Continuum Mechanics Exercises

1. **Strain rates** For each of the following velocity fields $\mathbf{u}(x, y, z)$, calculate the divergence $\nabla \cdot \mathbf{u}$ and the rate of strain tensor $\dot{\varepsilon}_{ij}$, and draw some example streamlines of the flow for $0 < z < H$. Suggest where on an ice sheet or glacier might each type of velocity field be found.

$$\text{(i) } \mathbf{u} = \begin{pmatrix} U \\ 0 \\ 0 \end{pmatrix}, \qquad \text{(ii) } \mathbf{u} = \lambda \begin{pmatrix} H^4 - (H-z)^4 \\ 0 \\ 0 \end{pmatrix},$$

$$\text{(iii) } \mathbf{u} = \frac{a}{H} \begin{pmatrix} \frac{1}{2}x \\ \frac{1}{2}y \\ -z \end{pmatrix}, \qquad \text{(iv) } \mathbf{u} = \begin{pmatrix} U \\ 0 \\ -\alpha \exp\left(\frac{z-H}{d}\right) \end{pmatrix}.$$

2. **Material derivatives** Imagine that you had attached a thermometer to the outside of your vehicle(s) while you travelled to Karthaus. Sketch the temperature it would have measured as a function of time, and label the periods where the material derivative DT/Dt (as measured by the thermometer) was dominated by the local component ($\partial T/\partial t$) and the advective component ($\mathbf{u} \cdot \nabla T$), respectively.

3. **Shallow ice approximation** The approximate form of the momentum equations for the shallow ice approximation in two dimensions is

$$0 = -\frac{\partial p}{\partial x} + \frac{\partial \tau}{\partial z}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g,$$

where $\tau = \tau_{xz}$ is the shear stress, p is the pressure, ρ is the density and g is gravity.

(i) Consider an ice sheet on a flat bed at $z = 0$, with surface at $z = h(x, t)$. What are the appropriate boundary conditions for p and τ at the surface? Hence calculate p and τ and show that they increase linearly with depth.

(ii) The xz component of Glen's flow law is approximately

$$\frac{\partial u}{\partial z} = 2A\tau^n.$$

Use this to show that the velocity profile assuming no slip at the bed is

$$u(x, z) = \frac{2A(\rho g)^n}{n+1} \left(-\frac{\partial h}{\partial x}\right)^n \left[h^{n+1} - (h-z)^{n+1}\right].$$

[You should assume that A is constant and that the surface slope is negative].

(iii) Suppose the ice thickness is given by $h = H(1 - x^2/L^2)$ where H and L are constant. Draw a sketch of the ice sheet, and label the point with (a) the highest velocity, and (b) the highest strain rate.

(iv) (Harder) Use the continuity equation $\nabla \cdot \mathbf{u} = 0$, together with the kinematic conditions at $z = 0$ and $z = h$, to derive the depth-integrated mass conservation equation

$$\frac{\partial h}{\partial t} + \frac{\partial q}{\partial x} = a,$$

where $q = \int_0^h u \, dz$ is the ice flux, and a is the net accumulation. Assuming the ice sheet in (iii) is in steady state, draw a sketch of $q(x)$ and $a(x)$, and show that the equilibrium line altitude (the surface elevation at which $a = 0$) is $\frac{2n+4}{3n+4}H$.