The Dirac–Higgs Bundle

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A thesis submitted for the degree of

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To Julie and Anton

Abstract

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The Dirac–Higgs bundle is a vector bundle with a natural connection on the moduli space of stable Higgs bundles on a compact Riemann surface. It is a vector bundle of null-spaces of a Dirac-operator coupled to stable Higgs bundles. In this thesis, we study various aspects of this bundle and its natural connection.

The Dirac–Higgs bundle is hyperholomorphic on the smooth hyperkähler moduli space of stable Higgs bundles. This property is a generalisation of the four-dimensional anti-self-duality equations to hyperkähler manifolds. One use of the Dirac–Higgs bundle is the construction of a Nahm transform for Higgs bundles. This transform produces hyperholomorphic bundles on the moduli space of rank one Higgs bundles.

The Higgs bundle moduli space is non-compact and we study the asymptotics of the connection in the Nahm transform of a Higgs bundle. We show that elements of the null-spaces concentrate at a finite number of points on the Riemann surface. This asymptotical behaviour naturally defines a frame for the Nahm transform, which is conjectured to be asymptotically unitary.

By considering only the holomorphic structure, the Nahm transform of a Higgs bundle extends to a holomorphic bundle on the natural compactification of the rank one Higgs bundle moduli space. We discuss various aspects of this extended holomorphic bundle. Most importantly, it is a sheaf extension in which the constituent sheaves and the extension class have natural interpretations in terms of the original Higgs bundle. Furthermore, the extended bundle is not fixed at the divisor at infinity; explicit examples show that it depends on the type of Riemann surface, for example.

The Dirac–Higgs bundle has a parabolic cousin. In the parabolic case the rank depends on the number of marked points and the total multiplicity of the zero weights in the parabolic structure. The moduli space of stable rank two parabolic Higgs bundles on the Riemann sphere with four marked points has complex dimension two. Furthermore, there is a combination of parabolic weights such that the Dirac–Higgs bundle is a line bundle with an instanton connection. We study the topology of this line bundle and find that the instanton does not have finite energy. As in the non-parabolic case we define a Nahm transform for parabolic Higgs bundles, and in the case of genus one Riemann surfaces use it to produce doubly-periodic instantons of finite energy.

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Contents

Introduction

The Dirac–Higgs bundle is a vector bundle with a natural connection on the moduli space of stable Higgs bundles on a compact Riemann surface of genus at least two. It is an example of a bundle of null spaces of a Dirac-operator; such bundles are often called Dirac bundles.

One of the first instances of a Dirac bundle is the ADHM construction of instantons on $\mathbb{R}^4$ [4]. In this case, the Dirac bundle provides a finite dimensional hyperkähler construction of the otherwise gauge theoretic moduli space of instantons on $\mathbb{R}^4$ [52].

In the algebraic formulation of the moduli space $\mathscr{M}$ of polystable Higgs bundles of fixed rank and degree on a compact Riemann surface of genus at least two, the vector bundle underlying the Dirac–Higgs bundle can be seen as the direct image of a universal Higgs bundle under the projection to $\mathscr{M}$. The moduli space was first constructed algebraically by Nitsure [63] using Geometric Invariant Theory, and showing that it is a coarse moduli space. Hausel [33] showed that if the rank and degree are coprime, then a universal Higgs bundle exists. When the degree is non-zero, the direct image is not concentrated in just one degree, so the Dirac–Higgs bundle is only a virtual bundle. Hausel directly used that the Dirac–Higgs bundle is a virtual bundle to prove that there are no topological L^2-harmonic forms on the moduli space of stable Higgs bundles of degree one with fixed determinant.

In Hausel's application the connection played no role. In most applications, however, the connection is of utmost importance. Following the ADHM construction of instantons Nahm [58] considered translation invariant instantons on $\mathbb{R}^4$ and obtained a correspondence between time-invariant instantons and instantons invariant by translations in three directions, i.e. an equivalence between solutions to what are now known as Nahm's equations and monopoles. The correspondence was formalised by Corrigan and Goddard [20], and later by Braam and van Baal [16] who coined the term *Nahm transform*. Today, there are a whole host of examples of Nahm transforms, all following Nahm's original recipe; see, for example the survey [48]. One of their main purposes is to find solutions to differential equations that are generally difficult to solve, by transforming solutions of 'equivalent' equations where solutions are known to exist for other reasons. For example, the ADHM construction of instantons using linear algebra data [4], or Hitchin's construction of monopoles [38] in which a spectral curve yields a solution to Nahm's equations that via the Nahm transform produces a monopole.

In the case of the self-duality equations, solutions which are translation invariant in two directions reduce to a pair consisting of a connection and a Higgs field on a vector bundle on $\mathbb{R}^2$ satisfying the Higgs bundle equations. The equations are conformally invariant and can thus be defined on a compact Riemann surface. This was first considered by Hitchin in his landmark paper [39], in which he showed (among many other things) that solutions to the Higgs bundle equations are provided by a stability condition for Higgs bundles. A Nahm transform for Higgs bundles should therefore produce solutions to equations analogous to the self-duality equations. This was first considered by Jardim [46], who constructed doubly-periodic instantons from singular Higgs bundles on a Riemann surface of genus one. Recently, Jardim's work and the subsequent work of Biquard and Jardim [10] have been greatly extended by Mochizuki [55], who gives an equivalence between doubly-periodic instantons of finite energy and wild harmonic bundles on a Riemann surface of genus one. Following Nahm's recipe, Corrigan and Goddard [20] suggest there is a Nahm transform for solutions to the Higgs bundle equations on $\mathbb{R}^2$ that produces solutions to the Higgs bundle equations on a 'dual' $\mathbb{R}^2$. Szabó [72] also considered this problem and constructed a Nahm transform between certain types of parabolic Higgs bundles on $\mathbb{P}^1$ and a 'dual' $\mathbb{P}^1$. The original conjecture of Corrigan and Goddard is still open.

The Nahm transform for ordinary Higgs bundles on compact Riemann surfaces of genus at least two was first studied by Bonsdorff [14]. Bonsdorff used the language of derived categories of coherent sheaves to assign a holomorphic vector bundle on the moduli space of degree zero, rank one Higgs bundles to every stable Higgs bundle of rank at least two and degree zero. In [15], he used twistor methods to obtain a connection on the transformed bundle. The moduli space of rank one Higgs bundles is a hyperkähler manifold, and if the curvature of a connection is of type $(1,1)$ with respect to all complex structures the pair consisting of the bundle and connection is called a *hyperholomorphic* bundle. If the hyperkähler manifold is four-dimensional, then the connection is an instanton. Bonsdorff's transform of a Higgs bundle is indeed a hyperholomorphic bundle. More generally, the moduli space of stable higher rank Higgs bundles is hyperkähler and the Dirac–Higgs bundle is also hyperholomorphic.

Hyperholomorphic bundles have come into fashion in recent years, most notably because of their role in mirror symmetry. Roughly speaking, one instance of the SYZ approach to mirror symmetry is a duality between a so-called BAA-brane in a hyperkähler manifold and a BBB-brane in a 'dual' hyperkähler manifold. The three-letter combination of a brane indicates its relationship with the symplectic and complex structures of the hyperkähler structure, with "A" standing for symplectic and "B" for holomorphic. To first approximation an A-brane is a Lagrangian submanifold with a flat vector bundle and a B-brane is a complex submanifold with a holomorphic bundle. A BBB-brane is therefore a holomorphic vector bundle on a complex submanifold for all complex structures, or equivalently a hyperkähler submanifold with a hyperholomorphic bundle.

The moduli space of Higgs bundles found its way into the world of mirror symmetry through the work of Hausel and Thaddeus [35], and Higgs bundle moduli spaces were used by Kapustin and Witten [49] to give a physical proof of the Geometric Langlands

Correspondence. Hitchin [42] has recently constructed examples of BBB-branes and their conjectured BAA-brane counterparts in which the Dirac–Higgs bundle plays the role of the hyperholomorphic bundle.

Returning to the Nahm transform, in good cases these are hyperkähler isometries between various moduli spaces. In the case of Higgs bundles, Biquard and Jardim [10] found appropriate boundary conditions on the doubly-periodic instantons to get a hyperkähler isometry between the moduli spaces of certain doubly-periodic instantons and certain singular Higgs bundles on a Riemann surface of genus one. In the case of ordinary Higgs bundles on a Riemann surface of genus at least two, Bonsdorff [14] showed that the holomorphic bundle of the Nahm transform extends to a holomorphic bundle on a natural compactification of the rank one Higgs bundle moduli space. He additionally showed that the association of this extended bundle to a stable Higgs bundle is injective. However, the essential image of Bonsdorff's transform is unknown, though it is expected to be determined by boundary conditions on the connection. One of the first corollaries to the Hitchin–Simpson Theorem (Theorem 1.1.4) is the existence of a Riemannian metric with negative constant curvature, and finding the essential image of Bonsdorff's transform would give new information about the uniformizing metrics on the Riemann surface. This is one of the reasons why this is an interesting, but also difficult, problem.

The Higgs bundles of most concern in this thesis are stable of degree zero and rank at least two. The moduli space of these is however only coarse and lacks a universal Higgs bundle. The Dirac–Higgs bundle is, strictly speaking, only defined locally. For the purpose of studying properties of the connection this is of course not an issue as connections are local objects. However, it would be interesting to understand the geometry of the Dirac–Higgs bundle where it is globally defined. This is where parabolic Higgs bundles enter the picture. Parabolic Higgs bundles often live very parallel lives to ordinary Higgs bundles, but need more complicated analytic arguments due to the singularities. The moduli space of parabolic Higgs bundles was first constructed by Konno [50], using weighted Sobolev spaces defined by Biquard [8] but otherwise following Hitchin's approach [39]. Despite the close analogies, there are differences in the parabolic story, one being that under very mild conditions on the parabolic structure, the moduli space is fine and has a universal parabolic Higgs bundle. This makes it possible to construct a globally defined Dirac–Higgs bundle when the parabolic degree is zero.

Results and overview of the thesis

This thesis revolves around the Dirac–Higgs bundle on the moduli space of stable Higgs bundles on a compact Riemann surface Σ of genus at least two with underlying topological bundle of degree zero and rank at least two. It is divided into seven chapters, the first of which contains the background material. The following is an overview of the results from each of the subsequent chapters.

In **Chapter 2** we define the Dirac–Higgs bundle by introducing the Dirac–Higgs operator and Hodge theory for Higgs bundles. Hodge theory gives an isomorphism between the analytical and holomorphic aspects of the theory, and is an important tool in the construction of the bundle. Ultimately, Hodge theory is the main ingredient of Theorem 2.6.3, which shows that the Dirac–Higgs bundle is hyperholomorphic. We consider the Dirac–Higgs equations for the trivial line bundle on $\mathbb{C}$ with monomial Higgs fields. In this case, the equations can be explicitly solved using modified Bessel functions. We find that if the monomial Higgs field has degree k, then there are k global solutions. We conjecture this to be true more generally for polynomial Higgs fields of degree k. We use the explicit calculations to give a toy model for the Dirac–Higgs connection and its curvature. If the conjecture is true, we can construct a Dirac–Higgs bundle of rank k. This Dirac–Higgs bundle can be used to construct a Nahm transform for Higgs line bundles on $\mathbb{R}^2$, as conjectured by Corrigan and Goddard [20].

In **Chapter 3** we use the Dirac–Higgs bundle to give an analytic construction of Bonsdorff's [14] transform, producing a hyperholomorphic bundle on $J \times H^0(K)$, where J is the Jacobian of Σ and K is the canonical bundle of Σ. The chapter primarily concerns the asymptotics of the Dirac–Higgs connection along rays defined by a holomorphic one-form. In the main theorem (Theorem 3.1.1), we consider square integrable solutions to the Dirac–Higgs equations. We see that these solutions concentrate around the zeros of a holomorphic one-form.

Following this, we return to the Higgs line bundles on $\mathbb{C}$, focusing on Higgs fields of degree one. In this explicit case, we observe the localisation of Theorem 3.1.1 and furthermore see that a solution converges to a delta-function as a distribution. Based on this observation, we conjecture it to be true also for Higgs line bundles on a compact Riemann surface. The remaining part of Chapter 3 is conjectural in the sense that it builds on the validity of this distributional conjecture.

We then shift focus and in **Chapter 4** consider the holomorphic aspects of the Nahm transform. Considered as a holomorphic bundle on the cotangent bundle of the Jacobian we obtain Bonsdorff's Nahm transform, which in algebraic terms may be thought of as a Fourier–Mukai transform. We are mainly concerned with the extension to a holomorphic bundle on the natural compactification. The Fourier–Mukai transform is defined by hypercohomology of a complex of sheaves. There are two spectral sequences converging to hypercohomology and Chapter 4 consists of an investigation of the properties of the Fourier–Mukai transform that can be extracted from these spectral sequences. The second hypercohomology spectral sequence recovers the transformed Higgs bundle as a sheaf extension. Generically, the constituent sheaves in the extension are locally free and are restrictions of Picard bundles determined by the vector bundle in the original Higgs bundle (Corollary 4.2.10). In this generic case, the extension class is completely determined by the Higgs field (Theorem 4.2.15). This is a way of extracting a Higgs bundle from an element in the image of Bonsdorff's transform. We formalise this using Beilinson's spectral sequence to give a different proof of Bonsdorff's injectivity result (Theorem 4.3.6). We also give a detailed analysis of the constituent sheaves in the non-generic situation and see that they are not locally free (Proposition 4.2.22).

The first hypercohomology spectral sequence shows that applying the Fourier–Mukai transform to a Higgs bundle results in a family of homogeneous vector bundles (Theorem 4.4.6). We construct a Fourier–Mukai transform based on spectral data (Theorem 4.5.5) and use the homogeneous bundle description to recover the spectral curve from the holomorphic structure of the Fourier–Mukai transform (Theorem 4.5.10).

In order to understand the boundary conditions on the connection in the Nahm transform of a Higgs bundle, it is natural to ask if the holomorphic structure of the extension to the natural compactification is fixed along the divisor at infinity. We answer this question in **Chapter 5**, in which Theorem 5.1.1 shows that on the big stratum consisting of holomorphic differentials without multiple zeros all transformed Higgs bundles are isomorphic. In Section 5.2 we consider the holomorphic structure at infinity in various examples of stable Higgs bundles. We see that the holomorphic structure not only depends on the underlying vector bundle but also, rather surprisingly, on the type of the Riemann surface. As a consequence, the holomorphic structure is not fixed on the whole divisor at infinity.

In **Chapter 6** we turn our attention to parabolic Higgs bundles. Contrary to the non-parabolic situation, only very mild conditions have to be imposed on the parabolic structure for the moduli space to have a universal parabolic Higgs bundle (Section 6.5). In Theorem 6.6.1 we construct the hyperholomorphic Dirac–Higgs bundle on the moduli space of parabolic Higgs bundles. The proof of Theorem 6.6.1 requires many technical results regarding weighted Sobolev spaces and the Fredholmness of certain elliptic operators. The centerpiece and most important theorem is a Hodge theorem for parabolic Higgs bundles (Theorem 6.3.1). The rank of the hyperholomorphic vector bundle depends on the number of parabolic points, as well as the total multiplicity of the zero-weights in the fixed parabolic structure. A minimal non-trivial example is the case of four parabolic points on $\mathbb{P}^1$ with parabolic structure such that the hyperholomorphic bundle has rank one. Throughout Section 6.7.2, we investigate the topology and use this to argue why the instanton most likely does not have finite energy (Theorem 6.7.14). Recently, the asymptotics of the L^2-metric on the moduli space of ordinary Higgs bundles has been investigated by Mazzeo, Swoboda, Weiss, and Witt [53] using so-called limiting configurations. Limiting configurations are a special type of parabolic Higgs bundles. In Section 6.8 we discuss the local shape of L^2-solutions to the Dirac–Higgs equations for limiting configurations.

In Section 6.9 we use the Hodge theorem to construct a Nahm transform for parabolic Higgs bundles. We finish the chapter by specifying to the case of genus one, and show that the Nahm transform constructs finite energy doubly-periodic instantons (Theorem 6.9.8).

Chapter 7 outlines a list of interesting problems related to the theory developed in the previous chapters.

1. Background material

In this chapter we recall general theory and basic facts that are used throughout the thesis.

We begin by giving an introduction to classical Higgs bundle theory: the foundation on which the thesis builds. The material presented here mainly follows Hitchin's original landmark paper [39], and is a rundown of the important theorems in this area which will be used in the thesis. The introduction is followed by a brief review of hypercohomology, and lastly, we recall the most important properties of direct images of sheaves and their higher derived versions.

1.1 Higgs bundles

Throughout this thesis, Σ is a compact Riemann surface and K denotes its canonical bundle. The genus g of Σ is assumed to be at least two, except in Chapter 6 discussing parabolic theory. We fix a metric in the conformal class of Σ and denote by ω the associated Kähler form.

Let $E \to \Sigma$ be a smooth complex vector bundle of degree zero. Let h be a Hermitian metric on E. We assume that (E, h) is fixed. Denote by $\mathscr{A}$ the space of unitary connections on (E, h); this is an infinite dimensional affine space modelled on $\Omega^1(\mathfrak{u}(E))$ the space of smooth 1-forms with values in $\mathfrak{u}(E)$ the bundle of skew-Hermitian endomorphisms with respect to h. Notice that $\Omega^1(\mathfrak{u}(E)) \simeq \Omega^{0,1}(\operatorname{End} E)$ the $(0,1)$-forms with values in $\operatorname{End} E$.

Definition 1.1.1. A *Higgs pair* (A, Φ) on (E, h) consists of a unitary connection A and a *Higgs field* $\Phi \in \Omega^{1,0}(\operatorname{End} E)$.

A connection on a trivial bundle on four-dimensional Euclidean space which is translation invariant in two directions can be seen as a Higgs pair on two-dimensional Euclidean space. If we in the same way consider the dimensional reduction of the four-dimensional self-duality equations, we obtain *the Higgs bundle equations* for pairs (A, Φ)

$$F_A + [\Phi, \Phi^*] = 0 \quad \text{and} \quad \bar{\partial}_A \Phi = 0 \tag{1.1}$$

where $[\Phi, \Phi^*] = \Phi\Phi^* + \Phi^*\Phi$ is the usual extension of the Lie bracket to Lie algebra valued forms. The dual Φ^* is taken with respect to the Hermitian metric h, and $\bar{\partial}_A$ is the $(0,1)$-part of the covariant derivative of the connection A. The equations are often also known as *Hitchin's self-duality equations* as they first appeared in [39].

The Hermitian metric on E defines a bundle $U(E, h)$ of unitary automorphisms of (E, h). We denote by $\mathscr{U}$ the *unitary gauge group* of sections of $U(E, h)$. The unitary gauge group acts on $\mathscr{A}$ by

$$d_A \mapsto d_{A^g} := g^{-1} \circ d_A \circ g = d_A + g^{-1} d_A g \quad \text{for all} \quad g \in \mathscr{U},$$

where d_A is the covariant derivative of a point $A \in \mathscr{A}$.

By choosing a local frame for E the covariant derivative has the form $d_A s = ds + As$ where we abuse notation and denote by A the connection matrix of the connection A. If the frame is unitary the connection matrix A is skew-Hermitian. The curvature F_A of A can be defined as $d_A \circ d_A$ where we naturally extend d_A to 1-forms with values in E. The curvature therefore transforms as

$$F_{A^g} = g^{-1} F_A g \quad \text{for all} \quad g \in \mathscr{U}.$$

The unitary gauge group $\mathscr{U}$ acts on $\Omega = \Omega^{1,0}(\operatorname{End} E)$ by conjugation $g^{-1}\Phi g$ for $g \in \mathscr{U}$. As g is unitary with respect to h the Higgs bundle equations are invariant under unitary gauge transformations. The group $\mathscr{U}$ acts thus not only on $\mathscr{A} \times \Omega$ but on the solution space to the Higgs bundle equations. The moduli space of Higgs bundles is the infinite dimensional quotient

$$\mathscr{M} = \{(A, \Phi) \in \mathscr{A} \times \Omega \mid (A, \Phi) \text{ satisfies } (1.1)\}/\mathscr{U}.$$

A Higgs pair is called *reducible* if there is a splitting of the underlying Hermitian bundle $(E, h) = (E_1, h_1) \oplus (E_2, h_2)$ which is carried to the Higgs pair $(A, \Phi) = (A_1, \Phi_1) \oplus (A_2, \Phi_2)$ such that A_i is a unitary connection on (E_i, h_i) and Φ_i is a Higgs field on E_i. A Higgs pair is *irreducible* if it is not reducible. The subset of $\mathscr{M}$ consisting of irreducible solutions is denoted $\mathscr{M}^*$.

Theorem 1.1.2. *The moduli space $\mathscr{M}^*$ of irreducible Higgs pairs satisfying the Higgs bundle equations is a smooth real manifold of dimension $4 + 4(\operatorname{rk} E)^2(g-1)$.*

The theorem was first proved by Hitchin [39] for rank two and generalised to the higher rank cases by Simpson [69].

Example 1.1.3. If (E, h) is a line bundle of topological degree zero and Hermitian metric h, then as the commutator of 1-forms vanish the Higgs bundle equations requires A to be a flat connection and Φ to be holomorphic with respect to $\bar\partial_A$, that is $\mathscr{M} \simeq \operatorname{Jac}(\Sigma) \times H^0(K)$ or by Serre duality $\mathscr{M} \simeq T^* \operatorname{Jac}(\Sigma)$.

1.1.1 Algebraic formulation

On a Riemann surface the $(0,1)$-part $\bar\partial_A$ of a covariant derivative of a connection A defines a $\bar\partial$-operator on the underlying vector bundle. If (A, Φ) is a Higgs pair satisfying the Higgs bundle equations the last of the equations requires Φ to be holomorphic with respect to the holomorphic structure defined by $\bar\partial_A$. We can in other words consider a Higgs pair as the holomorphic object $(E, \bar\partial_A, \Phi)$ where $\Phi \in H^0(\Sigma, \operatorname{End} E \otimes K)$. We call such a pair a *Higgs bundle.* We will often abuse notation and write (E, Φ) with E denoting a holomorphic vector bundle.

Let $\mathscr{C}$ denote the space of $\bar{\partial}$-operators on E. This space is an infinite dimensional affine space modelled on $\Omega^{0,1}(\operatorname{End} E)$. In a local frame of E, a $\bar{\partial}$-operator has the form $\bar{\partial}_A s = \bar{\partial} s + \beta s$ where $\beta \in \Omega^{0,1}(\operatorname{End} E)$. The frame is holomorphic if $\beta = 0$.

The spaces $\mathscr{A}$ and $\mathscr{C}$ are isomorphic by associating to a unitary connection A the $\bar{\partial}$-operator $\bar{\partial}_A$. Given a Hermitian metric, then to any $\bar{\partial}$-operator $\bar{\partial}_\beta$ there is a unique unitary connection $A = A(h, \bar{\partial}_\beta)$ called the Chern connection such that $\bar{\partial}_A = \bar{\partial}_\beta$. The inverse map is given by associating to $\bar{\partial}_\beta$ the Chern connection $A(h, \bar{\partial}_\beta)$. In terms of local connection matrices,

$$A(h, \bar{\partial}_\beta) = \beta - \beta^* \quad \text{for} \quad \beta \in \Omega^{0,1}(\operatorname{End} E).$$

The complex gauge group $\mathscr{G}$ of sections of the bundle of complex automorphisms of E acts naturally on $\mathscr{C}$ by

$$\bar{\partial}_\beta \mapsto \bar{\partial}_{\beta^g} = g^{-1} \circ \bar{\partial}_\beta \circ g = \bar{\partial}_\beta + g^{-1} \bar{\partial}_\beta g \quad \text{for all} \quad g \in \mathscr{G}.$$

The complex gauge group $\mathscr{G}$ also acts on $\mathscr{A}$ by $A(H, \bar{\partial}_\beta)^g = A(H, \bar{\partial}_{\beta^g})$ or in terms of covariant derivatives,

$$d_{A^g} = \bar{\partial}_{A^g} + \partial_{A^g} = g^{-1} \circ \bar{\partial}_A \circ g + g^* \circ \partial_A \circ g^{*-1}.$$

The first idea to construct the moduli space of Higgs bundles would be to consider the quotient space

$$\{(\bar{\partial}_\beta, \Phi) \in \mathscr{C} \times \Omega \mid \bar{\partial}_\beta \Phi = 0\}/\mathscr{G}$$

but due to jumping phenomena this quotient is not even Hausdorff. Also, we need a condition equivalent to the first Higgs bundle equation. This condition is stability.

A Higgs bundle (E, Φ) is *stable* if all subbundles $F \subset E$ preserved by Φ, that is $\Phi(F) \subset F \otimes K$, satisfy

$$\mu(F) = \frac{\deg F}{\operatorname{rk} F} < \frac{\deg E}{\operatorname{rk} E} = \mu(E),$$

and is semistable if the inequality is not strict. The fraction $\mu(E)$ is called the *slope* of E. A Higgs bundle is *polystable* if it is a direct sum of stable Higgs bundles of the same slope.

Using the notion of polystability we define the moduli space

$$\mathscr{M}' = \{(\bar{\partial}_\beta, \Phi) \in \mathscr{C} \times \Omega \mid \bar{\partial}_\beta \Phi = 0 \quad \text{and} \quad (E, \bar{\partial}_\beta, \Phi) \quad \text{is polystable}\}/\mathscr{G}.$$

One of the main theorems of [39] and [69] is identifying the moduli spaces.

Theorem 1.1.4 (Hitchin [39]/Simpson [69]). *Let (E, Φ) be a Higgs bundle of degree zero, then (E, Φ) is polystable if and only if E admits a Hermitian metric such that the Chern connection satisfies the Higgs bundle equations* (1.1). *Furthermore, the Hermitian metric is unique up to multiplication by a positive scalar. That is there is a homeomorphism $\mathscr{M} \simeq \mathscr{M}'$ giving a homeomorphism between the irreducible and stable loci $\mathscr{M}^* \simeq \mathscr{M}'^{st}$.*

The purely analytic Theorem 1.1.2 also has an algebraic formulation. Nitsure [63] used Geometric Invariant Theory to prove the following version.

Theorem 1.1.5. *The moduli space $\mathscr{M}'$ of polystable Higgs bundles of degree zero and rank $\operatorname{rk} E$ is a quasi-projective variety of complex dimension $2 + 2(\operatorname{rk} E)^2(g-1)$ containing the stable locus $\mathscr{M}'^{st}$ as an open smooth subvariety.*

Requiring the degree of the vector bundle E to vanish is not a requirement for the above theorems to work. Any degree will do, but the Higgs bundle equations must be altered slightly. If the degree is non-zero the right hand side of the first equation is $i\mu(E)\operatorname{Id}\omega$. When the degree and rank are coprime stability and semi-stability conincide and the whole moduli space $\mathscr{M}'$ is smooth. Also, the moduli space is initially just a coarse moduli space but Hausel showed [33] that if the degree and rank are coprime the moduli space is fine.

One of the advantages of working with degree zero is that the Higgs bundle equations are conformally invariant, as opposed to the non-zero case where the presence of ω on makes the equations non-conformal.

Example 1.1.6. If E is a stable vector bundle on Σ, then for any $\Phi \in H^0(\Sigma, \operatorname{End} E \otimes K)$ the corresponding Higgs bundle (E, Φ) is stable. If $\mathscr{N}$ is the moduli space of stable bundles of degree zero and rank $\operatorname{rk} E$ then by Serre duality $T^*\mathscr{N} \simeq \mathscr{N} \times H^0(\operatorname{End} E \otimes K)$ and thus $T^*\mathscr{N} \subset \mathscr{M}$. The complement has codimension at least two [40, Proof of Proposition 4.4].

Example 1.1.7. A non-trivial example is the *canonical Higgs bundle* of rank $n \geq 2$

$$E = K^{(n-1)/2} \oplus K^{(n-3)/2} \oplus \cdots \oplus K^{-(n-3)/2} \oplus K^{-(n-1)/2}$$

with Higgs field

$$\Phi = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \quad \text{where} \quad 1 : K^{(k-1)/2} \to K^{(k-3)/2} \otimes K \simeq K^{(k-1)/2}$$

is the identity section of $\mathcal{O}$ and $K^{1/2}$ is a choice of one of the 2^{2g} square roots of K. The only Φ-invariant subbundles are $\oplus_{j=1}^{k} K^{-(n-(2j+1))/2}$ with $k = 1, \ldots, n-2$ which all have negative degree. Since $\deg(E) = 0$ the canonical Higgs bundle is stable.

