 3$

$\alpha(G) = 3$

Independence number $\alpha(G)$:
Size of the largest independent vertex set (= set without edges).

Every colour class is an independent vertex set, so if there are $n$ vertices,

$$\chi(G) \geq \frac{n}{\alpha(G)}$$
Problem:
Determine the chromatic number of $G_{n,p}$, $0 < p < 1$ constant.

**Independence number:**
Let $0 < p < 1$ constant, $q = 1 - p$, $b = 1/(1 - p)$ and

$$\alpha_0 = 2 \log_b n - 2 \log_b \log_b n + 2 \log_b (e/2) + 1.$$ 

Then whp, $\alpha(G_{n,p}) = \lfloor \alpha_0 + o(1) \rfloor$.

**Theorem (Grimmett, McDiarmid 1975):**
Let $p \in (0, 1)$ constant, $b = 1/(1 - p)$, then whp,

$$(1 + o(1)) \frac{n}{2 \log_b n} \leq \chi(G_{n,p}) \leq (1 + o(1)) \frac{n}{\log_b n}.$$
Theorem (Bollobás 1987)

Let \( p \in (0, 1) \) constant, \( b = 1/(1 - p) \), then whp,

\[
\chi(G_{n,p}) \sim \frac{n}{2 \log_b n}.
\]

- **Lower bound:** \( \chi(G) \geq n/\alpha(G) \)
- **Upper bound:**
  - Show that every sufficiently large set of vertices contains an independent set of size \( \sim 2 \log_b n \)
  - select colour classes until few vertices remain
  - colour remaining vertices individually

- **Tool:** Martingale techniques / Azuma-Hoeffding inequality
There have been various refinements and improvements of Bollobás’ approach:

**Theorem (McDiarmid 1990)**

Let $p \in (0, 1)$ constant, then whp,

$$\chi(G_{n,p}) = \frac{n}{\alpha_0 + O(1)}.$$

The average colour class size $\frac{n}{\chi(G_{n,p})}$ is within constant distance of $\alpha(G)$. 
Previously best known bounds

**Theorem**

Let $p$ constant, $b = 1/(1 - p)$ and $\alpha_0$ as before, then whp,

$$\frac{n}{\alpha_0 - \frac{2}{\log b} - 1 + o(1)} \leq \chi(G_{n,p}) \leq \frac{n}{\alpha_0 - \frac{2}{\log b} - 2 + o(1)}.$$

- Upper bound: Fountoulakis, Kang, McDiarmid 2010
- Lower bound: Panagiotou, Steger 2009

\[
\frac{n}{\chi(G_{n,p})}
\]

is whp contained in an *explicit* interval of length $1 + o(1)$.

**Problem** (Kang, McDiarmid 2014): Find such an interval of length $o(1)$!
First moment threshold

\[
\gamma := \alpha_0 - \frac{2}{\log b} - 1 = 2 \log_b n - 2 \log_b \log_b n - 2 \log_b 2
\]

The expected number of *unordered* colourings turns infinite at about

\[
\frac{n}{\gamma + o(1)} \text{ colours.}
\]

- Fewer colours: Expected number of unordered colourings \( \to 0 \)
- More colours: Expected number of unordered colourings \( \to \infty \)

We know:

\[
\frac{n}{\gamma + o(1)} \leq \chi(G_{n,p}) \leq \frac{n}{\gamma - 1 + o(1)} \text{ whp}
\]

**Question:** Is \( \chi(G_{n,p}) = \frac{n}{\gamma + o(1)} \) ?
Theorem (H. 2016)

Fix \( p \in (0, 1) \), let \( b = \frac{1}{1-p} \), \( \gamma = 2 \log_b n - 2 \log_b \log_b n - 2 \log_b 2 \), and \( \Delta = \gamma - \lfloor \gamma \rfloor \). Then whp,

\[
\chi(G_{n,p}) = \frac{n}{\gamma - x_0 + o(1)},
\]

where \( x_0 \) is the smallest nonnegative solution of

\[
(1 - \Delta + x) \log_b (1 - \Delta + x) + \frac{(\Delta - x)(1 - \Delta)}{2} \leq 0. \tag{*}
\]

Determines average colour class size \( \frac{n}{\chi(G)} \) up to \( o(1) \).