Consider the rank 3 case

$$E = K \oplus \mathcal{O} \oplus K^{-1} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

The canonical Higgs bundle is of special interest since the Hermitian metric given by the Hitchin–Simpson Theorem induces a metric on the tangent bundle K^{-1}. The Higgs bundle equations for this metric reduce to the single equation

$$F_{K^{-1}} = -\omega$$

where ω is the Kähler form on Σ determined by the Hermitian structure on K^{-1} and the complex structure on Σ. If X is a local holomorphic section of the tangent bundle K^{-1} with pointwise length $h(X, X) = 1$, then the coefficient to the curvature $F_{K^{-1}}$ is

$$F_{K^{-1}}(X, JX) = -\omega(X, JX) = -h(X, X) = -1,$$

thus the metric h has constant sectional curvature -1. Here J is the complex structure on Σ. In other words, the Uniformization Theorem is a consequence of the Hitchin–Simpson Theorem.

1.1.2 Flat connections

Let (A, Φ) be an irreducible Higgs pair solving the Higgs bundle equations on a bundle E of degree zero. It is then easy to see that

$$D = d_A + \Phi + \Phi^*$$

is a flat irreducible complex connection on E.

Conversely, if D is a flat complex connection then a Hermitian metric h induces a splitting $D = d_h + \Psi$ into a unitary and a self-adjoint part. Splitting further $d_h = \partial_A + \bar{\partial}_A$ and $\Psi = \Phi + \Phi^*$ into types we can define the operators $D'' = \bar{\partial}_A + \Phi$ and $D' = \partial_A + \Phi^*$. Donaldson showed [22] (for rank two) that there is a natural one-to-one correspondence between irreducible Higgs pairs (A, Φ) satisfying the Higgs bundle equations and irreducible flat complex connections. The equivalence goes through the existence of a so-called harmonic metric: the operator D'' defined via the metric satisfy $(D'')^2 = \bar{\partial}_A(\Phi) = 0$. The proof was later generalised by Corlette [19] to higher rank and to a more general setting relating the existence of harmonic metrics for principal G-bundles with a flat semisimple complex connection, when G is a complex reductive algebraic group.

Let $\mathscr{X}$ be the infinite dimensional affine space of complex connections modelled on $\Omega^1(\Sigma, \operatorname{End} E)$, and let $\mathscr{X}_0$ be the subset of *flat* complex semisimple connections. The complex gauge group $\mathscr{G}$ acts on $\mathscr{X}$ in the usual way

$$D \mapsto D^g = g^{-1} \circ D \circ g = D + g^{-1} Dg \quad \text{for all} \quad g \in \mathscr{G}.$$

Donaldson and Corlette's theorem can be interpreted as giving a homeomorphism

$$\mathscr{X}_0/\mathscr{G} \simeq \mathscr{M}.$$

This is a generalisation of the classical theorem of Narasimhan and Seshadri [60] giving a homeomorphism between the moduli space of flat unitary connections and stable holomorphic vector bundles.

1.1.3 Hyperkähler structure

One of Hitchin's main motivations for studying the Higgs bundle equations and the moduli space of solutions was to find new examples of hyperkähler manifolds, and indeed $\mathscr{M}^*$ carries the structure of a smooth hyperkähler manifold [39].

In this section we review the definition of the three inequivalent complex structures I, J, K generating the hyperkähler structure.

As above, we fix a topologically trivial smooth Hermitian vector bundle (E, h) on a Riemann surface Σ.

The space $\Omega = \Omega^{1,0}(\operatorname{End} E)$ has a natural complex structure defined by multiplication by i, denoted I_Ω, and a natural symplectic structure

$$\omega_\Omega(\dot\Phi_1, \dot\Phi_2) = \int_\Sigma \operatorname{Tr}(\dot\Phi_1 \wedge \dot\Phi_2^* - \dot\Phi_2 \wedge \dot\Phi_1^*) \quad \text{for } \Phi \in \Omega \quad \text{and} \quad \dot\Phi_1, \dot\Phi_2 \in T_\Phi\Omega = \Omega^{1,0}(\operatorname{End} E).$$

The space $\mathscr{C} \simeq \mathscr{A}$ also has a natural complex structure $I_{\mathscr{C}}$ induced from the complex structure on Σ: as $\mathscr{C}$ is modelled on $\Omega^{0,1}(\operatorname{End} E)$ the complex structure is multiplication by i, $I_{\mathscr{C}}(\dot\beta) = i\dot\beta$ for $\bar\partial_A \in \mathscr{C}$ and $\dot\beta \in T_{\bar\partial_A}\mathscr{C} = \Omega^{0,1}(\operatorname{End} E)$. The natural symplectic structure on $\mathscr{C}$ is

$$\omega_{\mathscr{C}}(\dot\beta_1, \dot\beta_2) = \int_\Sigma \operatorname{Tr}(\dot\beta_2^* \wedge \dot\beta_1 - \dot\beta_1^* \wedge \dot\beta_2) \quad \text{for } \bar\partial_\beta \in \mathscr{C} \quad \text{and} \quad \dot\beta_1, \dot\beta_2 \in T_{\bar\partial_\beta}\mathscr{C} = \Omega^{0,1}(\operatorname{End} E).$$

The combination $(\omega_{\mathscr{C}} + \omega_\Omega, I_{\mathscr{C}} + I_\Omega)$ of the symplectic and complex structures on $\mathscr{C}$ and Ω defines a Kähler structure on $\mathscr{C} \times \Omega$ and hence on the subvariety

$$\mathscr{D}^{st} = \{(\bar\partial_A, \Phi) \in \mathscr{C} \times \Omega \,|\, \bar\partial_A \Phi = 0, (\bar\partial_A, \Phi) \text{ is stable}\}.$$

As we saw above, the moduli space of stable Higgs bundles $\mathscr{M}^{st}$ with smooth underlying vector bundle E is the quotient $\mathscr{D}^{st}/\mathscr{G}$.

The Riemannian metric defined by the symplectic structure $\omega_{\mathscr{C}} + \omega_\Omega$ and complex structure $I_{\mathscr{C}} + I_\Omega$ is the natural L^2-Hermitian metric

$$g((\dot\beta, \dot\Phi), (\dot\beta, \dot\Phi)) = 2i \int_\Sigma \operatorname{Tr}(\dot\beta^* \wedge \dot\beta + \dot\Phi \wedge \dot\Phi^*)$$

for $(\bar\partial_\beta, \Phi) \in \mathscr{C} \times \Omega$ and $(\dot\beta, \dot\Phi) \in T_{(\bar\partial_\beta, \Phi)}\mathscr{C} \times \Omega = \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$.

The space of complex connections $\mathscr{X}$ is an infinite dimensional affine space modelled on $\Omega^1(\Sigma, \operatorname{End} E) \simeq \Omega^0(\Sigma, T^*\Sigma \otimes_{\mathbb{R}} \operatorname{End} E)$ and has a natural complex structure $J = 1 \otimes i$ coming from the complex structure of E (and not the complex structure of Σ). The Hermitian metric on E, with the choice of a Riemannian metric from the conformal class on Σ, defines a Riemannian metric on $\mathscr{X}$ which is Kähler with respect to J. By defining the affine map $\mathscr{C} \times \Omega \to \mathscr{X}$ as $(\beta, \Phi) \mapsto \beta - \beta^* + \Phi + \Phi^*$ the complex structure J on $\mathscr{X}$ defines a complex structure also denoted J on $\mathscr{C} \times \Omega$. In the coordinates $(\dot\beta, \dot\Phi)$ on $T_{(\bar\partial_A, \Phi)}\mathscr{C} \times \Omega = \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$, J is $J(\dot\beta, \dot\Phi) = (i\dot\Phi^*, -i\dot\beta^*)$. The product $K = IJ$ defines another complex structure $K(\dot\beta, \dot\Phi) = (-\dot\Phi^*, \dot\beta^*)$. For later reference

$$I(\dot\beta, \dot\Phi) = (i\dot\beta, i\dot\Phi) \quad J(\dot\beta, \dot\Phi) = (i\dot\Phi^*, -i\dot\beta^*) \quad K(\dot\beta, \dot\Phi) = (-\dot\Phi^*, \dot\beta^*) \tag{1.2}$$

are three linearly inequivalent complex structures on $\mathscr{C} \times \Omega$.

If we denote by ω_J and ω_K the symplectic structures associated to (g, J) and (g, K), respectively, then the symplectic structure $\omega_J + i\omega_K$ is holomorphic with respect to the complex structure I. Explicitly the symplectic structure is

$$\omega_J((\dot\beta_1, \dot\Phi_1), (\dot\beta_2, \dot\Phi_2)) + i\omega_K((\dot\beta_1, \dot\Phi_1), (\dot\beta_2, \dot\Phi_2)) = 2\int_\Sigma \operatorname{Tr}(\dot\Phi_1 \wedge \dot\beta_2 - \dot\Phi_2 \wedge \dot\beta_1) \tag{1.3}$$

in the coordinates on $T_{(\bar\partial_A, \Phi)}\mathscr{C} \times \Omega$.

1.1.4 Spectral data and the Hitchin fibration

The Higgs bundle moduli space $\mathscr{M}$ has a very rich geometry. Besides the hyperkähler structure considered above, Hitchin [39] also showed that $\mathscr{M}$ is an algebraically completely integrable Hamiltonian system, more specifically that the map

$$\mathscr{M} \xrightarrow{H} \bigoplus_{n=1}^{\operatorname{rk} E} H^0(\Sigma, K^n) \quad \text{defined by} \quad (E, \Phi) \mapsto (\operatorname{Tr}\Phi, \operatorname{Tr}\wedge^2\Phi, \dots, \det\Phi)$$

is proper, surjective, and choosing a basis $\{\lambda_i\}_{i=1}^{\operatorname{rk} E^2(g-1)+1}$ for the dual of the base space, the functions $f_i = \lambda_i \circ H$ commute with respect to the Poisson bracket determined by the holomorphic symplectic structure (1.3) on $\mathscr{M}$. The map H is called *the Hitchin map.* In the remainder of this section we follow [7] and describe the generic fibre of the Hitchin map.

Let (E, Φ) be a stable degree zero Higgs bundle rank of rank r on Σ. Then, the characteristic polynomial of Φ defines a curve of eigenvalues of Φ in the total space of the canonical bundle $p : K \to \Sigma$

$$0 = \det(\Phi + \eta \operatorname{Id}) = \eta^r + \operatorname{Tr}\Phi\, \eta^{r-1} + \dots + \det\Phi$$

where η is the tautological section of p^*K on K. This curve is called *the spectral curve* of (E, Φ) and is denoted by S. The spectral curve is a r : 1-branched cover of Σ. The branch points are the points on Σ where Φ has multiple eigenvalues. The total space of the cotangent bundle has trivial canonical bundle. Combining this with the adjunction formula, the the canonical bundle of S is $K_S \simeq p^*K^r$, and thus the genus of S is $r^2(g-1)+1$.

On a smooth spectral curve S there is a line bundle L defined by the exact sequence of vector bundles [7]

$$0 \to L \otimes p^*K^{1-r} \to p^*E \xrightarrow{\Phi+\eta} p^*(E \otimes K) \to L \otimes p^*K \to 0.$$

The vector bundle E is recovered as the direct image sheaf p_*L of the line bundle L under the branching map p. The degree of L is $r(r-1)(g-1)$ and any line bundle of this degree pushes down to a vector bundle of degree zero. The Higgs field is recovered by pushing down

$$L \xrightarrow{\eta} L \otimes p^*K$$

under p. If S is smooth, then the Higgs bundle (E, Φ) is equivalent to (S, L).

Notice that a point in the base of the integrable system defines the equation of a spectral curve. For a generic point in the base the associated spectral curve S is smooth. By following the above recipe, pushing down any degree $r(r-1)(g-1)$ line bundle and tautological section from S to Σ produces a Higgs bundle of degree zero and rank r. By the Cayley–Hamilton Theorem the Higgs field will satisfy the equation defining S, and restricting the Higgs field to an invariant subbundle would divide the equation for S. However, S is smooth and hence irreducible and there are hence no invariant subbundles of this Higgs field. The constructed Higgs bundle is therefore stable.

The above shows that the generic fibre of the Hitchin map is the Jacobian of degree $r(r-1)(g-1)$ line bundles on the spectral curve determined by the point in the base. Notice that the genus of S is $1 + r^2(g-1)$, i.e. half the dimension of the moduli space, as it should be.

1.1.5 Higgs bundles and the derived category of coherent sheaves

In Chapter 4 we use the derived category of coherent sheaves as a convenient tool for some of the computations. A good introduction to derived categories for the purpose of Fourier–Mukai transforms can be found in [6] or [44]. In this section we describe how Higgs bundles can be treated using derived categories.

Let (E, Φ) be a Higgs bundle and consider it as a two term complex of locally free sheaves concentrated in degrees 0 and 1

$$E \xrightarrow{\Phi} E \otimes K,$$

that is an object in $D(\Sigma)$ the bounded derived category of coherent sheaves on Σ.

A homomorphism between two Higgs bundles (E, Φ) and (F, Ψ) is a homomorphism $\eta : E \to F$ making the following diagram commutative.

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\eta"] & E \otimes K \arrow[d, "\eta\otimes\mathrm{Id}"] \\ F \arrow[r, "\Psi"] & F \otimes K. \end{tikzcd}$$

Homomorphisms between objects in the derived category do not require the homomorphism $E \otimes K \to F \otimes K$ to be of the form $\eta \otimes \mathrm{Id}$ and it is therefore easy to see that for any non-zero $c \in \mathbb{C}$ there is a homomorphism between (E, Φ) and $(E, c\Phi)$ as elements of $D(\Sigma)$ but not as Higgs bundles in general. As c is non-zero, the homomorphism induce an isomorphism between the cohomology sheaves of the complexes, i.e. a quasi-isomorphism, and therefore an isomorphism in the derived category. This is problematic as in general (E, Φ) and $(E, c\Phi)$ are not isomorphic as Higgs bundles. To rectify this we extend a Higgs bundle (E, Φ) to a family of Higgs bundles.

Let (E, Φ) be a Higgs bundle on Σ. Consider the family of Higgs bundles

$$E \xrightarrow{\Theta} E \otimes K \otimes \mathcal{O}_{\mathbb{P}^g}(1)$$

parametrised by $\mathbb{P}^g = \mathbb{P}(\mathbb{C}\Phi \oplus H^0(\Sigma, K))$, such that the restriction to $\Sigma \times [a : \alpha]$ is the Higgs bundle

$$E \xrightarrow{a\Phi+\alpha\,\mathrm{Id}} E \otimes K.$$

We view this family of Higgs bundles as a two term complex of coherent sheaves on $\Sigma \times \mathbb{P}^g$ and denoted it by $\mathcal{C}(E)$. As above, $\mathcal{C}(E) \in D(\Sigma \times \mathbb{P}^g)$.

Proposition 1.1.8. *The category of stable Higgs bundles is a full subcategory of $D(\Sigma\times\mathbb{P}^g)$.*

Proof. Let $\mathcal{C}(E)$ and $\mathcal{C}(F)$ be two Higgs bundles considered as elements of $D(\Sigma \times \mathbb{P}^g)$ and let $f^\bullet$ be a homomorphism between $\mathcal{C}(E)$ and $\mathcal{C}(F)$, that is the diagram of locally free sheaves on $\Sigma \times \mathbb{P}^g$

$$\begin{tikzcd} E \arrow[r, "\Theta_E"] \arrow[d, "f_0"] & E \otimes K \otimes \mathcal{O}_{\mathbb{P}^g}(1) \arrow[d, "f_1"] \\ F \arrow[r, "\Theta_F"] & F \otimes K \otimes \mathcal{O}_{\mathbb{P}^g}(1) \end{tikzcd}$$

is commutative. When restricting to $\Sigma \times [0 : \alpha]$ for any $\alpha \in H^0(K)$ it follows that $f_1 = f_0 \otimes \mathrm{Id}$, and $f^\bullet$ therefore restricts to a homomorphism of Higgs bundles on $\Sigma \times [1 : 0]$.

Assume now that $f^\bullet$ is a quasi-isomorphism. It is not difficult to show that Θ_E and Θ_F are injective [14, Lemma 3.2.1.1], giving a commutative diagram of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E \arrow[r, "\Theta_E"] \arrow[d, "f_0"] & E \otimes K \otimes \mathcal{O}_{\mathbb{P}^g}(1) \arrow[r] \arrow[d, "f_0\otimes\mathrm{Id}"] & Q_E \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & F \arrow[r, "\Theta_F"] & F \otimes K \otimes \mathcal{O}_{\mathbb{P}^g}(1) \arrow[r] & Q_F \arrow[r] & 0 \end{tikzcd}$$

where f is the isomorphism induced on cohomology by the quasi-isomorphism $f^\bullet$. The short exact sequences combined with $f : Q_E \to Q_F$ being an isomorphism implies that $\mathrm{ch}(E) = \mathrm{ch}(F)$, and thus $\mu(E) = \mu(F)$. It now follows from [33, Theorem 4.3] that the restriction of $\mathcal{C}(E)$ and $\mathcal{C}(F)$ to $\Sigma \times [1:0]$ are isomorphic as Higgs bundles. □

Remark 1.1.9. Another way to rephrase Proposition 1.1.8 is to say that quasi-isomorphisms of complexes on $\Sigma \times \mathbb{P}^g$ associated to Higgs bundles are isomorphisms of Higgs bundles.

1.1.6 Kähler identities

A very useful but often overlooked fact is that Higgs pairs satisfies the usual Kähler identities.

Lemma 1.1.10. *Let (A, Φ) be a Higgs pair on a smooth Hermitian vector bundle (E, h) and denote by $D'' = \bar{\partial}_A + \Phi$ and $D' = \partial_A + \Phi^*$. We have the following Kähler identities*

1. $(D'')^\vee = -i[\Lambda, D']$
2. $(D')^\vee = i[\Lambda, D'']$

where $(D')^\vee, (D'')^\vee : \Omega^s(E) \to \Omega^{s-1}(E)$ are adjoints with respect to the L^2-inner product on $\Omega^s(E)$ defined by h, and Λ is contraction with the fixed Kähler form on Σ.

Proof. The lemma is proved in a similar fashion to the usual Kähler identities; see e.g. [71]. □

1.2 Hypercohomology

An essential tool used time and time again in the present thesis is hypercohomology of a complex of sheaves. In this section we set the notation straight and introduce the most important results.

Let $(\mathcal{E}^\bullet, \delta)$ be a complex of sheaves on a topological space X

$$\mathcal{E}^0 \to \cdots \to \mathcal{E}^p \xrightarrow{\delta} \mathcal{E}^{p+1} \to \ldots,$$

where $\mathcal{E}^p$'s are abelian sheaves and δ's are sheaf maps satisfying $\delta^2 = 0$.

For a complex of sheaves $(\mathcal{E}^\bullet, \delta)$ the notion of cohomology sheaves $\mathcal{H}^q = \mathcal{H}^q(\mathcal{E}^\bullet)$ is defined as the sheafication of the presheaves

$$U \mapsto \frac{\ker\{\delta : \mathcal{E}^q(U) \to \mathcal{E}^{q+1}(U)\}}{\delta\mathcal{E}^{q-1}(U)}$$

where $\mathcal{E}^q(U) = H^0(U, \mathcal{E}^q)$ and U is an open subset of X.

Let $U = \{U_\alpha\}$ be a covering of X and $C^p(U, \mathcal{H}^q)$ the Čech cochains of degree p with values in $\mathcal{H}^q$. The two coboundary operators

$$
\begin{aligned}
d &: C^p(U, \mathcal{H}^q) \to C^{p+1}(U, \mathcal{H}^q) \\
\delta &: C^p(U, \mathcal{H}^q) \to C^p(U, \mathcal{H}^{q+1}),
\end{aligned}
$$

anti-commute, $d\delta + \delta d = 0$, and square to zero, $d^2 = \delta^2 = 0$, thus giving rise to a double complex

$$(C^p(U, \mathcal{H}^q), d, \delta).$$

The hypercohomology is defined as the direct limit of the cohomology of the total complex $(C^\bullet(U), d + \delta)$ of $(C^p(U, \mathcal{H}^q), d, \delta)$,

$$\mathbb{H}^*(X, \mathcal{E}^\bullet) = \lim_{U} H^*(C^\bullet(U), d + \delta).$$

The hypercohomology can be calculated by either of two spectral sequences with second pages given by

$$ {}^{I}E_2^{p,q} = H^p(X, \mathcal{H}^q(\mathcal{E}^\bullet)) \quad \text{and} \quad {}^{II}E_2^{p,q} = H^p_\delta(H^q(X, \mathcal{E}^\bullet)), \tag{1.4}$$

where the first spectral sequence is the Čech cohomology of the cohomology sheaves $\mathcal{H}^*(\mathcal{E}^\bullet)$, and the second spectral sequence is the cohomology of the complex

$$H^*(X, \mathcal{E}^0) \xrightarrow{\delta} H^*(X, \mathcal{E}^1) \xrightarrow{\delta} \dots .$$

If a spectral sequence only has two non-zero adjacent rows on the second page, then there is a long exact sequence relating the groups in the spectral sequence to the limit, see e.g. [75, Exercise 5.2.2],

$$\cdots \to \mathbb{H}^p \to E_2^{p-1,1} \to E_2^{p+1,0} \to \mathbb{H}^{p+1} \to E_2^{p,1} \to E_2^{p+2,0} \to \mathbb{H}^{p+2} \to \dots \tag{1.5}$$

This can be generalised to the case of a spectral sequence with only two non-zero rows or two non-zero columns on the final page. For a spectral sequence with only non-zero groups in the first quadrant (1.5) contains the *five term exact sequence*,

$$0 \to E_2^{1,0} \to \mathbb{H}^1 \to E_2^{0,1} \to E_2^{2,0} \to \mathbb{H}^2. \tag{1.6}$$

For ordinary cohomology Serre duality is essential in any explicit computation; a generalisation to hypercohomology does exist and is stated here for reference. For a bounded complex $\mathcal{E}^\bullet$ of locally free sheaves on a Riemann surface Σ of the form

$$0 \to \mathcal{E}^0 \to \cdots \to \mathcal{E}^m \to 0$$

there exists a natural duality

$$\mathbb{H}^i(\mathcal{E}^\bullet)^* \simeq \mathbb{H}^{1-i+m}((\mathcal{E}^\bullet)^* \otimes K), \tag{1.7}$$

where $(\mathcal{E}^\bullet)^* \otimes K$ is the complex of twisted dual sheaves

$$0 \to (\mathcal{E}^m)^* \otimes K \to \cdots \to (\mathcal{E}^0)^* \otimes K \to 0.$$

Notice that in the case of a one step complex hypercohomology is just ordinary Čech cohomology, and Serre duality of hypercohomology is the classical Serre duality.

In the language of derived categories of coherent sheaves, hypercohomology of a complex of coherent sheaves is defined as the cohomology of the derived pushforward of the complex along the constant map. The hypercohomological Serre duality follows from the general Serre duality in the derived category of coherent sheaves.

1.3 The direct image functor

The direct image functor of sheaves plays an essential role in almost all of the succeeding chapters. The facts summarised here are from a host of different sources, especially [31].

Let $\mathcal{F}$ be a sheaf on a complex manifold X and $f : X \to Y$ a proper holomorphic map to a complex manifold Y. The *(zero'th) direct image* of $\mathcal{F}$ under f is the sheaf defined by

$$f_*(\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for each open set U of Y. The direct image sheaf $f_*(\mathcal{F})$ on Y is coherent if $\mathcal{F}$ is coherent on X. The functor $f_* : \mathrm{Coh}(X) \to \mathrm{Coh}(Y)$ is left exact but rarely right exact. The right derived functors of the direct image are called the *higher direct images* and are denoted $R^i f_*$. It can be shown that $R^i f_*(\mathcal{F})$ is the sheaf associated to the presheaf

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

When f is a proper holomorphic map the higher direct images of a coherent sheaf are again coherent.

The *projection formula* is a key tool when working with direct image sheaves:

$$R^i f_*(\mathcal{E}) \otimes_{\mathcal{O}_Y} \mathcal{F} \simeq R^i f_*(\mathcal{E} \otimes_{\mathcal{O}_X} f^*\mathcal{F}).$$

The projection formula is valid whenever $\mathcal{E}$ is a coherent sheaf, $\mathcal{F}$ is locally free and f is a proper holomorphic map.

Definition 1.3.1. Let X, Y be complex manifolds and $f : X \to Y$ a proper holomorphic map. A coherent sheaf $\mathcal{F}$ on X is flat over Y if all stalks $\mathcal{F}_x$ are flat $\mathcal{O}_{Y,f(x)}$-modules. The map f is called flat if $\mathcal{O}_X$ is flat over Y.

Example 1.3.2. Let X, Y be smooth complex manifolds. Then any proper holomorphic submersion $f : X \to Y$ is flat.

Example 1.3.3. A coherent sheaf on X is locally free if and only if it is flat over X. If $f : X \to Y$ is a flat map then locally free sheaves are flat over Y.

Theorem 1.3.4 (Base change). *Let $f : X \to Y$, $g : Z \to Y$ be holomorphic maps between smooth complex manifolds with g flat and $\mathcal{F}$ a coherent sheaf on X. Consider the push-out diagram*

$$\begin{array}{ccc} Z \times_Y X & \xrightarrow{g'} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow{g} & Y, \end{array}$$

then there is an isomorphism

$$g^* R^i f_*(\mathcal{F}) \simeq R^i f'_*(g'^* \mathcal{F}).$$

The cohomology groups $H^i(X_y, \mathcal{F}_y)$ where $X_y = f^{-1}(\{y\})$ and $\mathcal{F}_y = \mathcal{F}|_{X_y}$ are intimately related to the i'th higher direct image of $\mathcal{F}$ under f. The following theorem by Grauert summarises the relationship.

Theorem 1.3.5 (Grauert). *Let X, Y be smooth complex manifolds and $f : X \to Y$ a proper holomorphic map. If $\mathcal{F}$ is a coherent sheaf on X flat over Y the following is true:*

- *The Euler characteristic $\chi(X_y, \mathcal{F}_y) = \sum(-1)^i h^i(X_y, \mathcal{F}_y)$ is a locally constant function on Y.*
- *The functions $h^i(y, \mathcal{F}_y) = \dim H^i(X_y, \mathcal{F}_y)$ are upper semi-continuous in y for all $i \geq 0$.*
- *If $h^i(y, \mathcal{F}_y)$ is constant, then $R^i f_*(\mathcal{F})$ is locally free.*
- *If $h^i(y, \mathcal{F}_y)$ is constant, then $R^i f_*(\mathcal{F}) \otimes \mathbb{C}_y \to H^i(X_y, \mathcal{F}_y)$ is an isomorphism.*
- *If $h^i(y, \mathcal{F}_y)$ is constant, then $R^{i-1} f_*(\mathcal{F}) \otimes \mathbb{C}_y \to H^{i-1}(X_y, \mathcal{F}_y)$ is an isomorphism.*

The following well-known facts about torsion-free and reflexive sheaves and their direct images will be important in several instances in Chapter 4.

Lemma 1.3.6. *Let X and Y be smooth complex manifolds, $f : X \to Y$ a proper holomorphic submersion and $\mathcal{F}$ a coherent sheaf on X. If $\mathcal{F}$ is torsion free, then so is $f_*(\mathcal{F})$.*

Lemma 1.3.7. *Let X and Y be smooth complex manifolds, $\pi : X \times Y \to Y$ the projection, and $\mathcal{E}$ a reflexive sheaf on $X \times Y$, then $\pi_*(\mathcal{E})$ is reflexive.*

2. The Dirac–Higgs bundle

Let Σ be a Riemann surface of genus $g \geq 2$. The Dirac–Higgs bundle $(\mathbb{D}, \nabla)$ is a *hyperholomorphic* bundle on the smooth hyperkähler manifold $\mathscr{M}^{st}$ of stable Higgs bundles on Σ of degree zero and fixed rank at least two. A vector bundle with connection is hyperholomorphic if the curvature of the connection is of type $(1,1)$ with respect to all complex structures on the underlying hyperkähler manifold. If the real dimension of the manifold is four the connection of a hyperholomorphic bundle is an anti-self-dual connection.

Coupling the Dirac-operator to a Higgs bundle gives a new Dirac-type operator called the Dirac–Higgs operator. If the Higgs bundle is stable and of degree zero and rank at least two the kernel of the Dirac–Higgs operator is always trivial. The Dirac–Higgs vector bundle is thus the bundle of cokernels of the Dirac–Higgs operator. The connection is obtained by projecting the trivial connection to the bundle of cokernels. Technically speaking we do not get a vector bundle on $\mathscr{M}^{st}$ as this space does not have a universal bundle, but on every open set of $\mathscr{M}^{st}$ we do indeed get a bundle with a connection. The connection is of most interest and as this is a local object the lack of universal bundle poses no real problem.

The Dirac–Higgs operator and all its properties are introduced in Section 2.1. We give conditions for when the kernel vanishes and compute the index.

The cokernel of the Dirac–Higgs operator is equivalent to the solution set of a coupled set of differential equations called the Dirac–Higgs equations. In Section 2.2 we consider rank one Higgs bundles on $\mathbb{C}$ with the Higgs field being a polynomial. We see that for the monomials z^k the Dirac–Higgs equations have a k-dimensional space of global L^2-solutions. We conjecture this to be generally so for any degree k polynomial. For the trivial line bundle on $\mathbb{C}$ with monomial Higgs fields we can explicitly solve the Dirac–Higgs equations using modified Bessel functions.

In Section 2.3 we use the explicit calculations of Section 2.2 to construct a toy model for the Dirac–Higgs bundle on $\mathbb{C}^2$. We show that the natural connection is an instanton by explicitly computing its curvature.

In Section 2.4 we return to Higgs bundles on Σ. We use Hodge theory to identify the cokernel of the Dirac–Higgs operator with hypercohomology of the Higgs bundles considered as a two term complex of locally free sheaves. We also get an identification with the de Rham cohomology of the Higgs bundle.

In Section 2.5 we construct the Dirac–Higgs vector bundle $\mathbb{D}$. We examine more closely the obstructions for $\mathbb{D}$ to be a bundle on all of $\mathscr{M}^{st}$.

The Dirac–Higgs connection is constructed in Section 2.6 and we prove that $(\mathbb{D}, \nabla)$ is indeed hyperholomorphic as claimed above.

In Section 2.8 we use the toy model of Section 2.3 to construct a Nahm like transform for Higgs line bundles on $\mathbb{C}$, which constructs other Higgs bundles on $\mathbb{C}$. This Nahm type transform was conjectured by Corrigan and Goddard [20].

Throughout this chapter we let ω be a fixed Kähler form on Σ given by choosing a metric in the conformal class.

2.1 THE DIRAC–HIGGS OPERATOR

Let (E,h) be a fixed Hermitian vector bundle on Σ. Let (A,Φ) be a Higgs pair consisting of a unitary connection on E and Φ a section of $\operatorname{End} E \otimes K$. Denote by ∂_A and $\bar{\partial}_A$ the $(1,0)$ and $(0,1)$-parts of the covariant derivative of the connection A, respectively.

In [41] Hitchin defines a Dirac-operator coupled to a Higgs pair (A,Φ)

$$\mathcal{D}_{A,\Phi} = \begin{pmatrix} \partial_A & -\Phi \\ \Phi^* & -\bar{\partial}_A \end{pmatrix} : \Omega^0(E)^{\oplus 2} \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E), \tag{2.1}$$

called *the Dirac–Higgs operator*. The operator

$$\mathcal{D}^*_{A,\Phi} = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \to \Omega^{1,1}(E)^{\oplus 2} \tag{2.2}$$

is the adjoint of $\mathcal{D}_{A,\Phi}$ with respect to the L^2-inner product on $\Omega^1(E)$ in the following sense:

Lemma 2.1.1. *Let $s \in \Omega^0(E)^{\oplus 2}$ and $u \in \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$, then*

$$\langle u, \mathcal{D}_{A,\Phi} s \rangle_h = \langle i\Lambda \mathcal{D}^*_{A,\Phi} u, s \rangle_h$$

where $\langle\, , \,\rangle_h$ is the L^2-inner product on $\Omega^s(E)$ defined by the Hermitian metric h and the conformal structure on Σ, and Λ is the adjoint of wedging with the fixed Kähler form on Σ.

Proof. The lemma is a direct consequence of the Kähler identities for Higgs bundles, Lemma 1.1.10. □

Remark 2.1.2. Strictly speaking, the adjoint of $\mathcal{D}_{A,\Phi}$ should take values in the zero forms and not $(1,1)$-forms as above. However, by choosing a Kähler form on Σ and letting Λ be contraction with this form $\Omega^{1,1}(E)$ and $\Omega^{0,0}(E)$ are identified. The Dirac–Higgs bundle defined in Section 2.5 should only depend on the conformal structure of Σ which is why the above operator is preferred. Whilst doing calculations the choice of a metric in the conformal class is however often needed.

Remark 2.1.3. The Dirac–Higgs operator $\mathcal{D}_{A,\Phi}$ defined for a vector bundle E on $\mathbb{R}^2$ is the dimensional reduction of the usual Dirac-operator in the vector representation coupled to a connection on a vector bundle on $\mathbb{R}^4$, in the same way as the Higgs bundle equations are a dimensional reduction of the self-duality equations. In light of the above, $\mathcal{D}_{A,\Phi}$ ought to be $C^\infty(S^+ \otimes E) \to C^\infty(S^- \otimes E)$ with the spinor bundles $S^+ = \wedge^{0,0}T^*\Sigma \oplus \wedge^{1,1}T^*\Sigma$ and $S^- = \wedge^{1,0}T^*\Sigma \oplus \wedge^{0,1}T^*\Sigma$. As in Remark 2.1.2 we replace the $(1,1)$-forms by $(0,0)$-forms to make the adjoint conformally invariant.

Lemma 2.1.4. *Let* (A,Φ) *be an irreducible Higgs pair on a vector bundle of degree zero and rank at least two satisfying the Higgs bundle equations* (1.1)*, then* $\ker \mathcal{D}_{A,\Phi} = 0$.

Proof. The proof essentially follows from the irreducibility of (A,Φ) and that $\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}$ is a real operator due to the Higgs bundle equations.

Let $(s_1, s_2) \in \ker \mathcal{D}_{A,\Phi}$. The Higgs bundle equations specifically imply that

$$0 = \mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}(s_1, s_2) = -\begin{pmatrix} \partial_A\bar{\partial}_A s_1 + \Phi^*\Phi s_1 \\ \partial_A\bar{\partial}_A s_2 + \Phi^*\Phi s_2 \end{pmatrix}.$$

Pairing this with (s_1, s_2)

$$0 = \left\langle i\Lambda\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}(s_1, s_2), (s_1, s_2)\right\rangle_h = \|\bar{\partial}_A s_1\|_h^2 + \|\Phi s_1\|_h^2 + \|\bar{\partial}_A s_2\|_h^2 + \|\Phi s_2\|_h^2$$

shows that non-zero s_1, s_2 defines holomorphic embeddings of the trivial line bundle $\mathcal{O}$ in E which are Φ invariant.

Assume that one of the sections s_1, s_2 are non-zero. With respect to a smooth splitting of $E \simeq \mathcal{O} \oplus Q$ the Higgs field and $\bar{\partial}$-operator are

$$\Phi = \begin{pmatrix} 0 & \varphi \\ 0 & \Phi_Q \end{pmatrix} \quad \text{and} \quad \bar{\partial}_A = \begin{pmatrix} \bar{\partial} & \beta \\ 0 & \bar{\partial}_Q \end{pmatrix}$$

where $\beta \in \Omega^{0,1}(Q^*)$ is the second fundamental form. The connection A is

$$A = \begin{pmatrix} \nabla & \beta \\ -\beta^* & \nabla_Q \end{pmatrix}$$

where ∇ and ∇_Q are the induced connections on $\mathcal{O}$ and Q, respectively. Writing out the Higgs bundle equations explicitly with respect to the smooth splitting of E we get four equations. The equation corresponding to $\mathcal{O}$ is

$$0 = F(\nabla) - \beta \wedge \beta^* + \varphi \wedge \varphi^* = -\beta \wedge \beta^* + \varphi \wedge \varphi^*.$$

Integrating this identity over Σ gives

$$0 = \|\beta\|_h^2 + \|\varphi\|_h^2,$$

implying that β and φ vanish and hence that (A,Φ) is reducible. $\square$

Remark 2.1.5. From the proof of Lemma 2.1.4 we see that if the degree of E is not zero, then $\ker \mathcal{D}_{A,\Phi}$ might be non-zero.

Remark 2.1.6. If E is a line bundle of degree zero and $\Phi = 0$, then $\ker \mathcal{D}_{A,\Phi}$ consists of holomorphic sections of E and E^*. Unless E is trivial there are no holomorphic sections of E. We therefore see that $\dim \ker \mathcal{D}_{A,0} = 2$ if $A = 0$ and 0 otherwise.

Remark 2.1.7. If E is a line bundle of degree zero and $\Phi \neq 0$ the proof of Lemma 2.1.4 gives $\Phi s = 0$ which is only satisfied if $s = 0$ as Φ is a section of a line bundle. This shows that $\ker \mathcal{D}_{A,\Phi} = 0$.

Lemma 2.1.8. *Let (A, Φ) be a Higgs pair, then the index of*

$$\mathcal{D}_{A,\Phi} : \Omega^0(E)^{\oplus 2} \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \quad \textit{is} \quad -2r(g-1)$$

where r is the rank of E.

Proof. As Σ is compact the elliptic operator $\mathcal{D}_{A,\Phi}$ is Fredholm. Furthermore, the index only depends on the principal symbol and thus

$$\operatorname{ind}(\mathcal{D}_{A,\Phi}) = \operatorname{ind}(\partial_A : \Omega^0(E) \to \Omega^{1,0}(E)) + \operatorname{ind}(\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)).$$

From the ordinary Kähler identities and the Atiyah–Singer Index Theorem

$$\begin{aligned}\operatorname{ind}(\partial_A : \Omega^0(E) \to \Omega^{1,0}(E)) &= -\operatorname{ind}(\bar\partial_A : \Omega^{1,0}(E) \to \Omega^{1,1}(E)) \\ &= -\operatorname{ind}(\bar\partial_A : \Omega^0(EK) \to \Omega^{0,1}(EK)) = -\chi(EK).\end{aligned}$$

It follows that

$$\operatorname{ind}(\mathcal{D}_{A,\Phi}) = \chi(E) - \chi(EK) = -2r(g-1)$$

proving the lemma. □

Remark 2.1.9. Notice that the proof of Lemma 2.1.8 does not require the Higgs pair to satisfy the Higgs bundle equations. Neither the degree nor the rank requirements are needed.

Proposition 2.1.10. *Let (A, Φ) be an irreducible Higgs pair solving the Higgs bundle equations* (1.1) *on a Hermitian bundle (E, h) of degree zero and rank at least two. Then the dimension of* $\ker \mathcal{D}^*_{A,\Phi}$ *is*

$$\dim \ker \mathcal{D}^*_{A,\Phi} = 2 \operatorname{rk} E (g-1).$$

and is independent of (A, Φ).

Proof. The result follows directly from Lemmas 2.1.4 and 2.1.8. □

Remark 2.1.11. From remarks 2.1.6 and 2.1.7 we see that when E is the trivial line bundle the dimension of $\ker \mathcal{D}^*_{A,\Phi}$ jumps when Φ is the 0-section of K, and is otherwise constantly $2(g-1)$.

Definition 2.1.12. Let (A, Φ) be a Higgs pair, then the equations for $\ker \mathcal{D}^*_{A,\Phi}$

$$0 = \bar{\partial}_A \psi_1 + \Phi\psi_2 \quad \text{and} \quad 0 = \partial_A \psi_2 + \Phi^* \psi_1 \tag{2.3}$$

for $(\psi_1, \psi_2) \in \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$ are called *the Dirac–Higgs equations* for (A, Φ).

In the following section we study a special case in which the Dirac–Higgs equations can be explicitly solved.

2.2 Solutions to rank one Dirac–Higgs equations on $\mathbb{C}$

In this section we discuss the shape of solutions to the Dirac–Higgs equations for rank one Higgs bundles on $\mathbb{C}$. We explicitly solve the Dirac–Higgs equations for monomial Higgs fields and see that the number of solutions is the degree of the Higgs field monomial. We conjecture that this is true for general polynomial Higgs fields.

2.2.1 Square integrable solutions

Modified Bessel functions turn out to play an important role in solving the Dirac–Higgs equations on $\mathbb{C}$. For the benefit of the results to follow we recall here basic properties about modified Bessel functions. These can be found in e.g. [1].

Lemma 2.2.1. *The differential equation*

$$x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} - (\nu^2 + x^2) = 0 \tag{2.4}$$

for a real function f on the positive real half line has solutions $I_{\pm\nu}(x)$ and $K_\nu(x)$ called the modified Bessel functions of the second kind. Each Bessel function is a regular function with the following properties.

- *$I_\nu(x)$ is unbounded as $x \to \infty$ for all ν.*
- *$K_\nu(z) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}$ so $K_{-\nu}(x) = K_\nu(x)$.*
- *$K_0(x) \sim -\ln x$ for $x \to 0$.*
- *$K_\nu(x) \sim \frac{1}{2}\Gamma(\nu)(\frac{1}{2}x)^{-\nu}$ for $x \to 0$ when $\nu > 0$.*
- *$K_\nu(x) \sim \sqrt{\frac{2}{\pi x}} e^{-x}(1 + O(x^{-1}))$ for $x \to \infty$ with ν fixed.*
- *$K_{1/2}(x) = \sqrt{\frac{2}{\pi x}} e^{-x}$.*

Lemma 2.2.2. *There are no non-trivial L^2-solutions to the Dirac–Higgs equations for the Higgs bundle $(\mathcal{O}, dz)$ on $\mathbb{C}$.*

Proof. If $\psi \in \Omega^1$ is written as $\psi = \psi_1 dz + \psi_2 d\bar{z}$ the Dirac–Higgs equations for the Higgs bundle $(\mathcal{O}, dz)$ are equivalent to

$$0 = \partial^2_{z\bar{z}}\psi_1 - \psi_1 \quad \text{and} \quad \psi_2 = \partial_{\bar{z}}\psi_1.$$

We are only interested in L^2-solutions and expands ψ_1 as a Fourier series $\psi_1 = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$. The equations for the Fourier coefficients a_n are

$$0 = r^2 a_n'' + r a_n' - (n^2 + 4r^2)a_n.$$

Assuming solutions have the form $a_n(r) = f_n(2r)$ the functions f_n must satisfy the modified Bessel differential equation (2.4) with $\nu = n$. The function a_n is thus a modified Bessel function of the second kind $I_n(2r)$ or $K_n(2r)$.

For ψ to be L^2 the components ψ_i must be L^2. In the case of ψ_1, we must have

$$\sum_{n\in\mathbb{Z}} \int_0^\infty |a_n|^2 r dr < \infty.$$

It follows from Lemma 2.2.1 that $K_0(2r)$ is the only L^2-function on $\mathbb{C}$. This determines ψ_1 and ψ_2

$$\psi_1 = cK_0(2r) \quad \text{and} \quad \psi_2 = -cK_1(2r)e^{i\theta}$$

for $c \in \mathbb{C}$. From Lemma 2.2.1 the radial function $K_1(2r)$ on $\mathbb{C}$ is not L^2 around zero, giving that $c = 0$ is the only L^2-solution. □

Lemma 2.2.3. *The only L^2-solutions to the Dirac–Higgs equations for the rank one Higgs bundle $(\mathcal{O}, zdz)$ on $\mathbb{C}$ are*

$$ce^{-z\bar{z}}dz - ce^{-z\bar{z}}d\bar{z}$$

for any $c \in \mathbb{C}$. There are more generally k linearly independent L^2-solutions to the Dirac–Higgs equations for the rank one Higgs bundle $(\mathcal{O}, z^k dz)$ on $\mathbb{C}$, $k \geq 1$, all given by Bessel functions.

Proof. We proceed as in the proof of Lemma 2.2.2. If we write $\psi \in \Omega^1$ as $\psi = \psi_1 dz + \psi_2 d\bar{z}$ the Dirac–Higgs equations for $(\mathcal{O}, zdz)$ on $\mathbb{C}$ are equivalent to

$$0 = \partial^2_{z\bar{z}}\psi_1 - z^{-1}\partial_{\bar{z}}\psi_1 - |z|^2\psi_1 \quad \text{and} \quad \psi_2 = z^{-1}\partial_{\bar{z}}\psi_1.$$

We are only interested in solutions where ψ_1, ψ_2 are square integrable, and express them in terms of their Fourier series in polar coordinates $\psi_1 = \sum_{n\in\mathbb{Z}} a_n(r)e^{in\theta}$ and $\psi_2 = \sum_{n\in\mathbb{Z}} b_n(r)e^{in\theta}$. Then a_n is a solution to

$$0 = r^2 a_n'' - r a_n' - (n^2 - 2n + 4r^4)a_n \quad \text{and} \quad b_n(r) = \frac{1}{2}r^{-1}a_n'(r) - \frac{n}{2}r^{-2}a_n(r)$$

for each $n \in \mathbb{Z}$. Assuming that $a_n(r) = rf_n(r^2)$ the equation for a_n becomes

$$0 = x^2 f_n''(x) + x f_n'(x) - \left(\left(\frac{n-1}{2}\right)^2 + x^2\right) f_n(x)$$

which is the modified Bessel differential equation for $\nu = \frac{n-1}{2}$ and solutions are modified Bessel functions $I_{\pm\frac{n-1}{2}}(x)$ and $K_{\frac{n-1}{2}}(x)$. Lemma 2.2.1 shows that only $K_{\frac{n-1}{2}}(x)$ decays for large x. Combining this with the above, we see that the Fourier coefficients are

$$a_n(r) = c_n r K_{\frac{n-1}{2}}(r^2) \quad \text{for} \quad c_n \in \mathbb{C}.$$

The function ψ_1 is square integrable if and only if

$$\sum_{n\in\mathbb{Z}} \int_0^\infty |a_n|^2 r dr < \infty.$$

Firstly, we notice from Lemma 2.2.1 that for any $\varepsilon > 0$ the integrals $\int_\varepsilon^\infty |a_n|^2 r dr$ are all finite. However, we also see that the Bessel functions $K_{\frac{n-1}{2}}(x)$ have a singularity at zero for all n. The asymptotic expansion for small x shows that the only values of n for which $\int_0^\varepsilon |a_n|^2 r dr$ is finite are $0, 1, 2$.

The function b_n is

$$b_n(r) = \frac{1}{2} r^{-1} a_n'(r) - \frac{n}{2} r^{-2} a_n(r) = -c_n r K_{\frac{n+1}{2}}(r^2).$$

It follows from Lemma 2.2.1 that the radial function b_n on $\mathbb{C}$ is L^2 around zero only if n is $-2, -1, 0$. Along with the above, we conclude that the only L^2-solution of the Dirac–Higgs equations comes from the zeroth Fourier mode.

Combining all of the above, we see that all L^2-solutions to the Dirac–Higgs equations have the form

$$c e^{-z\bar{z}} dz - c e^{-z\bar{z}} d\bar{z} \quad \text{with} \quad c \in \mathbb{C}.$$

The case of monomials of degree k is similar to the derivations above. The Dirac–Higgs equations for $(\mathcal{O}, z^k dz)$ are equivalent to

$$0 = \partial^2_{z\bar{z}} \psi_1 - k z^{-1} \partial_{\bar{z}} \psi_1 - |z|^{2k} \psi_1 \quad \text{and} \quad \psi_2 = z^{-k} \partial_{\bar{z}} \psi_1.$$

Splitting $\psi_1 = \sum_{n\in\mathbb{Z}} a_n e^{in\theta}$ and $\psi_2 = \sum_{n\in\mathbb{Z}} b_n e^{in\theta}$ into Fourier series the coefficients a_n, b_n must satisfy

$$0 = r^2 a_n'' - (2k-1) r a_n' - (n^2 - 2kn + 4r^{2+2k}) a_n \quad \text{and} \quad b_{n-k+1} = \frac{1}{2} r^{-k} a_n' - \frac{n}{2} r^{-k-1} a_n.$$

Solutions to this differential equation are as above modified Bessel functions

$$r^k I_{\pm\frac{k-n}{k+1}}\left(\frac{2}{k+1} r^{k+1}\right) \quad \text{and} \quad r^k K_{\frac{k-n}{k+1}}\left(\frac{2}{k+1} r^{k+1}\right).$$

The asymptotic behaviour of modified Bessel functions (Lemma 2.2.1) shows that only $K_\nu(x)$ decays for large x, and that it is only when $n \in \{0,\dots,2k\}$ that $r^k K_{\frac{k-n}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right)$ is quadratic integrable on $\mathbb{C}$. If $a_n(r) = r^k K_{\frac{k-n}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right)$, then

$$b_{n-k+1}(r) = -r^k K_{\frac{n+1}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right)$$

is only L^2 around zero if $n \in \{-k-1,\dots,k-1\}$. The general solution to the Dirac–Higgs equations are $\psi_1 dz + \psi_2 d\bar{z}$ with ψ_1, ψ_2 determined by k complex numbers $c_0,\dots,c_{k-1}$

$$\psi_1 = c_0 r^k K_{\frac{k}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right) + \cdots + c_{k-1} r^k K_{\frac{1}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right) e^{i(k-1)\theta}$$
$$\psi_2 = -c_0 r^k K_{\frac{1}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right) e^{-i(k-1)\theta} - \cdots - c_{k-1} r^k K_{\frac{k}{k+1}}\left(\frac{2}{k+1}r^{k+1}\right).$$

This proves that there are k linearly independent solutions to the Dirac–Higgs equations for $(\mathcal{O}, z^k dz)$ all of which are given by modified Bessel functions. □

Based on the results from Lemma 2.2.3 we make the following conjecture regarding the solutions to the Dirac–Higgs equations for a polynomial Higgs field.

Conjecture 2.2.4. *If φ is a polynomial of degree k, then there are k linearly independent L^2-solutions to the Dirac–Higgs equations for the Higgs bundle $(\mathcal{O}, \varphi dz)$ on $\mathbb{C}$.*

A possible proof of the conjecture is discussed in Chapter 7.

Remark 2.2.5. The picture emerging from the global solutions to the Dirac–Higgs equations on $\mathbb{C}$ resembles that of the abelian vortex equations on $\mathbb{C}$ studied by Jaffe and Taubes [45] and extended to vortices on Riemann Surfaces by García-Prada [28].

2.3 Toy model for the Dirac–Higgs bundle

We study the Higgs line bundle $(\mathcal{O}, z dz)$ on $\mathbb{C}$, and use the result of Lemma 2.2.3 to construct a toy model of the Dirac–Higgs bundle.

If we deform the Higgs bundle by conjugating the $\bar{\partial}$-operator by the exponential of a linear map and the Higgs field by adding a constant one-form, we get a family of Higgs line bundles $(\bar{\partial} + v d\bar{z}, (z+u)dz)$ parametrised by $(u,v) \in \mathbb{C}^2$.

Lemma 2.2.3 is easily modified to show that for any (u,v) there is only a one-dimensional space of L^2-solutions to the Dirac–Higgs equations for this family. Furthermore, if a solution is written as $\psi(z,u,v) = \psi_1(z,u,v)dz + \psi_2(z,u,v)d\bar{z}$, then ψ_1 and ψ_2 are

$$\psi_1(z,u,v) = c e^{\bar{v}z - v\bar{z} - (z+u)(\bar{z}+\bar{u})} \quad \text{and} \quad \psi_2 = -\psi_1 \quad \text{with } c \in \mathbb{C}. \tag{2.5}$$

As the dimension of the L^2-solution space is independent of (u,v), the solution spaces collectively define a line bundle $\mathbb{L}$ on $\mathbb{C}^2$.

The line bundle $\mathbb{L}$ inherits a Hermitian metric by inclusion into the infinite-dimensional trivial bundle

$$\underline{\Omega} = \Omega^{1,0} \oplus \Omega^{0,1} \times \mathbb{C}^2.$$

The bundle $\underline{\Omega}$ is equipped with the L^2-metric induced by the standard complex structure and Kähler form on $\mathbb{C}$.

Additionally, $\mathbb{L}$ has a natural unitary connection ∇ given by projecting the trivial connection $\underline{d}$ onto the line bundle $\mathbb{L}$

$$\nabla = P\underline{d}i,$$

where P at a point (u, v) is projection of a one-form onto the L^2-solution space for the Dirac–Higgs equations for $(\bar{\partial} + v d\bar{z}, (z + u)dz)$, and i is the inclusion of $\mathbb{L}$ in $\underline{\Omega}$.

A simple calculation shows that a unitary frame for $\mathbb{L}$ is given by a section ψ with $c = \frac{1}{\sqrt{\pi}}$ in (2.5). Define $\psi = \psi_1 dz + \psi_2 d\bar{z}$ to be such a unitary frame for $\mathbb{L}$.

The explicit expression in (2.5) makes it possible to directly compute the covariant derivative. The following identities are immediate consequences of (2.5)

$$\partial_v \psi_1 = -\bar{z}\psi_1 \qquad \partial_{\bar{v}} \psi_1 = z\psi_1 \qquad \partial_u \psi_1 = -(\bar{z} + \bar{u})\psi_1 \qquad \partial_{\bar{u}} \psi_1 = -(z + u)\psi_1.$$

By definition of the connection, the derivative $\nabla_{\partial_v}\psi$ is $\partial_v\psi$ projected back onto ψ using the L^2-metric.

$$\begin{aligned}
\langle \partial_v \psi, \psi \rangle &= \int_{\mathbb{C}} \partial_v \psi_1 \bar{\psi_1} \,\mathrm{dVol} + \int_{\mathbb{C}} \partial_v \psi_2 \bar{\psi_2} \,\mathrm{dVol} \\
&= 2 \int_{\mathbb{C}} -\bar{z} |\psi_1|^2 \,\mathrm{dVol} \\
&= \frac{-2}{\pi} \int_{\mathbb{C}} \bar{z} e^{-2(z+u)(\bar{z}+\bar{u})} \,\mathrm{dVol} \\
&= \frac{-2}{\pi} \int_{\mathbb{C}} (\bar{z} - \bar{u}) e^{-2z\bar{z}} \,\mathrm{dVol} \\
&= \bar{u}
\end{aligned}$$

where the last equality follows from $\int_0^\infty 4e^{-2r^2} r dr = 1$ and $\int_{\mathbb{C}} \bar{z} f(r) \,\mathrm{dVol} = 0$ where f is a radial function. We can do the same calculations for the other partial derivatives and get

$$\nabla_{\partial_v} \psi = \bar{u}\psi \qquad \nabla_{\partial_{\bar{v}}} \psi = -u\psi \qquad \nabla_{\partial_u} \psi = 0 \qquad \nabla_{\partial_{\bar{u}}} \psi = 0. \tag{2.6}$$

This completely specifies the connection ∇, and the curvature $F(\nabla)$ can be obtained by computing the commutators

$$\begin{aligned}
&[\nabla_{\partial_u}, \nabla_{\partial_{\bar{u}}}] = 0 \qquad && [\nabla_{\partial_u}, \nabla_{\partial_v}] = 0 \qquad && [\nabla_{\partial_u}, \nabla_{\partial_{\bar{v}}}] = -1 \\
&[\nabla_{\partial_{\bar{u}}}, \nabla_{\partial_{\bar{v}}}] = 0 \qquad && [\nabla_{\partial_v}, \nabla_{\partial_{\bar{v}}}] = 0 \qquad && [\nabla_{\partial_{\bar{u}}}, \nabla_{\partial_v}] = 1
\end{aligned}$$

giving

$$F(\nabla) = d\bar{u} \wedge dv - du \wedge d\bar{v}.$$

The curvature is clearly of type $(1,1)$ and orthogonal to the standard symplectic form $\frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v})$ on $\mathbb{C}^2$. In other words, the connection ∇ on $\mathbb{C}^2$ is an instanton. When we in the following sections consider the Dirac–Higgs bundle on the hyperkähler moduli space of Higgs bundles on a compact Riemann surfaces, the anti-self-duality equations are generalised to the curvature being of type $(1,1)$ with respect to all complex structures, i.e. the Dirac–Higgs bundle is hyperholomorphic, see Theorem 2.6.3.

2.4 Hodge theory

We now return to Higgs bundles on a compact Riemann surface. As we saw in Section 1.1, Higgs bundles both have a description as a holomorphic object (E,Φ) and as a differential geometric object (A,Φ). This duality is also present for the Dirac–Higgs operator where it is reflected via Hodge theory.

Consider a Higgs bundle (E,Φ) as a two term complex of locally free sheaves

$$E \xrightarrow{\Phi} EK. \tag{2.7}$$

The hypercohomology $\mathbb{H}^*(E,\Phi)$ of (2.7) can be computed by choosing the standard Dolbeault resolution of (2.7), giving that the hypercohomology groups are isomorphic to the cohomology of

$$\Omega^0(E) \xrightarrow{\bar{\partial}_E+\Phi} \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \xrightarrow{\bar{\partial}_E+\Phi} \Omega^{1,1}(E). \tag{2.8}$$

Lemma 2.4.1. *Let (E,h,Φ) be a Higgs bundle with a Hermitian metric and (A,Φ) the associated Higgs pair, then $\mathbb{H}^1(E,\Phi) \simeq \ker(\mathcal{D}^*_{A,\Phi})$.*

Proof. The differential in (2.8) is $D'' = \bar{\partial}_A + \Phi$ where A is the Chern connection of (E,h). From the Kähler identities its adjoint is $D' = \partial_A + \Phi^*$. If $\psi_1 + \psi_2 \in \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$ satisfy both

$$0 = D''(\psi_1 + \psi_2) = \bar{\partial}_A \psi_1 + \Phi\psi_2 \quad \text{and} \quad 0 = D'(\psi_1 + \psi_2) = \partial_A \psi_2 + \Phi^* \psi_1$$

it is a harmonic representative of the cohomology $\mathbb{H}^1(E,\Phi) = \ker(D'')/\operatorname{im}(D'')$, but is also an element of $\ker(\mathcal{D}^*_{A,\Phi})$. Standard Hodge theory for elliptic differential operators on compact manifolds [76, Chap IV, Theorem 5.2] now identifies the spaces $\ker(\mathcal{D}^*_{A,\Phi}) \simeq \mathbb{H}^1(E,\Phi)$. □

Remark 2.4.2. The identification in Lemma 2.4.1 also extends to the other hypercohomology groups, $\mathbb{H}^0(E,\Phi) \oplus \mathbb{H}^2(E,\Phi) \simeq \ker \mathcal{D}_{A,\Phi}$, proving that these cohomology groups vanish when (E,Φ) is stable of degree zero and rank at least two. This vanishing result can also be obtained directly from the definition of hypercohomology and Serre duality.