- For \( p \leq 1 - 1/e^2 \approx 0.86 \), 0 is smallest solution of \((*)\).
- For \( p > 1 - 1/e^2 \), smallest solutions dense in \([0, 1 - \frac{2}{\log b}]\).
Corollary

- If $p \leq 1 - 1/e^2 \approx 0.86$, then whp $\chi(G_{n,p}) = \frac{n}{\gamma + o(1)}$.
- If $p > 1 - 1/e^2$, then whp
  \[
  \frac{n}{\gamma + o(1)} \leq \chi(G_{n,p}) \leq \frac{n}{\gamma - 1 + \frac{2}{\log b} + o(1)}.
  \]

What’s the difference between $p \leq 1 - 1/e^2$ and $p > 1 - 1/e^2$?
What happens when $p > 1 - 1/e^2 \approx 0.86$?

Recall that $\alpha(G_{n,p}) = \left\lceil \alpha_0 + o(1) \right\rceil$ whp, and that

$$\alpha_0 = \gamma + \frac{2}{\log b} + 1$$

<2 iff $p > 1 - 1/e^2$ (as $b = 1/(1 - p)$)

$\Rightarrow$ Just one or two integers between $\gamma$ and $\alpha_0$

**Problem:**
The average colour class size $\gamma$ is getting too close to the independence number $\alpha$. 
Two cases for $p > 1 - 1/e^2$

Case 1: **One integer between $\gamma$ and $\alpha_0$:**

- Few independent sets of size $\lceil \gamma \rceil = \alpha$ in $G_{n,p}$
- Almost all colour classes of size $\leq \lfloor \gamma \rfloor$.

$$
\chi(G_{n,p}) \geq \frac{n}{\lceil \gamma \rceil + o(1)} = \frac{n}{\gamma - \Delta + o(1)}
$$

$x_0 = \Delta$
Two cases for $p > 1 - 1/e^2$

Case 1: Two integers between $\gamma$ and $\alpha_0$

- How many colour classes of size $\lceil \gamma \rceil$ are needed for average colour class size $\gamma - x$? Say $m_x$.
- Need a partial colouring with $m_x$ colour classes of size $\lceil \gamma \rceil$.
- Expected number of (unordered) partial colourings:

$$-n\left( (1-\Delta+x) \log_b (1-\Delta+x) + \frac{1}{2} (1-\Delta)(\Delta-x) \right)$$

$$\approx b$$

Condition (*) in Theorem: $\leq 0$
Fix $p \in (0, 1)$, let $b = \frac{1}{1-p}$, $\gamma = 2 \log_b n - 2 \log_b \log_b n - 2 \log_b 2$, and $\Delta = \gamma - \lfloor \gamma \rfloor$. Then whp,

$$\chi(G_{n,p}) = \frac{n}{\gamma - x_0 + o(1)},$$

where $x_0$ is the smallest nonnegative solution of

$$(1 - \Delta + x) \log_b (1 - \Delta + x) + \frac{(\Delta - x)(1 - \Delta)}{2} \leq 0. \quad (\ast)$$

**Two different first moment thresholds:**

- $p \leq 1 - 1/e^2$: Expected number of (unordered) complete colourings
- $p > 1 - 1/e^2$: Expected number of (unordered) partial colourings with just the very large colour classes
The Second Moment Method

Want: Colouring with \( k = \frac{n}{\gamma - x_0 - \epsilon} \) colours.

**Paley-Zygmund inequality**

For a random variable \( Z \geq 0 \),

\[
P(Z > 0) \geq \frac{\mathbb{E}[Z]^2}{\mathbb{E}[Z^2]}
\]

In an ideal world:

\( Z = \text{number of } k\text{-colourings} \)

\[
\frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \to 1
\]

We can get away with:

\( Z = \text{number of equitable } k\text{-colourings} \)

\[
\frac{\mathbb{E}[Z^2]}{\mathbb{E}[Z]^2} \leq \exp \left( \frac{n}{\log^7 n} \right), \text{ so}
\]

\[
P(Z > 0) \geq \exp \left( -\frac{n}{\log^7 n} \right)
\]
The Goal

**Ordered equipartition:** Part sizes that vary by at most 1

\[ Z = \text{number of ordered } k\text{-equipartitions that induce valid colourings} \]

Want: \( \mathbb{E}[Z^2]/\mathbb{E}[Z]^2 \leq \exp \left( \frac{n}{\log^7 n} \right) \)