Lemma 2.4.3. *Let (E, Φ) be a Higgs bundle of degree zero and rank r with $\det \Phi$ having only simple zeros $z_1, \ldots, z_N$. Then*

$$\mathbb{H}^1(E, \Phi) \simeq \bigoplus_{i=1}^{N} \operatorname{coker}(E_{z_i} \xrightarrow{\Phi_{z_i}} EK_{z_i})$$

where $N = 2 \operatorname{rk} E(g-1)$.

Proof. From the first hypercohomology spectral sequence (1.4) the hypercohomology group $\mathbb{H}^i(E, \Phi)$ can be computed from the cohomology sheaves $\mathcal{H}^i(E \xrightarrow{\Phi} EK)$. Since $\det \Phi$ has simple zeros Φ is an injective sheaf map and thus $\mathcal{H}^0 = 0$. For the same reason $\mathcal{H}^1$ is a skyscraper sheaf supported on the $N = 2 \operatorname{rk} E(g-1)$ zeros of $\det \Phi \in H^0(K^{\operatorname{rk} E} \otimes \det E)$, each non-zero stalk having length one. The result follows by definition of the spectral sequence.

More concretely, identify $\mathbb{H}^1(E, \Phi)$ with the first Dolbeault cohomology group of $(E, \bar{\partial}_E + \Phi)$. A cohomology class $[\psi]$ is represented by a 1-form $\psi = \psi_1 + \psi_2 \in \Omega^1(E) = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$, with $D''\psi = \bar{\partial}_E \psi_1 + \Phi\psi_2 = 0$. Define

$$\rho' : \mathbb{H}^1(E, \Phi) \to \bigoplus_{i=1}^{N} \operatorname{coker}(E_{z_i} \xrightarrow{\Phi_{z_i}} EK_{z_i}) \quad \text{as} \quad \rho'(\psi) = (\psi_1(z_1), \ldots, \psi_1(z_N))$$

the evaluation of ψ_1 at the zeros of $\det \Phi$. The map ρ' is well-defined as a different representative for the cohomology class differ from ψ by $D''s = \bar{\partial}_E s + \Phi s$ for s a section of E and the $(1,0)$-parts therefore differ by Φs.

Given a tuple $(v_1, \ldots, v_N)$ of elements in the cokernel we can by choosing trivialisations of EK around each z_i and using appropriately chosen bump functions for these neighbourhoods construct a solution to $D''\psi = 0$ defining the inverse of ρ. The different choices involved in the construction will give different solutions to $D''\psi = 0$ but all differing by $D''s$ where s is a section of E. For the details see e.g. [72, Proposition 4.22]. □

2.4.1 de Rham cohomology

Let (A, Φ) be an irreducible Higgs pair solving the Higgs bundle equations on a vector bundle E of degree zero and rank at least two. Recall from Section 1.1.2 that

$$D = d_A + \Phi + \Phi^*$$

is a flat irreducible complex connection on E.

Proposition 2.4.4. *If (E, D) is a vector bundle of degree zero and rank at least two with an irreducible flat connection D, then the first cohomology group of the de Rham complex of D*

$$\Omega^0(E) \xrightarrow{D} \Omega^1(E) \xrightarrow{D} \Omega^2(E)$$

is isomorphic to $\ker \mathcal{D}^*_{A,\Phi}$*, where* (A, Φ) *is the Higgs pair from Donaldson's equivalence, and the other cohomology groups vanish.*

Proof. From Donaldson [22] there is a harmonic metric splitting D into operators $D' = \partial_A + \Phi^*$, $D'' = \bar{\partial}_A + \Phi$ where

$$D = d_A + \Phi + \Phi^* = D' + D''$$

is split into unitary and self-adjoint parts. It follows from the Higgs bundle Kähler identities Lemma 1.1.10 that $\psi \in \Omega^1(E)$ satisfying

$$0 = D\psi = D'\psi + D''\psi \quad \text{and} \quad 0 = D^*\psi = i\Lambda(D''\psi - D'\psi)$$

is equivalent to $0 = D'\psi$ and $0 = D''\psi$, that is $\psi \in \ker \mathcal{D}^*_{A,\Phi}$. Hodge theory therefore shows that the first cohomology group of the de Rham complex of the flat connection D is $\ker \mathcal{D}^*_{A,\Phi}$.

The remaining cohomology groups vanish as for $\psi \in \Omega^0(E)$ satisfying $0 = D\psi$ is equivalent to $\psi \in \ker \mathcal{D}_{A,\Phi}$. The equation $0 = D^*\psi$ for $\psi \in \Omega^2(E)$ is likewise equivalent to $i\Lambda\psi \in \ker \mathcal{D}_{A,\Phi}$. By Donaldson's result (A, Φ) satisfies the Higgs bundle equations. Furthermore, as E has degree zero and rank at least two the kernel of $\mathcal{D}_{A,\Phi}$ vanish by Lemma 2.1.4. □

Remark 2.4.5. If (A, Φ) satisfies the Higgs bundle equations, then for any $\zeta \in \mathbb{C}^*$

$$D_\zeta = d_A + \zeta\Phi + \zeta^{-1}\Phi^*$$

is a flat complex connection. By a proof equivalent to that of Proposition 2.4.4 the de Rham cohomology of D_ζ is concentrated in degree one and is $\ker \mathcal{D}^*_{A,\Phi}$.

2.5 The Dirac–Higgs bundle

Proposition 2.1.10 shows that the dimension of $\ker \mathcal{D}^*_{A,\Phi}$ is independent of the Higgs pair (A, Φ) on a compact Riemann surface Σ if the pair satisfies the Higgs bundle equations and the topology of the underlying vector bundle is trivial and has rank at least two.

If there were a universal Higgs bundle $E \xrightarrow{\Theta} E \otimes K$ on $\Sigma \times \mathscr{M}^{st}$ we would obtain a locally free sheaf on $\mathscr{M}^{st}$ by taking its first direct image along the projection to $\mathscr{M}^{st}$. The zeroth and higher direct images all vanish in this case. By Lemma 2.4.1, the fibre at each point is $\ker \mathcal{D}^*_{A,\Phi}$ where (A, Φ) is the Higgs pair solving the Higgs bundle equations associated to a stable Higgs bundle (E, Φ). This is indeed the way Hausel [33] constructs his *virtual Dirac bundle* on the moduli space of rank two and degree one Higgs bundles. As mentioned in Remark 2.1.5, the kernel of the Dirac–Higgs operator might be non-trivial which is reflected in the fact that Hausel has a *virtual* bundle, as in his case the direct image is not just concentrated in one degree.

For the moduli space of semi-stable bundles on a Riemann surface of genus $g \geq 2$, Ramanan [67] showed that when the rank and degree are not coprime there is no universal bundle. The same is true for Higgs bundles as it is not possible to construct a universal vector bundle parametrising the holomorphic vector bundles underlying the Higgs bundles. However, given an open covering $\{U_i\}$ of $\mathscr{M}^{st}$ there is a local universal vector bundle E_i on $U_i \times \Sigma$ and on intersections $U_i \cap U_j$ there is a line bundle L_{ij} such that $E_i \otimes L_{ij} \simeq E_j$ with extra compatibility conditions on the isomorphisms for triple intersections. This describes a so-called *gerbe*. For an open set U_i with a universal vector bundle E_i on $U_i \times \Sigma$ we define the sheaf

$$\mathbb{D}_i = \mathbb{R}^1 \pi_* (E_i \xrightarrow{\Theta} E_i \otimes \pi^* K)$$

on U_i. By the Hodge theory in Lemma 2.4.1, this is the bundle of cokernels of Dirac–Higgs operators. The topological assumptions on the underlying vector bundle Proposition 2.1.10 gives that $\mathbb{D}_i$ is locally free of rank $2 \operatorname{rk} E(g-1)$. On intersections $U_i \cap U_j$ the vector bundles $\mathbb{D}_i$ and $\mathbb{D}_j$ are related by the line bundle L_{ij}.

In what follows we will drop the index i and mainly treat $\mathbb{D}$ as a bundle on all of $\mathscr{M}^{st}$. In the next section we define a metric and a connection on $\mathbb{D}$, both of which are local concepts. The vector bundle $\mathbb{D}$ on $\mathscr{M}^{st}$ is called the *Dirac–Higgs vector bundle*.

Remark 2.5.1. Instead of defining the vector bundle $\mathbb{D}$ locally we could have defined it globally as a projective bundle. This projective bundle is not the projectivisation of a vector bundle; if it was, the gerbe above would be trivial and $\mathbb{D}$ could be globally defined.

Remark 2.5.2. In Section 2.6 we equip the Dirac–Higgs vector bundle with a Hermitian metric and a unitary connection. Considered as a projective bundle the Dirac–Higgs bundle is therefore a principal $\mathrm{PU}(N)$-bundle, $N = 2 \operatorname{rk} E(g-1)$, which is the closest one can get to a $\mathrm{U}(N)$-bundle.

Remark 2.5.3. In Chapter 6 we shift attention to parabolic Higgs bundles. The extra data of a parabolic structure allows us to construct a universal parabolic Higgs bundle under very mild assumptions on the parabolic structure (Section 6.5). We use this in Section 6.6 to globally define a Dirac–Higgs bundle for parabolic Higgs bundles.

Serre duality gives the cotangent bundle of the Jacobian of Σ a group structure $T^*J \simeq J \times H^0(K)$ and T^*J acts on $\mathscr{M}^{st}$ by

$$(\xi, \alpha) \cdot (E, \Phi) = (E \otimes L_\xi, \Phi + \alpha \operatorname{Id})$$

where L_ξ denotes the degree zero line bundle given by $\xi \in J$. The orbit $\operatorname{Orb}_{T^*J}(E, \Phi)$ gives an immersion of T^*J in $\mathscr{M}^{st}$ through the point given by the equivalence class of

(E, Φ). By choosing a base point z_0 on Σ we define an Abel–Jacobi map $\Sigma \to J$ to the dual Jacobian by mapping a point z to the divisor class of $z - z_0$. The Poincaré bundle $\mathcal{P}$ on $\hat{J} \times J$ defined by the choice of Abel–Jacobi map now pulls back to a universal line bundle on $\Sigma \times \hat{J} \simeq \Sigma \times J$ denoted $\tilde{\mathcal{P}}$.

Given a stable Higgs bundle (E, Φ) of degree zero there is a universal Higgs bundle on $\Sigma \times J \times H^0(K)$

$$E \otimes \tilde{\mathcal{P}} \xrightarrow{\Theta} E \otimes K \otimes \tilde{\mathcal{P}}$$

parametrising the stable Higgs bundles in the orbit $\mathrm{Orb}_{T^*J}(E, \Phi)$, that is restricting to $\Sigma \times \{\xi, \alpha\}$ is the Higgs bundle $(E \otimes L_\xi, \Phi + \alpha \,\mathrm{Id})$. The direct image of this universal Higgs bundle to $J \times H^0(K)$ is the same as finding an open set U in $\mathscr{M}^{st}$ containing $\mathrm{Orb}_{T^*J}(E, \Phi)$ and pulling back $\mathbb{D}$ defined on U to $J \times H^0(K)$ along the map defined by the action.

The pull-back of $\mathbb{D}$ to $J \times H^0(K)$ by the orbit map is how the Dirac–Higgs vector bundle is discussed in Chapters 3, 4, and 5.

2.6 The Dirac–Higgs connection

The vector bundle $\mathbb{D}$ has a natural Hermitian metric and unitary connection. Both are obtained by embedding $\mathbb{D}$ in the infinite dimensional trivial bundle

$$\underline{\Omega} = \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \times \mathscr{M}^{st}$$

on $\mathscr{M}^{st}$ by the inclusion $i_{A,\Phi} : \ker(\mathcal{D}^*_{A,\Phi}) \to \Omega^{1,0}(E) \oplus \Omega^{0,1}(E)$. The bundle $\underline{\Omega}$ is equipped with the L^2-metric induced by the fixed Hermitian metric on E and the Hodge-$*$ on Σ. Denote by $P_{A,\Phi}$ the orthogonal projection onto $\ker \mathcal{D}^*_{A,\Phi}$ with respect to the L^2-metric and by P the family of projection operators. We define a unitary connection ∇ on $\mathbb{D}$ by

$$\nabla = P\underline{d}i$$

where $\underline{d}$ is the trivial connection on $\underline{\Omega}$.

Definition 2.6.1. The bundle with connection $(\mathbb{D}, \nabla)$ on $\mathscr{M}^{st}$ is called the *Dirac–Higgs bundle* and ∇ the *Dirac–Higgs connection.*

Remark 2.6.2. The Hermitian metric on $\underline{\Omega}$ only depends on the conformal structure of Σ as the Hodge star is conformally invariant on 1-forms on a Riemann surface. The Dirac–Higgs operator is also conformally invariant and hence $(\mathbb{D}, \nabla)$ is conformally invariant.

Theorem 2.6.3. *The Dirac–Higgs connection is of type* $(1, 1)$ *with respect to all complex structures on the moduli space of stable Higgs bundles.*

Proof. The moduli space of stable Higgs bundles is a hyperkähler manifold with three inequivalent complex structures I, J, K defined in Section 1.1.3.

For the complex structure I consider the family of complexes

$$\Omega^0(E) \xrightarrow{\bar{\partial}_A+\Phi} \Omega^1(E) \xrightarrow{\bar{\partial}_A+\Phi} \Omega^2(E).$$

An infinitesimal deformation of the differentials around a point $(\bar{\partial}_A, \Phi)$ is $\dot{\beta} + \dot{\Phi}$ where $(\dot{\beta}, \dot{\Phi}) \in \Omega^{0,1}(\operatorname{End} E) \oplus \Omega^{1,0}(\operatorname{End} E)$ is in the tangent space to $\mathscr{C} \times \Omega$ at the point $(\bar{\partial}_A, \Phi)$, and thus I acts as multiplication by i on the derivative of $\bar{\partial}_A + \Phi$, in other words the complex varies holomorphically with respect to I. From Lemmas 2.1.4 and 2.4.1 the complex is exact with cohomology concentrated in degree one. Furthermore, it is split by the Green's operator for $(\bar{\partial}_A + \Phi)(\partial_A + \Phi^*)$. The complex is therefore a so-called infinite dimensional monad and from general theory [23, Section 3.1.3] the cohomology defines a Hermitian holomorphic bunde on $\mathscr{M}^{st}$. The bundle $\mathbb{D}$ therefore has a holomorphic structure with respect to I with which ∇ is compatible.

For the complex structure J consider the de Rham complex for the family of flat connections $d_A + \Phi + \Phi^*$,

$$\Omega^0(E) \xrightarrow{d_A+\Phi+\Phi^*} \Omega^1(E) \xrightarrow{d_A+\Phi+\Phi^*} \Omega^2(E).$$

As the complex structure J is multiplication by i on $\dot{\beta} - \dot{\beta}^* + \dot{\Phi} + \dot{\Phi}^*$, the complex varies holomorphically with respect to J. By Proposition 2.4.4 the family is an infinite dimensional monad, and furthermore that $\mathbb{D}$ has a holomorphic structure with respect to J with which ∇ is compatible.

The argument for the complex structure K is equivalent to that for J, but instead consider the family of complexes with differentials $d_A - i\Phi + i\Phi^*$ as K is multiplication by i on $\dot{\beta} - \dot{\beta}^* - i\dot{\Phi} + i\dot{\Phi}^*$. It follows from Remark 2.4.5 that the cohomology is concentrated in degree one and that $\mathbb{D}$ has a holomorphic structure with respect to K with which ∇ is compatible. □

2.7 Example

A more general type of Higgs bundle is defined via principal bundles for complex Lie groups. For G a complex Lie group a principal G-Higgs bundle is a pair (P, Φ) of a principal G-bundle P and Φ a section of $\operatorname{ad} P \otimes K$ where $\operatorname{ad} P$ is the vector bundle associated to the adjoint representation of G. If we assume G is a matrix group $G \subset \mathrm{GL}(n, \mathbb{C})$ and let $E = P \times_G \mathbb{C}^n$ be the vector bundle associated to P by the vector representation we recover the definition of Higgs bundles as pairs (E, Φ).

The ordinary Dirac-operator on a manifold with spin structure can be seen as an operator $\mathcal{D} : C^\infty(S^+) \to C^\infty(S^-)$ where $S^\pm$ are the spinor bundles. In Remark 2.1.3 we noted that the Dirac–Higgs operator is the ordinary Dirac-operator coupled to a Higgs bundle and that we can see it as an operator $\mathcal{D}_{(E,\Phi)} : C^\infty(S^+ \otimes E) \to C^\infty(S^- \otimes E)$. From this perspective it is clear that given any representation V of G we can couple the Dirac–Higgs operator to this representation $\mathcal{D}_V : C^\infty(S^+ \otimes V) \to C^\infty(S^- \otimes V)$. One natural example to consider is the Lie algebra $V = \mathrm{Lie}(G)$ with G acting by the adjoint representation. In this case, the Dirac–Higgs bundle is the tangent bundle of the moduli space along with the Levi-Civita connection.

2.8 Nahm transform from $\mathbb{C}$ to $\mathbb{C}$

In Chapter 3 we shall use the Dirac–Higgs bundle to construct a Nahm transform for Higgs bundles on a compact Riemann surface Σ. Before doing that, we extend the construction of the toy model to Higgs line bundles on $\mathbb{C}$ with polynomial Higgs field of degree k, and use it to construct a Nahm transform for these Higgs bundles. The transform will produce Higgs bundles on a 'dual' $\mathbb{C}$. This Nahm transform fits into the framework of a transform between solutions to the anti-self-duality equations on $\mathbb{R}^4$ which are translation invariant in n directions to those which are translation invariant in $4-n$ directions. This was formalised by Corrigan and Goddard [20], who also conjectured the existence of the transform when $n = 2$. Obtaining such a transform was one of Hitchin's original motivations for studying the Higgs bundle equations. The following is based on the validity of Conjecture 2.2.4

Firstly, observe from Section 2.3, and the the explicit computation (2.6) that the Dirac–Higgs connection is independent of v. The connection is furthermore an instanton, and as it is independent of v it descends to a rank one solution of the Higgs bundle equations on the copy of $\mathbb{C}$ parametrising deformations of the Higgs field. In this simple case, equation (2.6) also explicitly shows that the Higgs field is $-udu$.

We consider the same deformations for a Higgs bundle $(\mathcal{O}, \varphi dz)$ with φ a polynomial of degree k, that is a family of Higgs bundles

$$(\bar{\partial} + vd\bar{z}, (\varphi + u)dz) \tag{2.9}$$

parametrised by $(u, v) \in \mathbb{C}^2$.

If Conjecture 2.2.4 is true, the construction of the toy model in Section 2.3 immediately generalise to give a Dirac–Higgs bundle $(\mathbb{D}, \nabla)$ on $\mathbb{C}^2$ of rank k. That the connection is an instanton follows from the general argument in Theorem 2.6.3. Following the outline above, if the connection is independent of v, it descends to a Higgs pair on $\mathbb{C}$ parametrising u, and solves the Higgs bundle equations.

Writing $\psi_1 dz + \psi_2 d\bar{z} \in \Omega^1$, a solution to the Dirac–Higgs equations for the family (2.9)

$$0 = \bar{\partial}(\psi_1 dz) + v\psi_1 d\bar{z} \wedge dz - (\varphi + u)\psi_2 d\bar{z} \wedge dz$$
$$0 = \partial(\psi_2 d\bar{z}) - \bar{v}\psi_2 dz \wedge d\bar{z} - (\bar{\varphi} + \bar{u})\psi_1 dz \wedge d\bar{z}$$

can be written in the form

$$\psi_1 = e^{-v\bar{z}+\bar{v}z}s_1 \quad \text{and} \quad \psi_2 = e^{-v\bar{z}+\bar{v}z}s_2 \tag{2.10}$$

where $s_1dz + s_2d\bar{z}$ solves the Dirac–Higgs equations for $(\mathcal{O}, (\varphi + u)dz)$

$$0 = \bar{\partial}(s_1dz) - (\varphi + u)s_2d\bar{z} \wedge dz$$
$$0 = \partial(s_2d\bar{z}) - (\bar{\varphi} + \bar{u})s_1dz \wedge d\bar{z}.$$

The solutions $s_1dz + s_2d\bar{z}$ are independent of v and the solutions $\psi_1dz + \psi_2d\bar{z}$ depend on v as specified in (2.10). In other words, a section of $\mathbb{D}$ is of the form

$$\psi(u, v, z) = e^{-v\bar{z}+\bar{v}z}s(u, z)$$

where for each u the section $s(u, z)$ is a 1-form solving the Dirac–Higgs equations for $(\mathcal{O}, (\varphi + u)dz)$. The derivatives of ψ are

$$\partial_v\psi = -\bar{z}e^{-v\bar{z}+\bar{v}z}s \quad \partial_{\bar{v}}\psi = ze^{-v\bar{z}+\bar{v}z}s \quad \partial_u\psi = e^{-v\bar{z}+\bar{v}z}\partial_u s \quad \partial_{\bar{u}}\psi = e^{-v\bar{z}+\bar{v}z}\partial_{\bar{u}}s. \tag{2.11}$$

Assume $\{\eta_1, \ldots, \eta_k\}$ is a local frame for $\mathbb{D}$ and $\eta_i = e^{-v\bar{z}+\bar{v}z}\rho_i$. It then follows directly from (2.11) that the covariant derivatives are independent of v, e.g.

$$\langle\partial_v\psi, \eta_i\rangle = \int_{\mathbb{C}} -\bar{z}s(u, z)\overline{\rho(u, z)}_i \,\mathrm{dVol} \quad \text{and} \quad \langle\partial_u\psi, \eta_i\rangle = \int_{\mathbb{C}} \partial_u s(u, z)\overline{\rho(u, z)}_i \,\mathrm{dVol}\,.$$

This shows that the Dirac–Higgs connection ∇ is pulled back from $\mathbb{C}$ where it solves the Higgs bundle equations.

Remark 2.8.1. As mentioned above, Corrigan and Goddard [20] discusses a Nahm transform for Higgs line bundles. It is interesting to notice that their formula (5.16) for the connection and Higgs field are the same as those which immediately follow from (2.11).

Assuming Conjecture 2.2.4 is true, the above gives a Nahm transform taking rank one Higgs bundles $(\mathcal{O}, \varphi dz)$ on $\mathbb{C}$ with φ a polynomial of degree k, to a rank k Higgs bundles $(\mathcal{O}^k, \hat{\varphi}du)$ on $\mathbb{C}$. The construction of a Higgs line bundle given by the toy model is clearly invertible. An inverse Nahm transform should produce a Higgs line bundle with polynomial Higgs field from $(\mathcal{O}^k, \hat{\varphi}du)$. This puts constraints on what type of rank k Higgs bundles can be in the image our Nahm transform. The solutions of $\varphi(z) + u = 0$ on $\mathbb{C}^2$ parametrising (z, u) is a spectral curve, which for fixed u gives the k eigenvalues of $\hat{\varphi}(u)$. This spectral curve remain fixed under the transform, that is it can also be seen as the solutions of $\det(\hat{\varphi}(u) + z\,\mathrm{Id})$. For fixed z there must be exactly one u where $-z$ is an eigenvalue. As this should be true for all $z \in \mathbb{C}$ the determinant $\det\hat{\varphi}$ must be a degree one polynomial in u, and all other coefficients of the characteristic polynomial of $\hat{\varphi}$ must be constants. By counting parameters with these conditions on the Higgs field we get a

$k+1$-dimensional space of potential $\hat{\varphi}$'s, exactly the same as the dimension of the space of degree k-polynomials parametrising φ. We expect these rank k Higgs fields to be the image of the Nahm transform.

Compactified to $\mathbb{P}^1 \times \mathbb{P}^1$, the spectral curve is the zero locus of a section of $\mathcal{O}(k,1)$. Additionally, the zero locus only intersects either of the divisors at infinity at (∞,∞). As a $k:1$-branched cover of $\mathbb{P}^1$ the spectral curve is branched at ∞ to order k.

A very similar situation to the Nahm transform above was studied by Szabó [72], giving a Nahm transform for a certain type of parabolic Higgs bundles on $\mathbb{P}^1$ with a double pole at infinity and otherwise simple poles. Szabó's Nahm transform produces parabolic Higgs bundles of the same type. Szabó proves that the transform is invertible, and that there is a spectral curve mediating the transform.

3. Nahm transform for Higgs bundles

Let Σ be a Riemann surface of genus $g \geq 2$, and let $T^*J \simeq J \times H^0(K)$ be the cotangent bundle of the Jacobian of Σ. Given the standard group structure $J \times H^0(K)$ acts on $\mathscr{M}^{st}$ the moduli space of stable Higgs bundles of degree zero and rank at least two by

$$(\xi, \alpha) \cdot (E, \Phi) = (EL_\xi, \Phi + \alpha \operatorname{Id})$$

where L_ξ is the degree zero line bundle on Σ corresponding to the point $\xi \in J$ and $\alpha \in H^0(K)$. Denote by $\mathcal{N}_{(E,\Phi)} : J \times H^0(K) \to \mathscr{M}^{st}$ the orbit map defined by (E, Φ).

Definition 3.0.2. The *Nahm transform* of a Higgs bundle (E, Φ) of degree zero and rank at least two is the pull-back of the Dirac–Higgs bundle $(\mathbb{D}, \nabla)$ by the orbit map $\mathcal{N}_{E,\Phi}$ to a bundle with connection on $J \times H^0(K)$ denoted $(\hat{E}, \hat{\nabla})$.

The holomorphic version of the Nahm transform, called the Fourier–Mukai transform, is discussed to great extent in Chapters 4 and 5, and was first defined by Bonsdorff [14]. The analytical version constructed above, is discussed by Frejlich and Jardim [24], reproving main theorems from [14] by differential geometric methods.

An important property of the Nahm transform is that $\hat{E}$ considered as a holomorphic bundle on $J \times H^0(K)$ with the holomorphic structure induced by the complex structure I (see Section 1.1.3) extends to a holomorphic bundle (also denoted $\hat{E}$) on $\mathbb{P}(T^*J \oplus \mathcal{O}) \simeq J \times \mathbb{P}^g$. The main theorem of [14] is that the association of a holomorphic bundle on $J \times \mathbb{P}^g$ to a stable degree zero Higgs bundle of rank at least two is injective. However, the essential image of this association is not known. The Higgs bundles produced in Section 2.8 from Higgs line bundles on $\mathbb{C}$ with polynomial Higgs fields are quite simple, and it is possible to identify the image of the transform, or rather identify the boundary conditions on the Dirac–Higgs connection. When we extend to compact Riemann surfaces we generalise these boundary conditions. Determining the boundary conditions for $\hat{\nabla}$ should identify the essential image of Bonsdorff's Nahm transform.

The main motivation for this chapter is to shed a bit of light on the asymptotics of the connection. In this regard, the important parts of the cotangent bundle are the non-compact fibres. We therefore fix a point $\xi \in J$ and consider the Nahm transform restricted to the fibre of T^*J at ξ. In the remaining part of this chapter, $(\hat{E}, \hat{\nabla})$ denotes the Nahm transform of (E, Φ) restricted to the fibre over $\xi \in J$.

Throughout this chapter we fix $\alpha \in H^0(K)$ and restrict attention to the one-parameter family of Higgs bundles $(EL_\xi, \Phi + t\alpha \operatorname{Id})$, $t \in \mathbb{R}$, given by a Higgs bundle (E, Φ). If (E, Φ) is stable and of degree zero and rank at least two, the same is true for the whole one-parameter family. Furthermore, if h is the Hermitian metric on the holomorphic bundle EL_ξ such that the Chern connection satisfies the Higgs bundle equations, then the same metric solves the Higgs bundle equations for every other member of the one-parameter family as

$$[\Phi + t\alpha\,\mathrm{Id}, \Phi^* + t\bar{\alpha}\,\mathrm{Id}] = [\Phi, \Phi^*].$$

We denote by $\mathcal{D}_t$ the associated Dirac–Higgs operator.

In Section 3.1 we use an adaptation of Witten's proof of the holomorphic Morse inequalities to prove that any t-sequence of elements of $\ker \mathcal{D}_t^*$ with bounded L^2-norm vanish away from the zeros of α as $t \to \infty$.

In Section 3.2 we discuss the behaviour around the zeros of α. We consider a model solution to the Dirac–Higgs equations and show that under mild natural conditions the distributional limit of a sequence of solutions is a delta-function. We conjecture that this is true for all local solutions. Working under the assumption that the conjecture is true, the localisation can be enhanced to a distributional convergence to a sum of delta-functions supported at the zeros of α.

Throughout this chapter we let ω be a fixed Kähler form on Σ given by choosing a metric in the conformal class.

3.1 Localisation

Let (A, Φ) be a Higgs pair on a Hermitian vector bundle (E, h). Fix $\alpha \in H^0(K)$ and denote by $\mathcal{D}_t$ the Dirac–Higgs operator for the family $(A, \Phi + t\alpha\,\mathrm{Id})$, $t \in \mathbb{R}$. A detailed account of Witten's proof of the holomorphic Morse inequalities [77] is given in [37]. We use similar methods to understand the elements of $\ker \mathcal{D}_t^*$ as t tends to infinity.

Theorem 3.1.1. *Let (A, Φ) be a Higgs pair on a Hermitian vector bundle (E, h). Fix $\alpha \in H^0(K)$ and $C \subset \Sigma$ a compact subset of Σ not containing zeros of α. Then there exists a constant m depending on C, α, (E, h), (A, Φ), and Σ such that for t large and all $\psi \in \ker D_t^*$*

$$\int_C h(\psi, \psi)\omega \leq \frac{m}{t} \|\psi\|_h^2.$$

In this section it is more appropriate to use the definition of the adjoint Dirac–Higgs operator requiring the choice of a metric in the conformal class on Σ, cf. Remark 2.1.2. Recall the definition of the Dirac–Higgs operator $\mathcal{D}_t$ and its adjoint $\mathcal{D}_t^*$

$$\mathcal{D}_t = \begin{pmatrix} \partial_A & -\Phi - t\alpha\,\mathrm{Id} \\ \Phi^* + t\bar{\alpha}\,\mathrm{Id} & -\bar{\partial}_A \end{pmatrix} \quad \text{and} \quad \mathcal{D}_t^* = \begin{pmatrix} i\Lambda\bar{\partial}_A & i\Lambda\Phi + it\Lambda(\alpha\,\mathrm{Id}) \\ i\Lambda\Phi^* + it\Lambda(\bar{\alpha}\,\mathrm{Id}) & i\Lambda\partial_A \end{pmatrix}.$$

Denote by $\Phi_t = \Phi + t\alpha\,\mathrm{Id}$, then for every $\psi \in \ker \mathcal{D}_t^*$

$$0 = \mathcal{D}_t\mathcal{D}_t^*\psi = \begin{pmatrix} i\partial_A(\Lambda\bar{\partial}_A\psi_1) + i\partial_A(\Lambda(\Phi_t\psi_2)) - i\Phi_t\Lambda(\Phi_t^*\psi_1) - i\Phi_t\Lambda(\partial_A\psi_2) \\ i\Phi_t^*\Lambda(\bar{\partial}_A\psi_1) + i\Phi_t^*\Lambda(\Phi_t\psi_2) - i\bar{\partial}_A(\Lambda(\Phi_t^*\psi_1)) - i\bar{\partial}_A\Lambda\partial_A\psi_2 \end{pmatrix},$$

where $\psi = \psi_1 + \psi_2$ is the decomposition into $(1,0)$ and $(0,1)$-parts.

Taking the inner product with ψ

$$\begin{aligned} 0 = \langle \mathcal{D}_t \mathcal{D}_t^* \psi, \psi \rangle_h = \left\langle \bar{\partial}_A^* \bar{\partial}_A \psi_1, \psi_1 \right\rangle_h + \langle \partial_A^* \partial_A \psi_2, \psi_2 \rangle_h - \langle i\Phi_t \Lambda(\Phi_t^* \psi_1), \psi_1 \rangle_h \\ + \langle i\Phi_t^* \Lambda(\Phi_t \psi_2), \psi_2 \rangle_h + \left\langle \Phi_t \bar{\partial}_A^* \psi_2 + \bar{\partial}_A^* (\Phi_t \psi_2), \psi_1 \right\rangle_h \\ + \langle \partial_A^* (\Phi_t^* \psi_1) + \Phi_t^* \partial_A^* \psi_1, \psi_2 \rangle_h . \end{aligned} \tag{3.1}$$

The first two terms of the above are

$$\left\langle \bar{\partial}_A^* \bar{\partial}_A \psi_1, \psi_1 \right\rangle_h = \|\bar{\partial}_A \psi_1\|_h^2 = \|\bar{\partial}_A \psi\|_h^2 \quad \text{and} \quad \langle \partial_A^* \partial_A \psi_2, \psi_2 \rangle_h = \|\partial_A \psi_2\|_h^2 = \|\partial_A \psi\|_h^2.$$

We define the following operators

$$G(\psi, \Phi) = \Phi \bar{\partial}_A^* \psi + \bar{\partial}_A^* (\Phi \psi) + \partial_A^* (\Phi^* \psi) + \Phi^* \partial_A^* \psi \tag{3.2}$$

$$H(\psi, \Phi) = -i\Phi \Lambda(\Phi^* \psi) + i\Phi^* \Lambda(\Phi \psi). \tag{3.3}$$

Notice that G in some sense records the effect of not having a Leibniz rule for the adjoint differentials $\bar{\partial}_A^*$ and ∂_A^*. Using these operators (3.1) is

$$0 = \langle \mathcal{D}_t \mathcal{D}_t^* \psi, \psi \rangle_h = \|\partial_A \psi\|_h^2 + \|\bar{\partial}_A \psi\|_h^2 + \langle G(\psi, \Phi_t), \psi \rangle_h + \langle H(\psi, \Phi_t), \psi \rangle_h . \tag{3.4}$$

To prove Theorem 3.1.1 the following lemmas concerning properties of the operators defined above are needed.

Lemma 3.1.2. *The operators G and H defined in* (3.2) *and* (3.3)*, respectively, are $C^\infty(\Sigma)$-linear in the first entry.*

Proof. It is clear that H is $C^\infty(\Sigma)$-linear in the first entry as multiplication by Φ is a zero order differential operator. But as G is a first order differential operator a bit more effort is required. The first two terms of $G(f\psi, \Phi)$ are:

$$-i\Phi \Lambda \partial_A (f\psi) = -i\Phi \Lambda(\partial(f)\psi) - fi\Phi \Lambda \partial_A(\psi)$$

$$i\partial_A \Lambda(\Phi f \psi) = i\partial(f)\Lambda(\Phi\psi) + fi\partial_A \Lambda(\Phi\psi).$$

As $\Phi\Lambda(\partial(f)\psi) = \partial(f)\Lambda(\Phi\psi)$ the first terms in each line above cancel and the sum is thus $C^\infty(\Sigma)$-linear. In the same way, the sum of the last two terms in G are $C^\infty(\Sigma)$-linear. □

It is clear that G is complex linear in the second entry, so

$$G(\psi, \Phi_t) = G(\psi, \Phi) + tG(\psi, \alpha).$$

Expanding $H(\psi, \Phi_t)$ gives

$$\begin{aligned} H(\psi, \Phi_t) &= H(\psi, \Phi) + t^2(-i\alpha\Lambda(\bar{\alpha}\psi) + i\bar{\alpha}\Lambda(\alpha\psi)) \\ &\quad + it(-\Phi\Lambda(\bar{\alpha}\psi) - \alpha\Lambda(\Phi^*\psi) + \Phi^*\Lambda(\alpha\psi) + \bar{\alpha}\Lambda(\Phi\psi)) \\ &= H(\psi, \Phi) + itF(\psi, \alpha, \Phi) + t^2 H(\psi, \alpha) \end{aligned}$$

where $F(\psi, \alpha, \Phi)$ by definition is the coefficient of t,

$$F(\psi, \alpha, \Phi) = -\Phi\Lambda(\bar{\alpha}\psi) - \alpha\Lambda(\Phi^*\psi) + \Phi^*\Lambda(\alpha\psi) + \bar{\alpha}\Lambda(\Phi\psi).$$

Lemma 3.1.3.

$$\langle H(\psi, \alpha), \psi\rangle_h = \|\alpha\psi\|_h^2 + \|\bar{\alpha}\psi\|_h^2.$$

Proof. The proof is a simple calculation. □

Lemma 3.1.4. *Let $C \subset \Sigma$ be a compact set avoiding all zeros of α, and let $\psi \in \ker \mathcal{D}_t^*$. Then there exists $\varepsilon > 0$ such that*

$$0 \le \varepsilon \int_C h(\psi_1, \psi_1)\omega \le \|\alpha\psi\|_h^2 \quad \textit{and} \quad 0 \le \varepsilon \int_C h(\psi_2, \psi_2)\omega \le \|\bar{\alpha}\psi\|_h^2.$$

Proof. Locally $\alpha = adz$, and $\psi = \psi_1 + \psi_2 = \varphi_1 dz + \varphi_2 d\bar{z}$ is a vector valued differential form. In a local neighbourhood, the Kähler form on Σ is $2ifdz \wedge d\bar{z}$ with f a positive real-valued function. In that case,

$$h(\alpha\psi_2, \alpha\psi_2)\omega = 2if|a|^2 h(\varphi_2, \varphi_2) dz \wedge d\bar{z}.$$

Furthermore, on a compact set C not containing the zeros of α this is bounded below by a constant $\varepsilon > 0$. Thus

$$0 \le \varepsilon \int_C h(\psi_2, \psi_2)\omega \le \int_C h(\alpha\psi_2, \alpha\psi_2)\omega \le \int_\Sigma h(\alpha\psi_2, \alpha\psi_2)\omega = \|\alpha\psi_2\|_h^2 = \|\alpha\psi\|_h^2.$$

The other case follows by similar considerations. □

Proof of Theorem 3.1.1. Let $\psi \in \ker \mathcal{D}_t^*$. Then equation (3.4), the expansion of $H(\psi, \Phi_t)$, and Lemma 3.1.3 gives

$$\begin{aligned} \|\partial_A\psi\|_h^2 + &\|\bar{\partial}_A\psi\|_h^2 + t^2(\|\alpha\psi\|_h^2 + \|\bar{\alpha}\psi\|_h^2) \\ &= |\langle G(\psi, \Phi), \psi\rangle_h + \langle H(\psi, \Phi), \psi\rangle_h + t\,\langle G(\psi, \alpha), \psi\rangle_h + t\,\langle iF(\psi, \alpha, \Phi), \psi\rangle_h|. \end{aligned}$$

By Lemma 3.1.2 all operators on the right hand side are $C^\infty(\Sigma)$-linear in the first entry, meaning that the operators are endomorphisms of the fibers of $\wedge^1 T^*\Sigma \otimes E$. Since each of the operators at every point of Σ is a $C^\infty(\Sigma)$-linear operator of the fibre they are pointwise bounded, e.g.

$$h(G_z(\psi(z), \Phi_z), \psi(z)) \le \|G_z(\Phi_z)\|_h^2\, h(\psi(z), \psi(z)).$$

The operator norm of $G_z(-, \Phi_z)$ is continuous in z and since Σ is compact $G_z(-, \Phi_z)$ as a function of z is bounded by a constant $M > 0$ depending on (A, Φ) and h but independent of t, thus

$$\|G(\psi, \Phi)\|_h \le M\|\psi\|_h.$$

The same is true for the other three operators. For all $\psi \in \Omega^1(E)$

$$0 \leq \|\partial_A \psi\|_h^2 + \|\bar{\partial}_A \psi\|_h^2,$$

and with the constants from above, the Cauchy–Schwartz inequality gives

$$\begin{aligned} t^2(\|\alpha\psi\|_h^2 + \|\bar{\alpha}\psi\|_h^2) &\leq (\|G(\psi,\Phi)\|_h + \|H(\psi,\Phi)\|_h + t\|G(\psi,\alpha)\|_h + t\|F(\psi,\alpha,\Phi)\|_h)\|\psi\|_h \\ &\leq (M_1 + tM_2)\|\psi\|_h^2, \end{aligned}$$

where M_1, M_2 depend on (A, Φ) and h but are independent of t.

From Lemma 3.1.4 the t^2-term on the left hand side is bounded from below by the integral over a compact subset $C \subset \Sigma$ avoiding zeroes of α:

$$\|\alpha\psi\|_h^2 + \|\bar{\alpha}\psi\|_h^2 \geq \varepsilon \int_C h(\psi,\psi)\omega.$$

Combining the above,

$$\int_C h(\psi,\psi)\omega \leq \left(\frac{m}{t} + \frac{m'}{t^2}\right)\|\psi\|_h^2$$

where $m = \frac{M_2}{\varepsilon}$ and $m' = \frac{M_1}{\varepsilon}$ depend on (A, Φ) and h but are independent of t. For large t the first term is dominating. □

Remark 3.1.5. It is clear from the proof of Theorem 3.1.1 that the independence of the Hermitian metric of the parameter t is crucial. If the Hermitian metric depended on t the constants used to bound operators G, F and H would depend on t and a detailed analysis of this dependence would be required.

3.2 Distributional behaviour of solutions to Dirac–Higgs equations

In this section we discuss the distributional properties of solutions to the Dirac–Higgs equations for Higgs pairs on a Hermitian line bundle. We first consider the model Higgs pairs $(0, \varphi dz + t\alpha dz)$ where φ, α are degree one polynomials on $\mathbb{C}$. We study the behaviour of the elements of $\ker \mathcal{D}_t^*$ as t tends to infinity. This is a model for the local behaviour of elements of $\ker \mathcal{D}_t^*$ for Higgs pairs $(A, \Phi + t\alpha)$ on a Hermitian line bundle (L, h) on a Riemann surface of genus $g \geq 2$. We conjecture that the behaviour is the same.

Proposition 3.2.1. *Let $\varphi_t = \varphi + t\alpha$ be a degree one polynomial on $\mathbb{C}$ with $\alpha = a(z - z_\infty)$ and $\varphi = b(z - z_0)$ for $a, b \in \mathbb{C}$ and $z_0, z_\infty \in \mathbb{C}$. Write $\psi_t \in \ker \mathcal{D}^*_{0,\varphi_t dz}$ on $\mathbb{C}$ as $\psi_t = \psi_t^1 dz + \psi_t^2 d\bar{z}$. Let z_t be the zero of φ_t. If the sequence of complex numbers $\frac{\pi}{|a|t}\psi_t^1(z_t)$ converges to $\lambda \in \mathbb{C}$, then ψ_t converges as a distribution to $T_\lambda \in (\Omega^1)^*$ defined as*

$$T_\lambda\big(\eta_1(z)dz + \eta_2(z)d\bar{z}\big) = \lambda(\bar{\eta}_1(z_\infty) - \bar{\eta}_2(z_\infty)).$$

Proof. As $\varphi_t = \varphi + t\alpha$ is a degree one polynomial we know from Lemma 2.2.3 that there is a one dimensional space of global solutions to the Dirac–Higgs equations for the pair $(0, \varphi_t dz)$. The Dirac–Higgs equations are

$$0 = \partial_{\bar{z}}\psi_t^1 - \varphi_t\psi_t^2 \qquad \text{and} \qquad 0 = \partial_z\psi_t^2 - \varphi_t\psi_t^1.$$

Denote by $\hat{\Phi}_t$ an antiderivative of $2\varphi_t$ with respect to ∂_z vanishing to order two at z_t. It is now easy to see that

$$\psi_t^1 = c_t e^{-|\hat{\Phi}_t|} \qquad \text{and} \qquad \psi_t^2 = -\psi_t^1 \qquad \text{with} \qquad c_t \in \mathbb{C}$$

are the only global L^2-solution to the Dirac–Higgs equations.

As $\alpha(z) = a(z - z_\infty)$ and $\varphi(z) = b(z - z_0)$ with $a, b \in \mathbb{C}$, then $\hat{\Phi}_t = \frac{1}{at+b}(\varphi + t\alpha)^2$. It is now easy to check that $\frac{1}{at}\hat{\Phi}_t$ converges as a polynomial to $(z - z_\infty)^2$. For a test function η we therefore have

$$\lim_{t\to\infty}\int_{\mathbb{C}} \psi_t^1 \bar{\eta}\, \mathrm{dVol} = \lim_{t\to\infty}\int_{\mathbb{C}} c_t e^{-t|a||z-z_\infty|^2}\bar{\eta}\, \mathrm{dVol}\,.$$

By the assumption on the convergence of $\psi_t^1(z_t)$, the sequence $c_t \in \mathbb{C}$ is of the form $c_t = \frac{t|a|}{\pi}\lambda_t$ with λ the limit of λ_t. It is well-known that $\frac{t|a|}{\pi}e^{-t|a||z-z_\infty|^2}$ converges as a distribution to the delta-function supported at z_∞ and thus

$$\lim_{t\to\infty}\int_{\mathbb{C}} \psi_t^1\bar{\eta}\, \mathrm{dVol} = \lambda\bar{\eta}(z_\infty),$$

where λ is the limit of λ_t. The above shows that $\psi_t = \psi_t^1 dz + \psi_t^2 d\bar{z}$ converges as a distribution to $T_\lambda \in (\Omega^1)^*$ defined as

$$T_\lambda\big(\eta_1(z)dz + \eta_2(z)d\bar{z}\big) = \lambda(\bar{\eta}_1(z_\infty) - \bar{\eta}_2(z_\infty)),$$

proving the proposition. $\square$

For the general situation let (A, Φ) be a Higgs pair on a Hermitian line bundle (L, h) with A a flat connection and $\alpha \in H^0(K)$. Let z_0 be a simple zero of α. Choose a neighbourhood U of z_0 such that $\alpha = zdz$ and $\Phi = \varphi dz$, and choose a unitary frame f for (L, h). For t sufficiently large there is only one zero of $\varphi + tz$ in U for each t, denote this zero by z_t. Write $\psi_t \in \ker \mathcal{D}^*_{A,\Phi+t\alpha}$ on U as

$$\psi_t = \psi_t^1 f dz + \psi_t^2 f d\bar{z}.$$

Conjecture 3.2.2. *Let (A, Φ), $\alpha \in H^0(K)$, $\psi_t \in \ker \mathcal{D}_{A,\Phi+t\alpha}$ and (U, z_0) be as above. In the above notation, if the sequence of complex numbers $\frac{\pi}{t}\psi_t^1(z_t)$ converges to $\lambda \in \mathbb{C}$, then $\psi_t|_U$ converges as a distribution to $T_\lambda \in (\Omega^1(L|_U))^*$ defined as*

$$T_\lambda\big(a(z)f dz + b(z)f d\bar{z}\big) = \lambda(\bar{a}(0) - \bar{b}(0)).$$

Remark 3.2.3. On the bounded disk U the Dirac–Higgs equations for $(A, \varphi_t dz)$ has more L^2-solutions than just the radial solution from Lemma 2.2.3 centered at the zero of φ_t in U. If there were only the one solution we could use arguments similar to that of Proposition 3.2.1 to obtain convergence.

Based on the above conjecture, we extend the result to a global statement for rank one Higgs bundles. First, we establish some notation. Let z be a point on Σ and $\lambda \in (LK_z)^*$ a functional on the fibre of LK at z. If we consider K as $\wedge^{1,0}T^*\Sigma$ we denote by $\bar{\lambda}$ the complex conjugate of the form part of λ so that $\bar{\lambda} \in (\wedge^{0,1}T^*_z\Sigma \otimes L_z)^* = (L\bar{K}_z)^*$.

If (E, Φ) is a stable Higgs bundle with $\det \Phi$ having simple zeros $z_1, \ldots, z_N$, and (A, Φ) is the corresponding Higgs pair solving the Higgs bundle equations and h the corresponding Hermitian metric, we can extend the map ρ' from Lemma 2.4.3 identifying the hypercohomology with a sum of cokernels to

$$\rho : \ker \mathcal{D}^*_{A,\Phi} \to \bigoplus_{i=1}^{2r(g-1)} V_i \quad \text{as} \quad \rho(\psi) = (h(\psi_1(z_1)), \ldots, h(\psi_1(z_N))) \tag{3.5}$$

where $V_i \subset (EK_{z_i})^*$ is a one-dimensional subspace of the dual space of the fibre of EK at z_i, and ψ_1 is the $(1,0)$-part of a solution ψ to the Dirac–Higgs equations.

Theorem 3.2.4. *Let (A, Φ) be a Higgs pair on a Hermitian line bundle (L, h) with A a flat connection, and $\alpha \in H^0(K)$ having only simple zeros. Let $\psi_t \in \ker \mathcal{D}^*_t$ with $\|\psi_t\|_h = 1$ for all t and such that the sequence $\frac{\pi}{t}\rho(\psi_t) \in \bigoplus_{i=1}^{2g-2}(LK_{z_i(t)})^*$ converges to $\lambda = (\lambda_1, \ldots, \lambda_{2g-2}) \in \bigoplus_{i=1}^{2g-2}(LK_{x_i})^*$ when $t \to \infty$ where $z_i(t)$ are the zeros of $\Phi + t\alpha$, then ψ_t converges as a distribution to*

$$\sum_{i=1}^{2g-2} \lambda_i \delta_{x_i} - \bar{\lambda}_i \delta_{x_i}$$

where $x_1, \ldots, x_{2g-2}$ are the zeros of α.

Proof. Let $x_1, \ldots, x_{2g-2}$ be the zeros of α and let U_i be small disks around each x_i not containing other zeros of α. Let C be the complement of the union of the U_i's in Σ, $C = \Sigma \setminus \cup_{i=1}^{2g-2} U_i$. Let $\eta \in \Omega^1(L)$ be a test-form. From Theorem 3.1.1 it follows that

$$\Big| \int_C h(\psi_t, \eta)\omega \Big| \leq \int_C |\psi_t|_h^2 \omega \int_C |\eta|_h^2 \omega \leq \frac{m}{t} \int_C |\eta|_h^2 \omega$$

and thus the limit of $\langle \psi_t, \eta \rangle_h$ is determined by local considerations around the zeros of α.

By the assumptions on the convergence of $\frac{\pi}{t}\rho(\psi_t)$ the result now follows from Conjecture 3.2.2. □

Remark 3.2.5. The condition in Theorem 3.2.4 that $\frac{\pi}{t}\rho(\psi_t)$ converges as $t \to \infty$ is not implied by requiring $\|\psi_t\|_h = 1$ for all t. If the latter condition is satisfied we only know that $\frac{1}{t}\rho(\psi_t)$ has a convergent subsequence.

Theorem 3.1.1 is valid for any Higgs pair on a Hermitian vector bundle (E, h), also for higher rank. It is therefore natural to conjecture that the generalisation of Theorem 3.2.4 to higher rank is also true.

Conjecture 3.2.6. *Let* (A, Φ) *be an irreducible Higgs pair on a Hermitian vector bundle* (E, h) *satisfying the Higgs bundle equations, and* $\alpha \in H^0(K)$ *having only simple zeros. Let* $\psi_t \in \ker \mathcal{D}_t^*$ *with* $\|\psi_t\|_h = 1$ *for all* t *and such that the sequence* $\frac{\pi}{t}\rho(\psi_t) \in \bigoplus_{i=1}^N (EK_{z_i(t)})^*$ *converges to* $\lambda = (\lambda_1, \ldots, \lambda_N) \in \bigoplus_{i=1}^N (EK_{z_i})^*$ *when* $t \to \infty$ *where* $z_i(t)$ *are the zeros of* $\det(\Phi + t\alpha)$*, then* ψ_t *converges as a distribution to*

$$\sum_{i=1}^{N} \lambda_i \delta_{z_i} - \bar{\lambda}_i \delta_{z_i}$$

where $z_1, \ldots, z_N$ *are the zeros of* $\alpha^{\operatorname{rk} E}$ *repeated with multiplicity, and* $N = 2 \operatorname{rk} E(g-1)$*.*

Remark 3.2.7. If Conjecture 3.2.2 is true we can use the cokernel description of the hypercohomology to define a frame, which in the limit is unitary with respect to the L^2-metric on the Nahm transform. We discuss this in Chapter 7.

4. Fourier–Mukai transform for Higgs bundles

In this chapter we discuss the holomorphic aspects of the Nahm transform for Higgs bundles. We follow Bonsdorff's [14] naming convention and denote it the Fourier–Mukai transform. We do this to distinguish the holomorphic from the analytical Nahm transform of Chapter 3.

Bonsdorff's Fourier–Mukai transform [14] is described completely in algebraic terms. It is a holomorphic vector bundle on the cotangent bundle of the Jacobian T^*J which by Serre duality is $J \times H^0(K)$. The fibre of a transformed Higgs bundle is given by the first hypercohomology group of a certain complex associated to the point in $J \times H^0(K)$. In [14], the transformed bundle is shown to extend to a holomorphic bundle on the natural compactification $J \times \mathbb{P}^g$. As this chapter concerns solely holomorphic properties of the transformed bundle we denote the extension to $J \times \mathbb{P}^g$ the Fourier–Mukai transform. In [14], it is called *the Total Fourier transform.* This chapter builds on top of the foundations laid by Bonsdorff [14]. The two spectral sequences converging to the hypercohomology of a complex of sheaves discussed in Section 1.2 is used to provide new features of the Fourier–Mukai transform.

This chapter is organised as follows. In Section 4.1 we give the definition of the Fourier–Mukai transform from [14] and show that it is isomorphic to the analytical definition given in Chapter 3.

In Section 4.2 we investigate the second hypercohomology spectral sequence and see that it expresses the transformed Higgs bundle as a sheaf extension. In the generic situation of a stable bundle with any Higgs field, the constituent sheaves of the extension are locally free and solely determined by the stable vector bundle. Furthermore, the extension class is completely determined by the Higgs field. The remaining part of the section discusses properties of the constituent sheaves. The extension is very briefly mentioned by Bonsdorff in his thesis [13, Remark 3.1.13 (ii)] but is not given any further attention.

The Fourier–Mukai transform for Higgs bundles is essentially a relative version of Mukai's original transform. Since we obtain a vector bundle on $J \times \mathbb{P}^g$ it is natural to ask what information is contained in the relative Beilinson spectral sequence. In Section 4.3 we use the first hypercohomology spectral sequence to show that the relative Beilinson spectral sequence recovers the sheaf extension from Section 4.2.

In Section 4.4 we see how the first hypercohomology spectral sequence makes a transformed Higgs bundle into a family of homogeneous bundles on J parametrised by $\mathbb{P}^g$. We give a spectral data construction of the Fourier–Mukai transform, and show that when the spectral curve is smooth, the holomorphic structure of the family of homogeneous bundles recovers the spectral curve.

4.1 Definition

Let $z_0 \in \Sigma$ be a base point on the curve Σ of genus at least two and denote by $j : \Sigma \to J$ the corresponding Abel–Jacobi map, given by mapping z to the line bundle L_z on Σ given by the divisor class $(z - z_0)$. The Abel–Jacobi map induces a principal polarisation on J and an isomorphism between the Jacobian and its dual abelian variety $\hat{J}$, $\varphi_{z_0} : J \xrightarrow{\simeq} \hat{J}$. This identification is used throughout this chapter, except if distinction is need to clarify the argument.

Let $\mathcal{P}$ be the Poincaré bundle on $\hat{J} \times J$ normalised such that $\mathcal{P}|_{\hat{J}\times 0}$ and $\mathcal{P}|_{0\times J}$ are trivial. Let $\tilde{\mathcal{P}}$ be the pullback of $\mathcal{P}$ via $\operatorname{Id} \times j$ to a line bundle on $\hat{J} \times \Sigma \simeq J \times \Sigma$.

Given a Higgs bundle (E, Φ) on Σ consider the family of Higgs bundles

$$\mathcal{C}(E) = E \xrightarrow{\Theta} EK(1) \tag{4.1}$$

on $\Sigma \times \mathbb{P}^g$ defined such that the restriction to $\Sigma \times [a : \alpha]$, with $[a : \alpha] \in \mathbb{P}^g$, is the Higgs bundle

$$E \xrightarrow{a\Phi + \alpha \operatorname{Id}} EK.$$

The projective space $\mathbb{P}^g$ is the projectivisation of $\mathbb{C}\Phi \oplus H^0(\Sigma, K)$ and thus the above definition makes sense. The family can be seen as a two term complex of coherent sheaves concentrated in degree zero and one, and hence as an element of $D(\Sigma \times \mathbb{P}^g)$ the bounded derived category of coherent sheaves on $\Sigma \times \mathbb{P}^g$. This complex will be denoted $\mathcal{C}(E)$.