\( \mathbb{E}[Z] \): easy to analyse

\[ \mathbb{E}[Z^2] = \sum_{\pi_1, \pi_2 \text{ equitable } \atop \text{k-partitions}} \mathbb{P}(\text{both } \pi_1 \text{ and } \pi_2 \text{ induce proper colourings}) \]

\[ \text{depends on how similar } \pi_1 \text{ and } \pi_2 \text{ are!} \]

\( d(\pi_1, \pi_2) \): number of shared forbidden edges

→ **Main challenge:** Count pairs \( \pi_1, \pi_2 \) with \( d(\pi_1, \pi_2) = d \)
Three ranges of overlap

**Overlap block:** Intersection between a part in $\pi_1$ and a part in $\pi_2$

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<th>I. Typical case</th>
<th>II. Intermediate range</th>
<th>III. High overlap</th>
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<tbody>
<tr>
<td>mostly small overlap blocks</td>
<td>some large, many small overlap blocks</td>
<td>almost all vertices in large overlap blocks</td>
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**Contribution:**

- $\exp\left(\frac{n}{\log^8 n}\right)$ requires consequence of condition $(\ast)$
- $o(1)$ requires condition $(\ast)$
- $o(1)$ requires large expectation $\mathbb{E}[Z]$
I. Typical case

Overlap matrix $M = (M_{ij})$:

$M_{ij} = \#$ vertices in part $i$ of $\pi_1$
and part $j$ of $\pi_2$

Most entries 0 or 1

Idea:
- Place entries 2, 3, ... in the matrix first
- Rest is 0–1 matrix with row and columns sums about $\frac{n}{k}$

Tool:
Asymptotic number of 0-1 matrices with prescribed row and column sums given by theorem by McKay 1984.
II. Intermediate degree of overlap

Fix $\pi_1$ and generate $\pi_2$:

- Subdivide into overlap blocks and singletons
- Sort blocks and singletons into $k$ parts

Large overlap blocks decrease number of choices!
Fix $\pi_1$ and generate $\pi_2$:
- Subdivide into overlap blocks and singletons
- Sort blocks and singletons into $k$ parts

Large overlap blocks decrease number of choices!

$\rho$: Proportion of vertices in the overlap blocks (of size $\geq 2$).

$$
\begin{aligned}
&(1 - \rho)^{1-\rho}n b^{(1-\Delta)\rho n/2} = b^n((1-\rho)\log_b(1-\rho)+(1-\Delta)\rho/2) \\
\text{restricted choice if blocks large} &\quad \text{vertices in } \left\lceil \frac{n}{k} \right\rceil \text{-blocks}
\end{aligned}
$$

Condition ($\ast$): exponent $\leq 0$
III. High overlap

- Mostly large parts that are permuted. \( \text{Contribution} \approx \frac{1}{\mathbb{E}[Z]} \)
- Exceptional vertices are permuted amongst themselves. \( \rightarrow \) constant factor
- Vertices jump from larger to smaller parts \( \rightarrow \) bound by \( 2^k \)

**Overall:** \( O\left(\frac{2^k}{\mathbb{E}[Z]}\right) \rightarrow 0 \)
Is this the final word?

Bounds not yet optimal:

- Gap of size $\frac{n \log \log n}{\log^3 n}$
- Shamir, Spencer 1987: $\chi(G_{n,p})$ concentrated on an interval of size about $\sqrt{n}$ (or smaller)
- Gap unavoidable if upper bound comes from equitable colourings

Concentration?

- Wide open for $p$ constant
- Sparse random graphs: Two-point concentration
- Both any non-concentration result or concentration on $n^{1/2-\epsilon}$ values even on a subsequence of the integers would be very interesting for constant $p$. 
**Variant: Equitable chromatic number**

**Equitable colouring:** Colour classes as equal in size as possible.

**Equitable chromatic number** $\chi_e(G)$: Minimum number of colours.

Variant of proof gives:

$$\frac{n}{\gamma - x_0 + o(1)} \leq \chi_e(G_{n,p}) \leq \frac{n}{\gamma - \Delta + o(1)}.$$

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**Theorem (H. 2016+)**

Let $p < 1 - 1/e^2$ constant. There is a subsequence $(n_i)_{i \geq 1}$ of the integers such that if $G \sim G_{n_i,m_i}$ with $m_i = \lfloor p\binom{n_i}{2} \rfloor$, then whp

$$\chi_e(G) = \frac{n}{i}.$$
Thank you!