The following naming conventions for projections are used throughout the chapter:

$$\Sigma \xleftarrow{p} J \times \Sigma \xrightarrow{q} J \qquad \Sigma \times \mathbb{P}^g \xleftarrow{\pi_{23}} J \times \Sigma \times \mathbb{P}^g \xrightarrow{\pi_{13}} J \times \mathbb{P}^g \xrightarrow{\pi_1} J \qquad J \times \Sigma \times \mathbb{P}^g \xrightarrow{\pi_{12}} J \times \Sigma.$$

Definition 4.1.1. The Fourier–Mukai transform of a Higgs bundle (E, Φ) is the image of $\mathcal{C}(E)$ under the functor

$$\mathbb{F}_{\mathbb{P}^g} : D(\Sigma \times \mathbb{P}^g) \to D(J \times \mathbb{P}^g) \quad \text{defined by} \quad \mathcal{A}^\bullet \mapsto \mathbb{R}\pi_{13*}(\pi_{23}^*\mathcal{A}^\bullet \otimes \pi_{12}^*\tilde{\mathcal{P}})$$

The functor $\mathbb{F}_{\mathbb{P}^g}$ is often called the *relative* Fourier–Mukai functor as opposed to the *absolute* Fourier–Mukai functor:

Definition 4.1.2. The *absolute* Fourier–Mukai functor $\mathbb{F} : D(\Sigma) \to D(J)$ maps a complex $\mathcal{A}^\bullet$ to the complex $\mathbb{R}q_*(p^*(\mathcal{A}^\bullet) \otimes \tilde{\mathcal{P}})$.

Definition 4.1.3. A complex of sheaves $\mathcal{A}^\bullet \in D(X)$ is called $WIT(i)$ with respect to a Fourier–Mukai functor $F : D(X) \to D(\hat{X})$ between derived categories of an algebraic variety X and its partner $\hat{X}$ if the cohomology sheaves $\mathcal{H}^j(F(\mathcal{A}^\bullet))$ vanish for $j \neq i$ and is called $IT(i)$ if it is $WIT(i)$ and the i'th cohomology sheaf is locally free.

Theorem 4.1.4 (Theorem 3.1.12, [14])**.** *A stable Higgs bundle of degree zero and rank at least two is $IT(1)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$, i.e. the cohomology of the transformed complex is concentrated in degree one and is a locally free sheaf. This locally free sheaf is denoted $\hat{E}$.*

Remark 4.1.5. In [14, Section 3.3], Bonsdorff shows that the trivial Higgs bundle $(\mathcal{O}_\Sigma, 0)$ is not $IT(1)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$. Not only is it not just concentrated in degree 1, but it is also not locally free. The degree two cohomology sheaf of $\mathbb{F}_{\mathbb{P}^g}(\mathcal{O}_\Sigma, 0)$ is a skyscraper supported at $(0,0) \in J \times H^0(K) \subset J \times \mathbb{P}^g$. The requirement of having at least rank two cannot be removed if we want the transform to be locally free. This is analogous to the analytical Remark 2.1.6

To ease the notation, $X \times \mathbb{P}^g$ is denoted by $X_{\mathbb{P}^g}$ in the following.

4.1.1 An isomorphism of functors

The Abel–Jacobi map $j : \Sigma \to J$ is a closed embedding and its composition with a relative version of Mukai's original transform $\hat{S}_{\mathbb{P}^g} : D(J_{\mathbb{P}^g}) \to D(\hat{J}_{\mathbb{P}^g}) \simeq D(J_{\mathbb{P}^g})$ is isomorphic to the Fourier–Mukai transform in Definition 4.1.1.

Definition 4.1.6. The relative Fourier–Mukai transform $\hat{S}_{\mathbb{P}^g} : D(J_{\mathbb{P}^g}) \to D(J_{\mathbb{P}^g})$ of a complex $\mathcal{A}^\bullet$ is

$$\hat{S}_{\mathbb{P}^g}(\mathcal{A}^\bullet) = \mathbb{R}q_{13*}(q_{23}^*(\mathcal{A}^\bullet) \otimes q_{12}^*\mathcal{P}),$$

where q_{ij} is the canonical projection to the ij'th factor of $J \times J \times \mathbb{P}^g$

$$J_{\mathbb{P}^g} \xleftarrow{q_{23}} J \times J \times \mathbb{P}^g \xrightarrow{q_{13}} J_{\mathbb{P}^g} \qquad J \times J \times \mathbb{P}^g \xrightarrow{q_{12}} J \times J.$$

Lemma 4.1.7. *The functors $\mathbb{F}_{\mathbb{P}^g}$ and $\hat{S}_{\mathbb{P}^g} \circ (j_{\mathbb{P}^g})_* : D(\Sigma_{\mathbb{P}^g}) \to D(J_{\mathbb{P}^g})$ are isomorphic.*

Proof. As j is a closed embedding it has no higher direct images. Combining this with a simple base change and use of the projection formula the result follows. □

Corollary 4.1.8. *The absolute functors $\mathbb{F}$ and $\hat{S} \circ j_* : D(\Sigma) \to D(J)$ are isomorphic.*

4.1.2 Fourier–Mukai and Nahm transforms

The Fourier–Mukai transform of a Higgs bundle is a holomorphic bundle on $J \times \mathbb{P}^g$. Restricted to $J \times H^0(K)$ it is a holomorphic bundle on the same space as the Nahm transform bundle. The two bundles are isomorphic as holomorphic bundles.

Proposition 4.1.9. *The Nahm and Fourier–Mukai transform of a stable degree zero Higgs bundle of rank at least two are isomorphic as holomorphic bundles on $J \times H^0(K)$.*

Proof. Given the complex structure I the cotangent bundle T^*J identifies as $J \times H^0(K)$ via Serre duality. We know from Theorem 2.6.3 that the Nahm transform of (E, Φ), considered as a holomorphic bundle with respect to this complex structure, is the cohomology of the infinite dimensional monad

$$\Omega^{0,0}(E) \xrightarrow{\bar{\partial}_{E,\xi}+\Phi+\alpha\,\mathrm{Id}} \Omega^{1,0}(E) \oplus \Omega^{0,1}(E) \xrightarrow{\bar{\partial}_{E,\xi}+\Phi+\alpha\,\mathrm{Id}} \Omega^{1,1}(E) \tag{4.2}$$

varying holomorphically with respect to the coordinates $(\xi, \alpha) \in J \times H^0(K)$. The important ingredient is Hodge theory giving a harmonic representative of each first cohomology class.

The Fourier–Mukai transform on $J \times H^0(K)$ is defined as

$$\mathbb{R}^1\pi_{13*}(\pi_{23}^*\mathcal{C}(E) \otimes \pi_{12}^*\mathcal{P}),$$

where $\mathcal{C}(E)$ is the family of complexes (4.1) restricted to $\Sigma \times H^0(K)$. This bundle has as fibre at (ξ, α) the first hypercohomology group of the complex of locally free sheaves

$$EL_\xi \xrightarrow{\Phi+\alpha\,\mathrm{Id}} EL_\xi K.$$

One way of computing the hypercohomology is to take fine resolutions of the sheaves EL_ξ and $EL_\xi K$ and take cohomology of the total complex. The total complex of the standard Dolbeault resolutions of EL_ξ and $EL_\xi K$ is exactly the infinite-dimensional monad (4.2). The holomorphic structure of the Fourier–Mukai transform is also the one coming from (4.2) varying holomorphically in the parameters (ξ, α). Therefore, the two transforms are holomorphically isomorphic on $J \times H^0(K)$. □

4.2 Sheaf extension

In this section we use the second hypercohomology spectral sequence to show that the transformed Higgs bundle on $J \times \mathbb{P}^g$ is a sheaf extension. The following subsection includes useful homological results used for constructing the extension.

4.2.1 A homological intermezzo

The homological results in this section concerning a general abelian category $\mathfrak{A}$ and its bounded derived category $D(\mathfrak{A})$, can be found in e.g. [44], except Lemma 4.2.2 which is easy to prove.

Definition 4.2.1. Let $\mathcal{A}^\bullet, \mathcal{B}^\bullet$ be objects of $D(\mathfrak{A})$ and let $u : \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ be a morphism. The *cone* of u is the complex $(\mathcal{C}^\bullet(u), d_\mathcal{C})$ with $\mathcal{C}^q = \mathcal{B}^q \oplus \mathcal{A}^{q+1}$ and $d_\mathcal{C}(b, a) = (db + u(a), -da)$.

A special case is when $\mathcal{A}$ and $\mathcal{B}$ are elements of $\mathfrak{A}$ considered as one-term complexes of $D(\mathfrak{A})$ concentrated in degree 0. In that case, the cone $\mathcal{C}'$ of a morphism $u : \mathcal{A} \to \mathcal{B}$ is just the complex $\mathcal{A} \xrightarrow{u} \mathcal{B}$ concentrated in degrees -1 and 0.

A morphism between objects of $D(\mathfrak{A})$ can be completed to a distinguished triangle by use of the cone

$$\mathcal{A}^\bullet \xrightarrow{u} \mathcal{B}^\bullet \to \mathcal{C}^\bullet(u) \to \mathcal{A}^\bullet[1],$$

and especially for a morphism between objects of $\mathfrak{A}$ we consider the distinguished triangle

$$\mathcal{A} \xrightarrow{u} \mathcal{B} \to \mathcal{C}' \to \mathcal{A}[1].$$

If $F : D(\mathfrak{A}) \to D(\mathfrak{B})$ is an *exact functor* it maps distinguished triangles to distinguished triangles, hence

$$F(\mathcal{A}^\bullet) \to F(\mathcal{B}^\bullet) \to F(\mathcal{C}^\bullet(u)) \to F(\mathcal{A}^\bullet[1]).$$

Cohomology of this distinguished triangle naturally induces a long exact sequence of sheaves

$$\cdots \to \mathcal{H}^i(F(\mathcal{A}^\bullet)) \to \mathcal{H}^i(F(\mathcal{B}^\bullet)) \to \mathcal{H}^i(F(\mathcal{C}^\bullet(u))) \to \mathcal{H}^{i+1}(F(\mathcal{A}^\bullet)) \to \dots \tag{4.3}$$

In the special case of a morphism between two objects of $\mathfrak{A}$, (4.3) gives a long exact sequence relating the cohomology of the transformed two-term complex and the cohomology of the transform of the individual objects.

Lemma 4.2.2. *Suppose $\mathcal{A}, \mathcal{B}, \mathcal{C}$ are three objects of an abelian category $\mathfrak{A}$ with morphisms $a : \mathcal{A} \to \mathcal{B}$ and $b : \mathcal{B} \to \mathcal{C}$. The sequence*

$$0 \to \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \to 0$$

is exact if and only if

$$\mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \to \mathcal{A}[1]$$

is a distinguished triangle in $D(\mathfrak{A})$ for some morphism $\mathcal{C} \xrightarrow{c} \mathcal{A}[1]$.

Corollary 4.2.3. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be objects of an abelian category $\mathfrak{A}$ with morphisms $\mathcal{A} \xrightarrow{a} \mathcal{B}$ and $\mathcal{B} \xrightarrow{b} \mathcal{C}$ such that the sequence*

$$0 \to \mathcal{A} \xrightarrow{a} \mathcal{B} \xrightarrow{b} \mathcal{C} \to 0$$

is exact. Let $F : D(\mathfrak{A}) \to D(\mathfrak{B})$ be an exact functor, then

$$F(\mathcal{C}) \simeq F(\mathcal{C}'(a))$$

in $D(\mathfrak{B})$, where $\mathcal{C}'(a)$ is the cone of $\mathcal{A} \xrightarrow{a} \mathcal{B}$.

Remark 4.2.4. If the exact functor F in Corollary 4.2.3 is a Fourier–Mukai type functor $D(X) \to D(Y)$ given by a kernel sheaf on $X \times Y$, then the isomorphism $F(\mathcal{C}'(a)) \simeq F(\mathcal{C})$ induces an isomorphism of the fibres of the cohomology sheaves of $F(\mathcal{C}'(a))$ and $F(\mathcal{C})$ at each point in Y. The fibres are hypercohomology groups and the isomorphism is the convergence of the first hypercohomology spectral sequence. In this sense, Corollary 4.2.3 is a sheaf version of the first hypercohomology spectral sequence.

4.2.2 Fourier–Mukai transform is an extension of sheaves

The Fourier–Mukai functor is a composition of exact functors and the homological machinery from the previous section is directly applicable.

Proposition 4.2.5. *Let (E, Φ) be a stable degree zero Higgs bundle of rank at least two. Then there exists a canonical exact sequence of sheaves on $J \times \mathbb{P}^g$*

$$0 \to \mathcal{Q} \to \hat{E} \to \mathcal{R} \to 0$$

where $\mathcal{Q}$ and $\mathcal{R}$ are cokernel and kernel of sheaf morphisms,

$$0 \to q_*(p^*(E) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g} \to q_*(p^*(EK) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1) \to \mathcal{Q} \to 0$$

$$0 \to \mathcal{R} \to R^1 q_*(p^*(E) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g} \to R^1 q_*(p^*(EK) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1) \to 0.$$

Proof. Given a stable degree zero Higgs bundle (E, Φ) the complex $\mathcal{C}(E)$ in (4.1) is concentrated in degrees 0 and 1 and is therefore the shifted cone of the morphism $E \xrightarrow{\Theta} EK(1)$. Considered as complexes E and $EK(1)$ are concentrated in degree 0, and thus the cone $\mathcal{C}(\Theta)$ is concentrated in degrees -1 and 0, i.e. $\mathcal{C}(E) = \mathcal{C}(\Theta)[-1]$. Taking cohomology of the exact Fourier–Mukai functor on $\mathcal{C}(E)$ gives a long exact sequence by (4.3). As $\mathbb{F}^i(\mathcal{C}(\Theta)) = \mathbb{F}^{i+1}(\mathcal{C}(E))$ and as $\mathcal{C}(E)$ is $IT(1)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$, $\mathcal{C}(\Theta)$ is $IT(0)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$. The long exact sequence reduces to

$$\begin{aligned} 0 \to \mathcal{H}^0(\mathbb{F}_{\mathbb{P}^g}(E)) \to \mathcal{H}^0(\mathbb{F}_{\mathbb{P}^g}(EK(1))) \to \mathcal{H}^1(\mathbb{F}_{\mathbb{P}^g}(\mathcal{C}(E))) \\ \to \mathcal{H}^1(\mathbb{F}_{\mathbb{P}^g}(E)) \to \mathcal{H}^1(\mathbb{F}_{\mathbb{P}^g}(EK(1))) \to 0 \end{aligned}$$

By definition $R^i q_*(p^*(E) \otimes \tilde{\mathcal{P}}) = \mathcal{H}^i(\mathbb{F}_{\mathbb{P}^g}(E))$ when E is a complex concentrated in a single degree, so splitting this five term exact sequence into three short exact sequences and using the projection formula gives the statement. □

We often denote by $\mathbb{F}^i_{\mathbb{P}^g}(E)$ the cohomology sheaf $\mathcal{H}^i(\mathbb{F}_{\mathbb{P}^g}(E))$.

Remark 4.2.6. On the level of fibres, the extension in Proposition 4.2.5 is the second hypercohomology spectral sequence.

Remark 4.2.7. The extension in Proposition 4.2.5 also appears in [13] Remark 3.1.13.*ii*, but is not discussed in any detail.

Remark 4.2.8. The sheaves $\mathcal{Q}$ and $\mathcal{R}$ in Proposition 4.2.5 are a priori only coherent sheaves. However, if the underlying holomorphic vector bundle E of the stable Higgs bundle is itself stable they are locally free.

4.2.3 CONSTITUENT BUNDLES

In this section we follow up on Remark 4.2.8. We investigate the constituent sheaves in the generic situation where the vector bundle is stable.

Proposition 4.2.9. *If E is a stable vector bundle of degree zero and rank at least two, then the constituent sheaves $\mathcal{Q}$ and $\mathcal{R}$ associated to (E, Φ) for any Φ are locally free. Furthermore, the bundles $\mathcal{Q}$ and $\mathcal{R}$ are solely determined by E.*

Proof. As E is stable of degree zero it does not have any non-zero holomorphic sections. It therefore follows by Grauert's theorem and Serre duality that $\mathcal{Q}$ and $\mathcal{R}$ are locally free. In case E is stable, E is $IT(1)$ and $EK(1)$ is $IT(0)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$ and

$$\mathcal{Q} \simeq q_*(p^*(EK) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1) \quad \text{and} \quad \mathcal{R} \simeq R^1 q_*(p^*(E) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g}.$$

From the isomorphism above it follows that $\mathcal{Q}$ and $\mathcal{R}$ are completely determined by E. □

Corollary 4.2.10. *If E is a stable vector bundle of degree zero and rank at least two, then for any Higgs field the constituent sheaves $\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^g}(-1)$ and $\mathcal{R}^*$ are Picard bundles on the moduli space of stable bundles restricted to an orbit of the Jacobian.*

Proof. Denote by $\mathscr{N}_{d,r}$ the moduli space of stable bundles of degree d and rank r. A Picard bundle is the direct image of a universal bundle on $\mathcal{U} \to \Sigma \times \mathscr{N}_{d,r}$ along the projection to the moduli space $\mathscr{N}_{d,r}$. As E is stable of degree zero EK is stable of degree $2r(g-1)$. Although the moduli space of degree $2r(g-1)$ and rank r bundles does not have a global universal bundle (as discussed in Section 2.5) there is a universal bundle on the orbit of the Jacobian through EK given by the Poincaré bundle.

From the proof of Proposition 4.2.9 the constituent bundle $\mathcal{Q} \otimes \mathcal{O}_{\mathbb{P}^g}(-1)$ is exactly the Picard bundle restricted to the mentioned Jacobian orbit.

From the Relative Duality Theorem (see e.g. [11, Theorem D.10, p 590])

$$\mathcal{R}^* \simeq q_*(p^*(E^* \otimes K) \otimes \tilde{\mathcal{P}}^*) \boxtimes \mathcal{O}_{\mathbb{P}^g}$$

and by the same reasoning as above $\mathcal{R}^*$ is a Picard bundle restricted to a Jacobian orbit. □

Corollary 4.2.11. *If E is a stable vector bundle of degree zero and rank at least two, then for any Higgs field the sheaf extension in Proposition 4.2.5 reduces to*

$$0 \to q_*(p^*(EK) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1) \to \hat{E} \to R^1 q_*(p^*(E) \otimes \tilde{\mathcal{P}}) \boxtimes \mathcal{O}_{\mathbb{P}^g} \to 0 \tag{4.4}$$

an extension of holomorphic vector bundles.

4.2.4 Extension class completely determined by the Higgs field

If the holomorphic bundle underlying a stable Higgs bundle is itself stable, the extension from Proposition 4.2.5 is completely determined by the Higgs field. The crucial part in proving this is the isomorphism between $\mathbb{F}$ and $\hat{S} \circ j_*$.

Lemma 4.2.12. *Let $\mathcal{F}, \mathcal{G}$ be coherent sheaves on Σ. The Abel–Jacobi map $j : \Sigma \to J$ induces an isomorphism*

$$j_* : \mathrm{Hom}_\Sigma(\mathcal{F}, \mathcal{G}) \xrightarrow{\simeq} \mathrm{Hom}_J(j_*(\mathcal{F}), j_*(\mathcal{G})).$$

Proof. The lemma follows directly from the definition of j_* and that j is a closed embedding. □

In [57, Corollary 2.5], Mukai proves the following *Parseval formula* for $\hat{S}$. The proof is rather simple and is included for later reference.

Theorem 4.2.13 (Parseval formula)**.** *Assume $\mathcal{F}$ is a $WIT(i)$-sheaf and $\mathcal{G}$ a $WIT(j)$-sheaf on J with respect to the Fourier–Mukai functor $\hat{S} : D(J) \to D(\hat{J})$, then for every integer k*

$$\mathrm{Ext}^k_J(\mathcal{F}, \mathcal{G}) \simeq \mathrm{Ext}^{k+i-j}_{\hat{J}}(\hat{\mathcal{F}}, \hat{\mathcal{G}})$$

and especially

$$\mathrm{Ext}^k_J(\mathcal{F}, \mathcal{F}) \simeq \mathrm{Ext}^k_{\hat{J}}(\hat{\mathcal{F}}, \hat{\mathcal{F}}).$$

The isomorphism is given by the Fourier–Mukai functor $\hat{S}$.

Proof. The proof is a simple application of the Fourier–Mukai functor $\hat{S}$.

$$\begin{aligned}
\mathrm{Ext}^k_J(\mathcal{F}, \mathcal{G}) &\simeq \mathrm{Hom}_{D(J)}(\mathcal{F}, \mathcal{G}[k]) \\
&\simeq \mathrm{Hom}_{D(\hat{J})}(\hat{S}(\mathcal{F}), \hat{S}(\mathcal{G})[k]) \\
&\simeq \mathrm{Hom}_{D(\hat{J})}(\hat{\mathcal{F}}[-i], \hat{\mathcal{G}}[k-j]) \\
&\simeq \mathrm{Hom}_{D(\hat{J})}(\hat{\mathcal{F}}, \hat{\mathcal{G}}[k-j+i]) \\
&\simeq \mathrm{Ext}^{k-j+i}_{\hat{J}}(\hat{\mathcal{F}}, \hat{\mathcal{G}}).
\end{aligned}$$

The third isomorphism follows from the $WIT(i)$ and $WIT(j)$ conditions, as $\hat{S}(\mathcal{F}) \simeq \hat{\mathcal{F}}[-i]$ in $D(\hat{J})$ if $\mathcal{F}$ is $WIT(i)$ with respect to $\hat{S}$. □

Remark 4.2.14. The Parseval formula is valid whenever the functor is *fully faithful* and integral [6, p. 23], so especially also for the relative Fourier–Mukai functor $\hat{S}_{\mathbb{P}^g}$. By Bondal and Orlov's theorem [44, Proposition 7.1] the absolute Fourier–Mukai functor $\mathbb{F} : D(\Sigma) \to D(J)$ is not fully faithful. As $\hat{S}$ is fully faithful the deficiency in $\mathbb{F}$ of not being fully faithful can be circumvented by the isomorphism $\mathbb{F} \simeq \hat{S} \circ j_*$. This is used to prove the following theorem.

Theorem 4.2.15. *If E is a stable bundle on Σ, then*

$$\mathrm{Ext}^1(\mathbb{F}^1(E), \mathbb{F}^0(EK)) \simeq H^0(\Sigma, \mathrm{End}(E)K).$$

The isomorphism is given by $\hat{S} \circ j_*$.

Proof. The proof is an application of the Parseval formula. From Corollary 4.1.8 the functor $\mathbb{F} : D(\Sigma) \to D(J)$ is isomorphic to $\hat{S} \circ j_*$. As E is stable it is $IT(1)$ by Remark 4.2.8 and EK is similarly $IT(0)$. As the $\hat{S}$-transform of $j_*(E)$ and $j_*(EK)$ are isomorphic to $\mathbb{F}(E)$ and $\mathbb{F}(EK)$, respectively, the sheaves $j_*(E)$ and $j_*(EK)$ are $IT(1)$ and $IT(0)$ with respect to $\hat{S}$, respectively. By the Parseval formula and Lemma 4.2.12

$$\mathrm{Ext}^1(\mathbb{F}^1(E), \mathbb{F}^0(EK)) \simeq \mathrm{Ext}^0(j_*(E), j_*(EK)) \simeq H^0(\Sigma, \mathrm{End}(E)K).$$

From Theorem 4.2.13 the first isomorphism is induced by $\hat{S}$. By Lemma 4.2.12 the second isomorphism is induced by the Abel–Jacobi map. □

Theorem 4.2.16. *Let (E, Φ) be a stable degree zero Higgs bundle of rank at least two with stable underlying holomorphic bundle E. For any $A = [a : \alpha] \in \mathbb{P}^g$, the extension class of* (4.4) *restricted to J_A is completely determined by the Higgs field $a\Phi + \alpha \,\mathrm{Id}$.*

Proof. For $A = [a : \alpha]$ the restriction of $\hat{E}$ to the slice J_A is the absolute Fourier–Mukai transform of the Higgs bundle $E \xrightarrow{a\Phi + \alpha \,\mathrm{Id}} EK$ [14, Proposition 3.1.10]. The image of the extension class under the string of isomorphisms in the proof of Theorem 4.2.15 is $a\Phi + \alpha \,\mathrm{Id}$. □

Proposition 4.2.17. *If E is a stable vector bundle of degree zero and rank at least two, then for any Higgs field the extension class of* (4.4)*, as an element of*

$$\mathrm{Ext}^1_{J \times \mathbb{P}^g}(\mathbb{F}^1_{\mathbb{P}^g}(E), \mathbb{F}^0_{\mathbb{P}^g}(EK(1))) \simeq \mathrm{Ext}^1_J(\mathbb{F}^1(E), \mathbb{F}^0(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}(1)),$$

is

$$a_0 c_\Phi + \sum_{i=1}^{g} a_i c_{\alpha_i \,\mathrm{Id}}.$$

Here c_Φ is the class of the extension restricted to $J \times [1 : 0 : \cdots : 0]$; $c_{\alpha_i \,\mathrm{Id}}$ is likewise the class of the extension restricted to $J \times [0 : \cdots : 0 : 1 : 0 \cdots : 0]$ with 1 *in the i'th position, and a_i, $i = 0, \ldots, g$ are coordinate sections of $\mathbb{P}^g$.*

Proof. First, notice that as $\mathbb{F}^1_{\mathbb{P}^g}(E)$ and $\mathbb{F}^0_{\mathbb{P}^g}(EK(1))$ are locally free sheaves

$$\begin{aligned}\operatorname{Ext}^1_{J_{\mathbb{P}^g}}(\mathbb{F}^1_{\mathbb{P}^g}(E), \mathbb{F}^0_{\mathbb{P}^g}(EK(1))) &\simeq H^1(J_{\mathbb{P}^g}, (\mathbb{F}^1(E)^* \otimes \mathbb{F}^0(EK)) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)) \\ &\simeq H^1(J, \mathbb{F}^1(E)^* \otimes \mathbb{F}^0(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)) \\ &\simeq \operatorname{Ext}^1_J(\mathbb{F}^1(E), \mathbb{F}^0(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)),\end{aligned}$$

where the second isomorphism is the Künneth formula combined with the vanishing of the first cohomology of $\mathcal{O}_{\mathbb{P}^g}(1)$.

Secondly, it follows from the proof of the Parseval formula the isomorphism induced between extension groups is the relative functor $\hat{S}_{\mathbb{P}^g}$

$$\operatorname{Ext}^0_{J_{\mathbb{P}^g}}(j_{\mathbb{P}^g *}(E), j_{\mathbb{P}^g *}(EK(1))) \xrightarrow{\hat{S}_{\mathbb{P}^g}} \operatorname{Ext}^1_{J_{\mathbb{P}^g}}(\mathbb{F}^1_{\mathbb{P}^g}(E), \mathbb{F}^0_{\mathbb{P}^g}(EK(1))).$$

Thirdly, notice that $j_{\mathbb{P}^g *}(EK(1)) \simeq j_*(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)$ on $J \times \mathbb{P}^g$, and that the zeroth extension group is

$$\begin{aligned}\operatorname{Ext}^0_{J_{\mathbb{P}^g}}(j_{\mathbb{P}^g *}(E), j_{\mathbb{P}^g *}(EK(1))) &\simeq H^0(J_{\mathbb{P}^g}, \mathcal{E}xt^0(j_{\mathbb{P}^g *}(E), j_{\mathbb{P}^g *}(EK(1)))) \\ &\simeq H^0(J_{\mathbb{P}^g}, \mathcal{E}xt^0(j_*(E), j_*(EK)) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)) \\ &\simeq H^0(J, \mathcal{E}xt^0(j_*(E), j_*(EK))) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)) \\ &\simeq \operatorname{Ext}^0_J(j_*(E), j_*(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1))\end{aligned}$$

where $\mathcal{E}xt^i$ is the i'th Ext-sheaf, and the second isomorphism is [31, Proposition III.6.7].

Combining all of the above gives a commutative diagram

$$\begin{tikzcd}
\operatorname{Ext}^0_{J_{\mathbb{P}^g}}(j_{\mathbb{P}^g *}(E), j_{\mathbb{P}^g *}(EK(1))) \arrow[r, phantom, "\simeq"] \arrow[d, "\hat{S}_{\mathbb{P}^g}"] & \operatorname{Ext}^0_J(j_*(E), j_*(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)) \arrow[d, "\hat{S} \otimes \mathrm{Id}"] \\
\operatorname{Ext}^1_{J_{\mathbb{P}^g}}(\mathbb{F}^1_{\mathbb{P}^g}(E), \mathbb{F}^0_{\mathbb{P}^g}(EK(1))) \arrow[r, phantom, "\simeq"] & \operatorname{Ext}^1_J(\mathbb{F}^1(E), \mathbb{F}^0(EK)) \otimes H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(1)).
\end{tikzcd}$$

The proposition now follows from the fact that $j_{\mathbb{P}^g *}$ induces an isomorphism between $H^0(\Sigma_{\mathbb{P}^g}, \operatorname{End}(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1))$ and $\operatorname{Ext}^0_{J_{\mathbb{P}^g}}(j_{\mathbb{P}^g *}(E), j_{\mathbb{P}^g *}(EK(1)))$ by a relative version of Lemma 4.2.12, and from the definition of the family of Higgs bundles (4.1). □

Corollary 4.2.18. *The extension* (4.4) *is non-split.*

Proof. If the extension was split all classes c_Φ, c_{α_i} must be zero, but this is equivalent to Φ and all α_i being zero by Theorem 4.2.15. □

Remark 4.2.19. Notice that only an $\operatorname{rk} E^2(g-1)+1$-dimensional subspace of the possible $g(\operatorname{rk} E^2(g-1)+1)$-dimensional space of extensions is used. This subspace is determined by the classes c_Φ.

4.2.5 Constituent sheaves in rank two and genus two

For rank two bundles on a genus two curve Hitchin [39] gives an explicit list of the possible holomorphic vector bundles supporting a stable Higgs bundle. We use this classification to get a closer look at the constituent sheaves $\mathcal{Q}$, $\mathcal{R}$ in this special case.

Proposition 4.2.20 ([39] Proposition 3.3)**.** *Let Σ be a compact curve of genus two. A rank holomorphic vector bundle E of rank two and degree zero supports a stabilising Higgs field if and only if one of the following holds:*

- *E is stable;*
- *$E \simeq U \otimes L$ where $\deg(L) = 0$ and U is a non-trivial extension of $\mathcal{O}$ by itself;*
- *$E \simeq L \oplus N$ where $\deg(L) = \deg(N) = 0$;*
- *E is decomposable as $E = (L \oplus L^{-1}) \otimes N$ where $L^2 \simeq K$ and $N^2 \simeq \det(E)$.*

Proposition 4.2.21. *Let (E, Φ) be a degree zero rank two Higgs bundle on a Riemann surface of genus two, then the constituent sheaf $\mathcal{Q}$ from Proposition 4.2.5 is locally free of rank two.*

Proof. If $\mathbb{F}^0(E) = 0$, then $\mathcal{Q} \simeq \mathbb{F}^0(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)$. By Lemma 1.3.7 the sheaf $\mathbb{F}^0(EK)$ is locally free as J has dimension two. The rank is easily calculated using Riemann–Roch.

The proof that $\mathbb{F}^0(E)$ is the zero sheaf is a case by case study using Hitchin's classification. The main tool used is Grauert's theorem combined with Lemma 1.3.6. In all cases but the last, the dimension of $H^0(\Sigma, EL_\xi)$ is zero for all but finitely many $\xi \in J$. Grauert's theorem gives local freeness away from these points, thus $\mathbb{F}^0(E)$ vanish on J except for finitely many points. However as $\mathbb{F}^0(E)$ is torsion-free it must be the zero sheaf on all of J, thus $\mathcal{Q} \simeq \mathbb{F}^0(EK) \otimes \mathcal{O}(1)$.

In the last case, let $K^{1/2}$ be a square root of the canonical bundle. As $K^{-1/2}$ has degree -1 neither it nor multiples of it by degree zero line bundles have global sections. The natural Θ-divisor on the Jacobian of degree one line bundles $J^1(\Sigma)$ is the subset

$$\Theta = \{L \in J^1(\Sigma) \mid h^0(\Sigma, L) > 0\}.$$

The degree zero and degree one line bundles on Σ can be identified by tensoring with a square root $K^{1/2}$ and thus the direct image of $K^{1/2} \otimes \tilde{\mathcal{P}}$ to J is supported on a divisor, but as $\mathbb{F}^0(K^{1/2})$ is torsion-free (Lemma 1.3.6) it must be zero. □

Proposition 4.2.22. *Let (E, Φ) be a Higgs bundle of degree zero and rank two on a curve of genus two. The constituent sheaf $\mathcal{R}$ from Proposition 4.2.5 is a sheaf of rank two with extra properties depending on the case:*

- *locally free if E is stable;*
- *strictly torsion-free if E is the non-trivial extension of a flat line bundle by itself or a direct sum of flat line bundles;*
- *torsion sheaf if $E = (L \oplus L^{-1}) \otimes N$, $L^2 \simeq K$ and* $\deg(N) = 0$.

Proof. The following table containing dimensions of sheaf cohomology groups will be important for the proof of the proposition. To avoid confusion the line bundle L in the last case, is written as $K^{1/2}$.

stable	$U \otimes L$	$L \oplus N$	$(K^{1/2} \oplus K^{-1/2}) \otimes N$
0 for all L_ξ	0 if $L_\xi \neq L^{-1}$ 1 if $L_\xi = L^{-1}$	0 if $L_\xi \neq L^{-1}$ and $L_\xi \neq N^{-1}$ 1 if $L_\xi = L^{-1}$ or $L_\xi = N^{-1}$ 2 if $L_\xi = L^{-1}$ and $L_\xi = N^{-1}$	0 if $L_\xi N \notin \Theta_{K^{1/2}}$ 1 if $L_\xi N \in \Theta_{K^{1/2}}$

Dimensions of the cohomology group $H^0(EL_\xi)$ for the four different cases of Hitchin's classification in Proposition 4.2.20. The divisor $\Theta_{K^{1/2}}$ is the translate of the Θ-divisor in $J^1(\Sigma)$ to $J^0(\Sigma)$ using $K^{1/2}$. The dimensions are easily computable, expect for the third case where Clifford's theorem must be applied.

When E is stable it is clear from Grauert's theorem that $\mathcal{R}$ is locally free. It has rank two by Riemann–Roch.

In the case of a non-trivial extension of a trivial line bundle and the direct sum of trivial line bundles, the jumping locus of the function $h^i(\xi, E) = \dim H^i(\Sigma, EL_\xi)$ on J is of codimension two in J. As $\mathbb{F}^0(E) = 0$, it follows from [73, Corollary 1.4] that $\mathbb{F}^1(E)$ is torsion free. As $\mathcal{R}$ is a subsheaf of $\mathbb{F}^1(E) \boxtimes \mathcal{O}_{\mathbb{P}^2}$ it is torsion free. Since $\mathbb{F}^2(E) = 0$ it follows from Grauert's theorem that $\mathbb{F}^1(E) \otimes \mathbb{C}_\xi \simeq H^1(\Sigma, EL_\xi)$ for all $\xi \in J$. From the table above, the dimension of the fibre jumps and thus $\mathbb{F}^1(E)$ is torsion-free but not reflexive, as if it was reflexive it would be locally free contradicting the jumps in the fibre dimension. The points in J giving rise to jumps in $h^0(\xi, E)$ is the locus where the stalks are not free. It is also evident from the table and Grauert's theorem that $\mathbb{F}^1(EK)$ is a pure torsion sheaf supported on the jumping locus of $h^0(\xi, E)$. Again, it follows from the last bullet of Grauert's theorem that $\mathbb{F}^1(EK)$ is not the zero-sheaf. Notice that $\mathcal{R}$ cannot be locally free as it would then be the double dual of $\mathbb{F}^1(E)$, implying that $\mathbb{F}^1(E)$ was locally-free.

In the last case, $\mathbb{F}^1(K^{-1/2} \otimes N)$ is locally free of rank two, $\mathbb{F}^1(K^{3/2} \otimes N) = 0$ and $\mathbb{F}^1(K^{1/2} \otimes N)$ is a pure torsion sheaf supported on the divisor $\Theta_{K^{1/2}} \subset J$. It follows from [31, Exercise II.5.8] that $\mathbb{F}^1(K^{1/2}N) \simeq i_*(N')$ where $i : \Theta_{K^{1/2}} \to J$ is the inclusion and N' is a line bundle on $\Theta_{K^{1/2}}$. Therefore $\mathbb{F}^1(E)$ is the direct sum of a locally free rank two sheaf and a pure torsion sheaf while $\mathbb{F}^1(EK) = \mathbb{F}^1(K^{1/2}N)$. Recall that $\mathcal{R}$ is the kernel of $\mathbb{F}^1(E) \boxtimes \mathcal{O}_{\mathbb{P}^2} \to \mathbb{F}^1(EK) \boxtimes \mathcal{O}_{\mathbb{P}^2}(1)$. At the divisor at infinity this map respects the direct sum, and maps the torsion part $\mathbb{F}^1(K^{1/2}N)$ to $\mathbb{F}^1(K^{3/2}N) = 0$. It therefore follows that $\mathcal{R}$ has torsion. $\square$

Remark 4.2.23. Propositions 4.2.21 and 4.2.22 show that in the case of stable Higgs bundles of degree zero and rank two on a genus two curve, the constituent sheaf $\mathcal{Q}$ is locally free but its counterpart $\mathcal{R}$ gets increasingly more complicated as the underlying bundle gets further away from being stable. Notice also that the Parseval formula is not applicable in other cases than E being stable as $EK(1)$ is not $WIT(i)$ with respect to $\mathbb{F}_{\mathbb{P}^g}$.

Remark 4.2.24. When the rank and genus is two and the holomorphic bundle is stable, then the Chern characters of $\mathcal{R}$ and $\mathcal{Q}(-1)$ are $2+2t$ and $2-2t$, where t is the first Chern class of the Θ-divisor. This is a simple application of Grothendieck–Riemann–Roch. The reason for the 2Θ-divisor cropping up is explained by the following proposition.

Proposition 4.2.25. *Let E be a stable bundle of degree zero and rank two on a curve of genus two and Φ any Higgs field. Then the constituent bundle $\mathcal{Q}(-1) \simeq \mathbb{F}^0(EK)$ is an elementary modification of a Fourier–Mukai transformed line bundle on Σ, and the constituent bundle $\mathcal{R} \simeq \mathbb{F}^1(E)$ is an elementary modification of a Fourier–Mukai transformed line bundle on Σ, up to a twist by a line bundle on J.*

Proof. Narasimhan and Ramanan [59] showed that the moduli space of stable vector bundle of rank two with fixed determinant on a genus two curve is isomorphic to $\mathbb{P}^3 = \mathbb{P}(H^0(J^1(\Sigma), \mathcal{O}(2\Theta)))$, where $\mathcal{O}(\Theta)$ is the line bundle defined by the Θ-divisor. This identification is given by associating to a stable bundle E the set

$$C_E = \{L \in J^1(\Sigma) \,|\, h^0(EL) \neq 0\}.$$

Narasimhan and Ramanan prove that all C_E are divisors linearly equivalent to 2Θ. In other words, a stable rank two bundle with trivial determinant can be defined as an extension

$$0 \to L^{-1} \to E \to L \to 0 \tag{4.5}$$

for some line bundle L of degree one. Multiplying this sequence by the canonical bundle gives

$$0 \to L^{-1}K \to EK \to LK \to 0. \tag{4.6}$$

As E is stable $\mathbb{F}^1(EK) = 0$. Furthermore, as $L^{-1}K$ has degree one the sheaf $\mathbb{F}^0(L^{-1}K)$ is supported on the divisor $\Theta_{L^{-1}K}$. However, $\mathbb{F}^0(L^{-1}K)$ is torsion free by Lemma 1.3.6 and thus $\mathbb{F}^0(L^{-1}K) = 0$. By Grauert's theorem $\mathbb{F}^1(LK) = 0$ and $\mathbb{F}^0(LK)$ and $\mathbb{F}^0(EK)$ are rank two bundles. The Fourier–Mukai transform of (4.6) is therefore

$$0 \to \mathbb{F}^0(EK) \to \mathbb{F}^0(LK) \to \mathbb{F}^1(L^{-1}K) \to 0.$$

The degree of $L^{-1}K$ is one which is the critical degree for Riemann–Roch in genus two. If a line bundle of degree one on a genus two curve has global sections it can only have a one dimensional space of such by Clifford's theorem. From the last bullet of Grauert's theorem $\mathbb{F}^1(L^{-1}K)$ is a torsion sheaf of pure dimension one and as the fibre dimension is constant it is a line bundle on $\Theta_{L^{-1}K}$ [31, Exercise II.5.8], i.e. $\mathbb{F}^1(L^{-1}K) \simeq i_*(N)$ where $i : \Theta_{L^{-1}K} \to J$ is the inclusion and N is a line bundle on $\Theta_{L^{-1}K}$. Therefore $\mathbb{F}^0(EK)$ is an elementary modification of $\mathbb{F}^0(LK)$.

It follows by similar arguments that the Fourier–Mukai transform of (4.5) is

$$0 \to \mathbb{F}^1(L^{-1}) \to \mathbb{F}^1(E) \to \mathbb{F}^1(L) \to 0$$

with $\mathbb{F}^1(L^{-1})$ and $\mathbb{F}^1(E)$ rank two bundles, and $\mathbb{F}^1(L) \simeq i_*(N')$ where $i : \Theta_{L^{-1}K} \to J$ is the inclusion and N' is a line bundle on $\Theta_{L^{-1}K}$. This shows that $\mathbb{F}^1(L^{-1})$ is an elementary modification of $\mathbb{F}^1(E)$ which by [25, Page 42] is equivalent to $\mathbb{F}^1(E)$ being an elementary modification of $\mathbb{F}^1(L^{-1}) \otimes \mathcal{O}(\Theta_{L^{-1}K})$. □

4.3 Relation to Beilinson's spectral sequence

For a holomorphic vector bundle on a projective space the Beilinson spectral sequence can sometimes give extra information about the bundle. Depending on the explicit terms in the spectral sequence the vector bundle might turn out to be given by a monad or sit in an extension. In this section, a relative version of the Beilinson spectral sequence is applied to a transformed Higgs bundle, and combined with the first hypercohomology spectral sequence the result is shown to be equivalent to the sheaf extension in Proposition 4.2.5.

Theorem 4.3.1 (Beilinson)**.** *Let Z be a complex manifold, $\pi : \mathbb{P}^g \times Z \to Z$ the projection. For every holomorphic vector bundle E on $\mathbb{P}^g \times Z$ there is a spectral sequence with E_1-term*

$$E_1^{-p,q} = \mathcal{O}_{\mathbb{P}^g}(-p) \boxtimes R^q\pi_*(E \otimes \Omega^p_{\mathbb{P}^g \times Z/Z}(p))$$

which converges to

$$E^i = \begin{cases} E & i = 0 \\ 0 & \textit{otherwise.} \end{cases}$$

Here $\Omega_{\mathbb{P}^g \times Z/Z}$ denotes the relative cotangent bundle of the projection to Z, i.e. the cotangent bundle of $\mathbb{P}^g$ pulled back to $\mathbb{P}^g \times Z$, and Ω^p the p'th wedge power of $\Omega_{\mathbb{P}^g \times Z/Z}$.

Remark 4.3.2. There is a version of Beilinson's theorem on any manifold X with a good resolution of the diagonal in $X \times X$. In Beilinson's original version $X = \mathbb{P}^n$ and the resolution is provided by the Koszul complex of a canonical section of

$$\mathcal{O}_{\mathbb{P}^n}(1) \boxtimes Q \simeq \mathcal{H}om(p_1^*\mathcal{O}_{\mathbb{P}^n}(-1), p_2^*Q)$$

where the p_i's are projections from $\mathbb{P}^n \times \mathbb{P}^n$ and $Q \simeq T\mathbb{P}^n(-1)$. The same kind of resolution exist in the relative case. For the application in this thesis the geometric description of projective space as $\mathbb{P}^g = \mathbb{P}(\mathbb{C}\Phi \oplus H^0(K))$ implies that the canonical section in question is exactly multiplication by $a\Phi + \alpha \operatorname{Id}$ on a slice $A \times \mathbb{P}^g$ where $A = [a : \alpha]$.

For a stable Higgs bundle (E, Φ) of degree zero and rank at least two Proposition 4.2.5 expresses the transformed bundle as an extension. This extension is related to the Beilinson Spectral Sequence of $\hat{E}(-1)$.

Lemma 4.3.3. *On $J \times \mathbb{P}^g$ denote by Ω^p the p'th wedge power of the relative cotangent bundle of the projection $\pi : J \times \mathbb{P}^g \to J$. Then*

$$\begin{aligned} R^q\pi_*(\Omega^p(p)) &\simeq 0 \quad \text{if } (p,q) \neq (0,0) & \pi_*(\mathcal{O}_{J\times\mathbb{P}^g}) &\simeq \mathcal{O}_J \\ R^q\pi_*(\Omega^p(p-1)) &\simeq 0 \quad \text{if } (p,q) \neq (1,1) & R^1\pi_*(\Omega) &\simeq \mathcal{O}_J \\ R^q\pi_*(\Omega^p(p+1)) &\simeq 0 \quad \text{if } q \neq 0 & \pi_*(\Omega^p(p+1)) &\simeq \mathcal{O}_J^{\binom{g+1}{p+1}} \end{aligned}$$

Proof. The vanishing results follow from Grauert's theorem by showing the vanishing of the cohomology groups $H^q(\mathbb{P}^g, \Omega^p(p))$, $H^q(\mathbb{P}^g, \Omega^p(p-1))$, and $H^q(\mathbb{P}^g, \Omega^p(p+1))$ for the appropriate q's. Their dimensions are calculated using Bott's rule [64, p. 8].

What remains is to show that the non-zero bundles are in fact trivial bundles. Only the argument for $\pi_*(\Omega^p(p+1))$ is given, as the others are parallel to this.

The dimension of the global holomorphic sections of $\pi_*(\Omega^p(p+1))$ is again computed by Bott's rule since

$$H^0(J, \pi_*(\Omega^p(p+1))) = H^0(J \times \mathbb{P}^g, \Omega^p(p+1)) \simeq H^0(\mathbb{P}^g, \Omega^p(p+1)) \simeq \binom{g+1}{p+1}.$$

The Chern character of $\pi_*(\Omega^p(p+1))$ is computed by Grothendieck–Riemann–Roch as all higher direct images vanish,

$$\operatorname{ch}(\pi_*(\Omega^p(p+1))) = \int_{\mathbb{P}^g} \operatorname{ch}(\Omega^p(p+1)) \operatorname{Td}(\mathbb{P}^g) = \chi(\Omega^p(p+1)) = \binom{g+1}{p+1}.$$

As the rank and the number of linearly independent holomorphic sections match up, $s_1 \wedge \cdots \wedge s_n$ is a non-zero holomorphic section of $\det \pi_*(\Omega^p(p+1))$ where $\{s_1, \ldots, s_n\}$ is a basis of global holomorphic sections. From the Chern character above, this section is non-zero at every point of J, implying that $s_i(x) \neq 0$ for all x, i.e. $\{s_1, \ldots, s_n\}$ constitute a global holomorphic frame of $\pi_*(\Omega^p(p+1))$. □

Theorem 4.3.4. *Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two. The Beilinson spectral sequence for $\hat{E}(-1)$ recovers $\hat{E}(-1)$ as a sheaf extension, which when twisted with $\mathcal{O}_{\mathbb{P}^g}(1)$ reproduces the sheaf extension for $\hat{E}$ found in Proposition 4.2.5.*

Proof. The proof of this theorem is an exercise in identifying the terms in Beilinson's spectral sequence using the definition of $\hat{E}$ via the first spectral sequence for hypercohomology. The proof is unfortunately long and technical.

The definition of the Fourier–Mukai transform of a stable Higgs bundle of degree zero uses the family (4.1). The sheaf morphism Θ is injective [14, Lemma 3.2.1.1], thus defining a short exact sequence of sheaves on $\Sigma \times \mathbb{P}^g$

$$0 \to E \xrightarrow{\Theta} EK(1) \to \mathcal{Q}_\Theta \to 0,$$

where $\mathcal{Q}_\Theta$ is a sheaf supported on the zero divisor of $\det(\Theta)$. By Corollary 4.2.3 the Fourier–Mukai transform of (E, Φ) is the Fourier–Mukai transform of the corresponding cokernel sheaf $\mathcal{Q}_\Theta$ up to a shift.

Pulling back the sheaf sequence to $J \times \Sigma \times \mathbb{P}^g$ and tensoring by $\Omega^k(k-1)\tilde{\mathcal{P}}$ gives the exact sequence

$$0 \to E\Omega^k(k-1)\tilde{\mathcal{P}} \to EK\Omega^k(k)\tilde{\mathcal{P}} \to \mathcal{Q}_\Theta\Omega^k(k-1)\tilde{\mathcal{P}} \to 0. \tag{4.7}$$

The terms in the relative Beilinson spectral sequence for $\hat{E}(-1)$ are derived images of $\hat{E}\Omega^k(k-1)$ under the projection $\pi : J \times \mathbb{P}^g \to J$,

$$R^n\pi_*(\hat{E}\Omega^k(k-1)) \boxtimes \mathcal{O}_{\mathbb{P}^g}(-k).$$

These sheaves can be computed by (4.7) as $\hat{E} \simeq \pi_{13*}(\pi_{23}^*(\mathcal{Q}_\Theta) \otimes \pi_{12}^*\tilde{\mathcal{P}})$ and

$$\hat{E}\Omega^k(k-1) \simeq \pi_{13*}(\pi_{23}^*(\mathcal{Q}_\Theta\Omega^k(k-1)) \otimes \pi_{12}^*\tilde{\mathcal{P}}),$$

by the projection formula. Recall that π_{ij} and π_i denote projections from $J \times \Sigma \times \mathbb{P}^g$ to the ij'th or i'th factor, respectively. The pullbacks needed to make sense of tensor product of $\mathcal{Q}_\Theta$ and $\Omega^k(k-1)$ are suppressed for ease of notation. As $\mathcal{Q}_\Theta$ is $IT(0)$ with respect to the relative Fourier–Mukai transform, the higher direct images of $\pi_{23}^*(\mathcal{Q}_\Theta) \otimes \pi_{12}^*\tilde{\mathcal{P}}$ under π_{13} vanish. Furthermore, by the projection formula the higher direct images of $\pi_{23}^*(\mathcal{Q}_\Theta\Omega^k(k-1)) \otimes \pi_{12}^*\tilde{\mathcal{P}}$ also vanish. This gives

$$R^n\pi_*(\hat{E}\Omega^k(k-1)) \simeq R^n\pi_{1*}(\pi_{23}^*(\mathcal{Q}_\Theta\Omega^k(k-1)) \otimes \pi_{12}^*\tilde{\mathcal{P}}).$$

The terms in Beilinson's spectral sequence also appear in the long exact sequence associated to (4.7) under $\pi_1 : J \times \Sigma \times \mathbb{P}^g \to J$. The other sheaves in the long exact sequence are

$$R^n\pi_{1*}(E\Omega^k(k-1)\tilde{\mathcal{P}}) \quad \text{and} \quad R^n\pi_{1*}(EK\Omega^k(k)\tilde{\mathcal{P}}).$$

These sheaves are easier to handle, and the rest of the proof argues why all terms in the spectral sequence are sheaves of the above form.

Combining Grauert's theorem, Bott's rule [64, p. 8], and the Künneth Formula for sheaf cohomology gives the following vanishing result

$$\begin{aligned} R^n\pi_{1*}(E\Omega^k(k-1)\tilde{\mathcal{P}}) &= 0 \quad \text{for } (k,n) \notin \{(0,0),(0,1)\} \\ R^n\pi_{1*}(EK\Omega^k(k)\tilde{\mathcal{P}}) &= 0 \quad \text{for } (k,n) \notin \{(1,1),(1,2)\}. \end{aligned}$$

In the case $k=0$, the only non-vanishing sheaves are

$$\pi_*(\hat{E}(-1)) \simeq \pi_{1*}(E(-1)\tilde{\mathcal{P}}) \quad \text{and} \quad R^1\pi_*(\hat{E}(-1)) \simeq R^1\pi_{1*}(E(-1)\tilde{\mathcal{P}}).$$

In the case $k=1$, the only non-vanishing sheaves are

$$\pi_*(\hat{E}\Omega) \simeq R^1\pi_{1*}(E\Omega\tilde{\mathcal{P}}) \quad \text{and} \quad R^1\pi_*(\hat{E}\Omega) \simeq R^2\pi_{1*}(E\Omega\tilde{\mathcal{P}}).$$

Consider the diagram of projections

$$\begin{tikzcd} J\times\Sigma\times\mathbb{P}^g \arrow[r, "\pi_{13}"] \arrow[d, "\pi_{12}"] & J\times\mathbb{P}^g \arrow[d, "\pi"] \\ J\times\Sigma \arrow[r, "q"] & J. \end{tikzcd}$$

Regard Ω on $J\times\Sigma\times\mathbb{P}^g$ as being pulled back by π_{13} from $J\times\mathbb{P}^g$. As $\pi_1 = \pi\circ\pi_{13} = q\circ\pi_{12}$ we revisit the above direct images, but now using $q\circ\pi_{12}$. The direct image sheaves $R^n\pi_{12*}(\pi_{13}^*(\Omega))$ vanish if $n\neq 1$ by Bott's rule, hence

$$\begin{aligned} R^n(q\circ\pi_{12})_*(\pi_{12}^*(E\tilde{\mathcal{P}})\otimes\pi_{13}^*(\Omega)) &\simeq R^{n-1}q_*R^1\pi_{12*}(\pi_{12}^*(E\tilde{\mathcal{P}})\otimes\pi_{13}^*\Omega) \\ &\simeq R^{n-1}q_*(E\tilde{\mathcal{P}}\otimes R^1\pi_{12*}\pi_{13}^*\Omega) \\ &\simeq R^{n-1}q_*(E\tilde{\mathcal{P}}\otimes q^*R^1\pi_*(\Omega)) \\ &\simeq R^{n-1}q_*(E\tilde{\mathcal{P}})\otimes R^1\pi_*\Omega \\ &\simeq R^{n-1}q_*(E\tilde{\mathcal{P}}). \end{aligned}$$

The last isomorphism is Lemma 4.3.3. Combining the above calculations finally gives

$$\begin{aligned} \pi_*(\hat{E}(-1)) &\simeq q_*(EK\tilde{\mathcal{P}}) & R^1\pi_*(\hat{E}(-1)) &\simeq R^1q_*(EK\tilde{\mathcal{P}}) \\ \pi_*(\hat{E}\Omega) &\simeq q_*(E\tilde{\mathcal{P}}) & R^1\pi_*(\hat{E}\Omega) &\simeq R^1q_*(E\tilde{\mathcal{P}}). \end{aligned}$$

The sheaves on the right hand sides are exactly $\mathbb{F}^i(E)$ and $\mathbb{F}^i(EK)$ for $i=0,1$. Inserting this on the first page of the Beilinson spectral sequence

$$\begin{array}{cccc|c}
 & & & & q \\
0 & 0 & 0 & & 2 \\
0 & \mathbb{F}^1(E) \boxtimes \mathcal{O}_{\mathbb{P}^g}(-1) & \mathbb{F}^1(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g} & & 1 \\
0 & \mathbb{F}^0(E) \boxtimes \mathcal{O}_{\mathbb{P}^g}(-1) & \mathbb{F}^0(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g} & & 0 \\
\hline
-2 & -1 & 0 & & p
\end{array}$$

with differentials on a slice J_A being multiplication by $a\Phi + \alpha \operatorname{Id}$ where $A \in \mathbb{P}^g$ corresponds to the line generated by $a\Phi + \alpha \operatorname{Id} \in \mathbb{C}\Phi \oplus H^0(K)$. The second page is stable and by Theorem 4.3.1, $\hat{E}(-1)$ is an extension

$$\begin{aligned}
&0 \to \mathcal{Q}(-1) \to \hat{E}(-1) \to \mathcal{R}(-1) \to 0 \\
&0 \to \mathbb{F}^0(E) \boxtimes \mathcal{O}_{\mathbb{P}^g}(-1) \to \mathbb{F}^0(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g} \to \mathcal{Q}(-1) \to 0 \\
&0 \to \mathcal{R}(-1) \to \mathbb{F}^1(E) \boxtimes \mathcal{O}_{\mathbb{P}^g}(-1) \to \mathbb{F}^1(EK) \boxtimes \mathcal{O}_{\mathbb{P}^g} \to 0.
\end{aligned}$$

which is exactly the extension obtained in Proposition 4.2.5 twisted by $\mathcal{O}_{\mathbb{P}^g}(-1)$. □

Remark 4.3.5. The Beilinson spectral sequence for $\hat{E}$ itself is rather complicated as it includes $R^q\pi_*(\Omega^p(p+1))$ which by Lemma 4.3.3 vanish for $q \neq 0$ and for $q = 0$ are trivial bundles of rank $\binom{g+1}{p+1}$.

The first page of the spectral sequence contains only two non-zero rows $q = 0$ and $q = 1$ with non-zero entries from $p = 0$ to $p = -g$. The second page is not stable, but the third is.

4.3.1 An alternative view upon injectivity

In this section we use the first hypercohomology spectral sequence and Beilinson's spectral sequence to give an alternative proof of Bonsdorff's injectivity result [14, Theorem 3.2.1].

Theorem 4.3.6. *Let (E, Φ), (F, Ψ) be two degree zero Higgs bundles of rank at least two on a Riemann surface of genus at least two. If $\hat{E} \simeq \hat{F}$ as holomorphic bundles on $J \times \mathbb{P}^g$, then $(E, \Phi) \simeq (F, \Psi)$ as Higgs bundles.*

Proof. As in [14, Theorem 3.2.1], the proof is a procedure for recovering a Higgs bundle from its transform $\hat{E}$ on $J \times \mathbb{P}^g$.

Let q_{ij} be the canonical projections from $\hat{J} \times J \times \mathbb{P}^g$ to the ij'th factors. If $\tilde{\mathcal{Q}}$ is a sheaf on $J_{\mathbb{P}^g}$ which is $IT(0)$ with respect to $\hat{S}_{\mathbb{P}^g}$, then as $\hat{S}_{\mathbb{P}^g}$ is an equivalence of categories

$\hat{S}_{\mathbb{P}^g}(\tilde{\mathcal{Q}})$ on $\hat{J}_{\mathbb{P}^g}$ is $WIT(g)$ with respect to $S_{\mathbb{P}^g}$. The sheaf $\tilde{\mathcal{Q}}$ is $IT(0)$ with respect to $\hat{S}_{\mathbb{P}^g}$ if by definition

$$R^s q_{13*}(q_{23}^*\tilde{\mathcal{Q}} \otimes q_{12}^*\mathcal{P}) = 0 \quad \text{for} \quad s \neq 0, \quad \text{and} \quad q_{13*}(q_{23}^*\tilde{\mathcal{Q}} \otimes q_{12}^*\mathcal{P}) \quad \text{is locally free.}$$

From [6, Chap. 3, Cor. 3.4]

$$\tilde{\mathcal{Q}} \simeq R^g q_{23*}(q_{13}^*(q_{13*}(q_{23}^*(\tilde{Q}) \otimes q_{12}^*\mathcal{P})) \otimes q_{12}^*\tilde{\mathcal{P}}^*).$$

It follows from Lemma 4.1.7 that $\hat{E} \simeq \hat{S}_{\mathbb{P}^g}({j_{\mathbb{P}^g}}_*\mathcal{Q}_\Theta)$ and that ${j_{\mathbb{P}^g}}_*\mathcal{Q}_\Theta$ is an $IT(0)$ sheaf on $J_{\mathbb{P}^g}$ with respect to $\hat{S}_{\mathbb{P}^g}$. We can therefore recover $\mathcal{Q}_\Theta$ on $\Sigma_{\mathbb{P}^g}$ by

$$\mathcal{Q}_\Theta \simeq j_{\mathbb{P}^g}^*(R^g q_{23*}(\hat{E} \otimes q_{12}^*\mathcal{P}^*)).$$

The data of a Higgs bundle is contained in the sheaf $\mathcal{Q}_\Theta$ on $\Sigma \times \mathbb{P}^g$. We can recover the Higgs bundle by applying the Beilinson spectral sequence to $\mathcal{Q}_\Theta(-1)$ with respect to the projection $\pi : \Sigma \times \mathbb{P}^g \to \Sigma$. The sheaf $\mathcal{Q}_\Theta$ is defined by a short exact sequence. Tensored by $\Omega^p(p-1)$

$$0 \to E \otimes \Omega^p(p-1) \to EK \otimes \Omega^p(p) \to \mathcal{Q}_\Theta \otimes \Omega^p(p-1) \to 0$$

its direct images under π vanish by Lemma 4.3.3 except for $p = 0$ and $p = 1$. In these cases, the exact sequence gives

$$\pi_*(\mathcal{Q}_\Theta(-1)) \simeq EK \quad \text{and} \quad \pi_*(\mathcal{Q}_\Theta \otimes \Omega) \simeq E.$$

The first page of Beilinson's spectral sequence only has non-zero terms $E_1^{-1,0} \simeq E(-1)$ and $E_1^{0,0} \simeq EK$. By definition of the spectral sequence, the differential is Θ as $\mathbb{P}^g = \mathbb{P}(\Phi\mathbb{C} \oplus H^0(K))$. The second page is stable, giving a short exact sequence

$$0 \to E_1^{-1,0} \simeq E(-1) \xrightarrow{\Theta} E_1^{0,0} \simeq EK \to \mathcal{Q}_\Theta(-1) \to 0$$

from which we recover the Higgs field by restriction. □

Remark 4.3.7. In the proof of Theorem 4.3.6 we silently use Remark 1.1.9.

Remark 4.3.8. The difference between the proof given above and that of [14, Theorem 3.2.1] is the use of the Beilinson spectral sequence. Bonsdorff use an argument adapted to the situation at hand, while the above is using a standard spectral sequence.

4.4 A family of homogeneous bundles

In this section we study the first hypercohomology spectral sequence and its implication for a transformed Higgs bundle considered as a family of holomorphic bundles on J parametrised by $\mathbb{P}^g$.

Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two, and let $\mathcal{C}(E, A)$ be the complex

$$E \xrightarrow{a\Phi+\alpha\,\mathrm{Id}} EK$$

on Σ concentrated in degrees zero and one with $A = [a : \alpha] \in \mathbb{P}^g$.

Lemma 4.4.1. *For a stable Higgs bundle (E, Φ) of degree zero and rank at least two, the complex $\mathcal{C}(E, A)$ is $IT(1)$ with respect to the absolute Fourier–Mukai transform for all $A \in \mathbb{P}^g$.*

Proof. Assume first that $A = [1 : \alpha]$ for $\alpha \in H^0(\Sigma, K)$. For every $\xi \in J$ the hypercohomology of the stable Higgs bundle

$$EL_\xi \xrightarrow{\Phi+\alpha\,\mathrm{Id}} EL_\xi K$$

is concentrated in degree one with a dimension independent of ξ. For $A = [1 : \alpha]$ the complex $\mathcal{C}(E, A)$ is therefore $IT(1)$ with respect to the absolute Fourier–Mukai transform.

Let $A = [0 : \alpha]$. The complex $E(\alpha, \xi)$

$$EL_\xi \xrightarrow{\alpha\,\mathrm{Id}} EL_\xi K$$

might not be a stable Higgs bundle and the previous argument does not work; however, the hypercohomology groups can be computed by the first hypercohomology spectral sequence (1.4) for $E(\alpha, \xi)$

$${}^{I}E_2^{p,q} = H^p(\Sigma, \mathcal{H}^q(E(\alpha, \xi))) \simeq H^p(\Sigma, \mathcal{H}^q(E(\alpha, 0)) \otimes L_\xi).$$

As $\alpha\,\mathrm{Id}$ is an injective sheaf map the only non-zero cohomology sheaf is in degree one where it is the cokernel sheaf

$$\mathcal{Q}_\alpha = \mathrm{coker}(E \xrightarrow{\alpha\,\mathrm{Id}} EK) = \mathcal{H}^1(E(\alpha, 0)).$$

The cokernel is a sum of skyscrapers on Σ, each length of which is the vanishing multiplicity of α at its zeros. As $\mathcal{Q}_\alpha$ is a sum of skyscrapers $\mathcal{Q}_\alpha \otimes L_\xi \simeq \mathcal{Q}_\alpha$ and hence higher cohomology groups of $\mathcal{Q}_\alpha \otimes L_\xi$ vanish. The only non-zero entry on the second page is $(p, q) = (0, 1)$, proving that the second page is stable. It follows immediately that $\mathbb{H}^i(E(\alpha, \xi)) = 0$ for $i \neq 1$, and since $\mathcal{Q}_\alpha \otimes L_\xi \simeq \mathcal{Q}_\alpha$ the dimensions of ${}^{I}E_2^{0,1}$ are independent of ξ, proving that $\mathbb{F}^1(\mathcal{C}(E, A))$ is locally free, and that $\mathcal{C}(E, A)$ is $IT(1)$ with respect to the absolute Fourier–Mukai transform when $A = [0 : \alpha]$. $\square$

Definition 4.4.2. Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two. For any $A \in \mathbb{P}^g$ denote by $\widehat{E_A}$ the only non-zero cohomology sheaf of the absolute Fourier–Mukai transform of $\mathcal{C}(E, A)$.

Lemma 4.4.3 ([14] Proposition 3.1.10)**.** *For any $A \in \mathbb{P}^g$ restriction to J_A commutes with the absolute Fourier–Mukai transform:*

$$\hat{E}|_{J_A} \simeq \widehat{E_A}.$$

4.4.1 Homogeneous bundles

Definition 4.4.4. A holomorphic vector bundle U on an abelian variety X which is invariant under pullback by translations $\tau_x : X \to X$, i.e. $\tau_x^* U \simeq U$ for all $x \in X$, is called *homogeneous.*

Definition 4.4.5. For $A = [a : \alpha] \in \mathbb{P}(\mathbb{C}\Phi \oplus H^0(K))$ define $\det \Phi_A := \det(a\Phi + \alpha \operatorname{Id})$.

Theorem 4.4.6. *Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two, and let $A \in \mathbb{P}^g$. If the holomorphic section $\det \Phi_A$ of K^r does not vanish identically on Σ, then $\hat{E}|_{J_A}$ is a homogeneous vector bundle.*

Proof. As $\det \Phi_A$ is a non-zero section of K^r the differential $a\Phi + \alpha \operatorname{Id}$ in the complex $\mathcal{C}(E, A)$ is an injective sheaf morphism. This extends $\mathcal{C}(E, A)$ to a short exact sequence of coherent sheaves on Σ

$$0 \to E \xrightarrow{a\Phi + \alpha \operatorname{Id}} EK \to \mathcal{Q}_A \to 0.$$

The sheaf $\mathcal{Q}_A$ is the cokernel sheaf supported on the vanishing locus of $\det \Phi_A$ and is thus a direct sum of skyscraper sheaves of various lengths. By the first hypercohomology spectral sequence (or rather its sheaf analogue Corollary 4.2.3) the Fourier–Mukai transform of the complex $\mathcal{C}(E, A)$ is isomorphic to the Fourier–Mukai transform of $\mathcal{Q}_A$, up to a shift. As $j : \Sigma \to J$ is an embedding, $j_*(\mathcal{Q}_A)$ is also a direct sum of skyscraper sheaves. It follows from the isomorphism of functors in Corollary 4.1.8 that $\widehat{E_A}$ is the image of a sum of skyscraper sheaves under Mukai's original transform $\hat{S} : D(J) \to D(J)$.

Let $x \in J$ be a point and consider the pullback of $\widehat{E_A}$:

$$\tau_x^*(\widehat{E_A}) \simeq \tau_x^* \hat{S}(j_*(\mathcal{Q}_A)) \simeq \hat{S}(j_*(\mathcal{Q}_A) \otimes \mathcal{P}_x) \simeq \hat{S}(j_*(\mathcal{Q}_A)) \simeq \widehat{E_A}.$$

The second isomorphism is the projection formula and third isomorphism follows from the support of $j_*(\mathcal{Q}_A)$ being points in J. $\square$

Given a stable Higgs bundle (E, Φ) of degree zero and rank at least two, consider the open set $V \subset \mathbb{P}^g$ defined by

$$V = \{A \in \mathbb{P}^g \mid \det \Phi_A \neq 0\}.$$

Theorem 4.4.6 shows that $\hat{E}|_{J \times V}$ is a family of homogeneous vector bundles on J parametrised by V.

Lemma 4.4.7. *The complement of $V \subset \mathbb{P}^g$ is at most* $\operatorname{rk} E$ *points.*

Proof. Let $r = \operatorname{rk} E$, and $A \in \mathbb{P}^g$ be of the form $[0 : \alpha]$, then $\det(\Phi_A) = \alpha^r$ and the only solution to $\det \Phi_A = 0$ is $\alpha = 0$.

Assume $A = [1 : \alpha]$, then

$$\det \Phi_A = \alpha^r + \operatorname{Tr} \Phi \, \alpha^{r-1} + \cdots + \det \Phi.$$

The characteristic polynomial of Φ defines an equation in the total space of the canonical bundle on Σ,

$$0 = \det(\Phi + \eta \operatorname{Id}) = \eta^r + \operatorname{Tr} \Phi \, \eta^{r-1} + \cdots + \det \Phi,$$

with η the tautological section on K. The solutions of the characteristic polynomials constitute a curve S in the total space of K. If there exists a point $[1 : \alpha] \in \mathbb{P}^g$ such that $\det \Phi_A = 0$, then as α is a section of K the image of this section in the total space is a component of S. As there can be at most r such components the complement $\mathbb{P}^g \setminus V$ is empty or at most a r points. □

4.4.2 Unipotent bundles

Definition 4.4.8. A vector bundle U on J is unipotent if U has a filtration of holomorphic subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \simeq \mathcal{O}_J$ for all $i = 1, \ldots, n$.

Theorem 4.4.9. *For (E, Φ) and $A \in \mathbb{P}^g$ as in Theorem 4.4.6 let $\{z_1^A, \ldots, z_n^A\}$ be the zeros of* $\det \Phi_A$ *then $\hat{E}|_{J_A}$ is a direct sum of twisted unipotent vector bundles*

$$\hat{E}|_{J_A} \simeq \bigoplus_{i=1}^{n} L_i \otimes U_i \tag{4.8}$$

where L_i are the degree zero line bundles given by the image of z_i^A in J by the Abel–Jacobi map and U_i are unipotent bundles determined by the cokernel $\mathcal{Q}_A$ at z_i^A. The rank of U_i is the vanishing multiplicity of $\det \Phi_A$ *at z_i^A.*

Proof. The theorem follows directly from Theorem 4.4.6 and Mukai's Theorem 4.17 in [56] □

Corollary 4.4.10. *Let $A \in \mathbb{P}^g$ be a point where* $\det \Phi_A$ *has simple zeros, then $\hat{E}|_{J_A}$ is a sum of line bundles.*

Proof. If $\det \Phi_A$ only has simple zeros, there are $2 \operatorname{rk} E(g-1)$ distinct zeros and all U_i are trivial line bundles. □

4.5 Spectral data transform for Higgs bundles

In this section we consider a stable Higgs bundle (E, Φ) of degree zero and rank r at least two. We furthermore assume that (E, Φ) has smooth spectral data (S, L), meaning that the curve of eigenvalues of Φ in the total space of the canonical bundle $p : K \to \Sigma$

$$0 = \det(\Phi + \eta \,\mathrm{Id})$$

with η the tautological section of p^*K on K is a smooth r-sheeted branched cover of Σ, and $p_*L \simeq E$.

Define a hypersurface $Y \subset S \times \mathbb{P}^g$ by

$$Y = \{(x, A) \in S \times \mathbb{P}^g \,|\, a_0\eta(x) + \sum a_i p^*(\alpha_i)(x) = 0\}$$

where $A = [a_0 : a_1 : \cdots : a_g]$ and $\{\alpha_1, \ldots, \alpha_g\}$ is a chosen basis for $H^0(K)$. The hypersurface is the zero locus of a section of $p^*K \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)$ on $S \times \mathbb{P}^g$. Denote by $i : Y \hookrightarrow S \times \mathbb{P}^g$ the inclusion, by $\pi : S \times \mathbb{P}^g \to \mathbb{P}^g$ the canonical projection, and by $\pi' = \pi \circ i$ the composition.

Lemma 4.5.1. *If $S \xrightarrow{p} \Sigma$ is a smooth r-sheeted branched cover, then $Y \xrightarrow{\pi'} \mathbb{P}^g$ is a smooth $2r(g-1)$-sheeted branched cover.*

Proof. The hypersurface Y is defined by the section

$$s = a_0\eta + a_1 p^*\alpha_1 + \cdots + a_g p^*\alpha_g$$

where $\{\alpha_1, \ldots, \alpha_g\}$ is a basis for $H^0(K)$ and $[a_0 : a_1 : \cdots : a_g]$ are coordinates on $\mathbb{P}^g$. If $(x', A') \in S \times \mathbb{P}^g$ is a singular point of Y, then

$$s(x', A') = 0 \quad \text{and} \quad \frac{\partial s}{\partial x}(x', A') = 0 \quad \text{and} \quad \frac{\partial s}{\partial a_i}(x', A') = 0 \quad \text{for all } i.$$

As

$$\frac{\partial s}{\partial a_i} = \begin{cases} \eta & i = 0 \\ p^*\alpha_i & i \neq 0, \end{cases}$$

a singular point $(x', A') \in Y$ has $p^*(\alpha_i)(x') = \alpha_i(p(x')) = 0$ for all $i = 1, \ldots, g$. Such a point does not exist as $H^0(K)$ is base point free when the genus is at least two, proving that Y is smooth.

The line bundle p^*K has degree $2r(g-1)$ on S, and Y is the zero locus of a section of $p^*K \boxtimes \mathcal{O}_{\mathbb{P}^g}(1)$ it is a $2r(g-1)$-sheeted branched cover over $\mathbb{P}^g$. □

Remark 4.5.2. The result in Lemma 4.5.1 is contrasted by the support of the cokernel sheaf $\mathcal{Q}_\Theta$ not being smooth at its intersection with the divisor at infinity $\Sigma \times \mathbb{P}^g$ where the points have high multiplicity.

Lemma 4.5.3. *The branch locus of $Y \to \mathbb{P}^g$ corresponds to $[1 : \alpha]$ with $\alpha \in H^0(K)$ taking values in a ramification point of $S \to \Sigma$ or $[0 : \alpha]$ with $\alpha \in H^0(K)$ having zeros at branch points of $S \to \Sigma$.*

Proof. By definition of the spectral curve, $x = (z, a(z)) \in T^*\Sigma$ is in S if and only if $-a(z)$ is an eigenvalue of Φ_z. The branch points of $S \xrightarrow{p} \Sigma$ are the points $z \in \Sigma$ where Φ_z has multiple eigenvalues.

Let $A = [a_0 : \alpha] \in \mathbb{P}^g$ be short hand for $[a_0 : a_1 : \cdots : a_g]$ with $\alpha = a_1\alpha_1 + \cdots + a_g\alpha_g$. Assume $a_0 \neq 0$, then $\pi'^{-1}(\{A\})$ consists of $(z, a(z))$ such that $a(z) = -\alpha(z)$, or the points $z \in \Sigma$ where $\alpha(z)$ is an eigenvalue of Φ_z. Assume $a_0 = 0$, then $\pi'^{-1}(\{A\})$ consists of points $(z, a(z)) \in S$ where $\alpha(z) = 0$.

The branch points of $Y \xrightarrow{\pi'} \mathbb{P}^g$ are therefore the $A = [1 : \alpha]$ such that there is $z \in \Sigma$ such that $-\alpha(z)$ is a multiple eigenvalue of Φ_z, or the $A = [0 : \alpha]$ where α vanish at a branch point of $S \xrightarrow{p} \Sigma$. □

Definition 4.5.4. Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two with smooth spectral data (S, L). Define the spectral transform of (E, Φ) to be the sheaf

$$\check{E} = \pi_{13*}(\pi_{23}^*(L \otimes p^*K \otimes \mathcal{O}_{\mathbb{P}^g}(1)) \otimes \mathcal{O}_{J\times Y} \otimes \pi_{12}^*(\mathrm{Id} \times p)^*\tilde{\mathcal{P}})$$

where π_{ij} are the canonical projections on the ij'th factor of $J \times S \times \mathbb{P}^g$ and $\mathcal{O}_{J\times Y}$ is the structure sheaf of the divisor $J \times Y$ in $J \times S \times \mathbb{P}^g$.

Theorem 4.5.5. *Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two with smooth spectral data. Then the spectral data transform and the Fourier–Mukai transform are equivalent, i.e.*

$$\hat{E} \simeq \check{E}$$

as holomorphic bundles on $J \times \mathbb{P}^g$.

Proof. The spectral line bundle L together with the tautological section η defines the Higgs bundle on Σ by pushing down the sequence $L \xrightarrow{\eta} L \otimes p^*K$. We extend (L, η) to a family

$$\mathcal{C}(L) = L \xrightarrow{\Psi} L \otimes p^*K \otimes \mathcal{O}_{\mathbb{P}^g}(1)$$

on $S \times \mathbb{P}^g$ which when restricted to $S \times [a_0 : a_1 : \cdots : a_g]$ is

$$L \xrightarrow{a_0\eta + p^*\alpha} L \otimes p^*K \quad \text{with} \quad \alpha = a_1\alpha_1 + \cdots + a_g\alpha_g.$$

This family pushes down to the family (4.1) on $\Sigma \times \mathbb{P}^g$. Let $\mathcal{P}_S$ be the pullback of the Poincaré bundle to $J \times S$, i.e. $\mathcal{P}_S = (\mathrm{Id} \times p)^*\tilde{\mathcal{P}} = (\mathrm{Id} \times (j \circ p))^*\mathcal{P}$.

Since the morphism Ψ in $\mathcal{C}(L)$ is a non-zero section of a line bundle it is an injective sheaf map. It therefore follows from the first hypercohomology spectral sequence that the hyper direct image of $\pi_{23}^*\mathcal{C}(L)\otimes\pi_{12}^*\mathcal{P}_S$ along π_{13} is concentrated in degree one where it is locally free. We furthermore know from the first hypercohomology spectral sequence that

$$\mathbb{R}^1\pi_{13*}(\pi_{23}^*\mathcal{C}(L)\otimes\pi_{12}^*\mathcal{P}_S)\simeq\check{E}.$$

It now directly follows from base-change that

$$\mathbb{R}^1\pi_{13*}(\pi_{23}^*\mathcal{C}(L)\otimes\pi_{12}^*\mathcal{P}_S)\simeq\mathbb{F}_{\mathbb{P}^g}(\mathcal{C}(E))=\hat{E}$$

proving the theorem. □

Proposition 4.5.6. *Let S be a smooth spectral curve and Y the associated hypersurface in $S\times\mathbb{P}^g$, then*

$$\mathrm{Pic}(Y)\simeq\mathrm{Pic}(S)\times\mathbb{Z}.$$

Proof. Let $x_0=(z_0,\alpha_0(z_0))\in S$ be expressed as a point in the total space of the canonical bundle, then the fibre of the projection $Y\to S$ at x_0 is given by the solutions to the equation

$$0=a_0\alpha_0(z_0)+a_1\alpha_1(z_0)+\cdots+a_g\alpha_g(z_0)$$

on $\mathbb{P}^g$. The equation is the vanishing locus of a section of $\mathcal{O}_{\mathbb{P}^g}(1)$ and is therefore a projective space of dimension $g-1$. As $x_0\in S$ was arbitrary Y is a $\mathbb{P}^{g-1}$-bundle on S. The result now follows from [31, Exercise III.12.5]. □

A simple application of Grothendieck–Riemann–Roch shows that the restriction of $\hat{E}$ to a $\mathbb{P}^g$-slice of $J\times\mathbb{P}^g$ has Chern character

$$r(g-1)+r(g-1)\,\mathrm{ch}(\mathcal{O}_{\mathbb{P}^g}(1)).\tag{4.9}$$

Together with the above proposition this determines which line bundles on Y can be used to construct transformed Higgs bundles.

Corollary 4.5.7. *Let S be a smooth spectral curve and $Y\xrightarrow{\pi'}\mathbb{P}^g$ the associated $2r(g-1)$-sheeted branched cover of $\mathbb{P}^g$. Then only line bundles of degree $r(r+1)(g-1)$ on S and degree 1 on $\mathbb{P}^g$ have Chern character* (4.9) *under the direct image to $\mathbb{P}^g$.*

Proof. From Proposition 4.5.6 every line bundle N on Y is of the form

$$N\simeq i^*M\otimes i^*\mathcal{O}_{\mathbb{P}^g}(m)$$

for $i:Y\hookrightarrow S\times\mathbb{P}^g$ the inclusion and M a line bundle on S and m an integer. By a simple, but long and tedious calculation using Grothendieck–Riemann–Roch, we obtain

$$\mathrm{ch}(\pi'_*N)=\mathrm{ch}\big(\mathcal{O}_{\mathbb{P}^g}(m-1)\big)\big(2r(g-1)+(g-1)-\deg M\big)+\mathrm{ch}\big(\mathcal{O}_{\mathbb{P}^g}(m)\big)\big(\deg M-(g-1)r^2\big).$$

This Chern character agrees with (4.9) only if $\deg M=r(r+1)(g-1)$ and $m=1$. □

Remark 4.5.8. Notice that if M has degree $r(r+1)(g-1)$ on S the degree of $M \otimes p^*K^{-1}$ is $r(r-1)(g-1)$. Line bundles of this degree exactly push down to a degree zero vector bundle on Σ, and as the spectral curve is smooth the corresponding Higgs bundle is stable. The spectral transform of any line bundles of $\mathrm{Pic}^{r(r+1)(g-1)}(S) \times \{1\} \subset \mathrm{Pic}(Y)$ is therefore in the image of the Fourier–Mukai transform for Higgs bundles.

4.5.1 Spectral transform and homogeneous bundles

In this section we investigate the consequences for $\hat{E}$ of (E, Φ) having smooth spectral data. First of all, we will see that the holomorphic structure of the homogeneous bundles $\hat{E}|_{J_A}$ is tractable, and furthermore that information contained in the whole family of homogeneous vector bundles recovers the spectral curve.

First, notice that when (E, Φ) has smooth spectral curve, the set

$$V = \{A \in \mathbb{P}^g \mid \det \Phi_A = 0\} = \mathbb{P}^g.$$

This follows as an $A \in \mathbb{P}^g$ with $0 = \det \Phi_A = \det(a\Phi + \alpha\,\mathrm{Id})$ implies that $\frac{\alpha}{a}$ defines an irreducible component of the spectral curve, assuming $a \neq 0$; if $a = 0$, then $0 = \det \Phi_A$ implies $\alpha = 0$.

Therefore $\hat{E}$ on $J \times \mathbb{P}^g$ is a family of homogeneous bundles on J parametrised by $\mathbb{P}^g$.

Proposition 4.5.9. *If (E, Φ) is a stable Higgs bundle with smooth spectral data, then for every $A = [1 : \alpha] \in \mathbb{P}^g$ the unipotent bundles in Theorem 4.4.9 are successive* non-trivial *extensions of the trivial bundle.*

Proof. The result is clear from Corollary 4.4.10 if $\det \Phi_A$ only has simple zeros. Assume $z_0 \in \Sigma$ is a zero of multiplicity k of $\det(\Phi_A)$, and $A = [1 : \alpha]$. Then $-\alpha(z_0)$ is an eigenvalue of multiplicity k of Φ_{z_0} and z_0 is therefore a branch point of $S \to \Sigma$ where k sheets come together. On S the section $\eta + p^*\alpha$ vanishes at $(z_0, -\alpha(z_0)) \in S$ with multiplicity k.

As in the proof of Theorem 4.4.6, we can show that

$$\check{E}|_{J \times [1:\alpha]} = \pi_{1*}(\pi_2^* \mathcal{Q}_\alpha \otimes \mathcal{P}_S)$$

where π_i is the projection onto the $i'th$ component of $J \times S$, and $\mathcal{Q}_\alpha$ is cokernel sheaf of $\mathcal{C}(L)$ restricted to $S \times [1 : \alpha]$

$$0 \to L \xrightarrow{\eta + p^*\alpha} L \otimes p^*K \to \mathcal{Q}_\alpha \to 0.$$

Since $\eta + p^*\alpha$ vanishes with multiplicity k at $(z_0, -\alpha(z_0))$ the cokernel sheaf $\mathcal{Q}_\alpha$ is locally $\mathcal{O}/w^k\mathcal{O}$ where w is a coordinate on S. The unipotent bundle corresponding to the zero $z_0 \in \Sigma$ is therefore the Fourier–Mukai transform of $\mathcal{O}/w^k\mathcal{O}$ which is a rank k bundle defined as $k-1$ successive non-trivial extensions of $\mathcal{O}$ by itself. The extensions are non-trivial as a trivial extension would induce a splitting of the unipotent bundle such a splitting would correspond to

$$\mathcal{O}/w^k\mathcal{O} \simeq \mathcal{O}/w^i\mathcal{O} \oplus \mathcal{O}/w^j\mathcal{O} \quad \text{with} \quad i+j=k \quad \text{and} \quad i,j \neq 0,$$

which is not the case. □

Theorem 4.5.10. *Let (E,Φ) be a stable Higgs bundle of degree zero and rank at least two with smooth spectral data. Then the holomorphic structure of the family of homogeneous bundles $\hat{E} \to J \times \mathbb{P}^g$ recovers the spectral curve from $\hat{E}$.*

Proof. We restrict our attention to the restriction $\hat{E}|_{J\times H^0(K)}$ and denote it by $\hat{E}$ as well. For every $\alpha \in H^0(K)$ the homogeneous bundle $\hat{E}|_{J_\alpha}$ is

$$\hat{E}|_{J_\alpha} \simeq \bigoplus_{i=1}^{n} L_{z_i^\alpha} \otimes U_i^\alpha$$

where the z_i^α's are the zeros of $\det(\Phi + \alpha\,\mathrm{Id})$ and $\mathrm{rk}\,U_i^\alpha$ the corresponding multiplicities.

Define a map into the symmetric product of Σ

$$H^0(K) \xrightarrow{\Psi} S^{2r(g-1)}(\Sigma) \quad \text{by} \quad \alpha \mapsto \{z_1^\alpha, \ldots, z_{2r(g-1)}^\alpha\}$$

with the z_i^α's repeated with the appropriate multiplicities. This map is holomorphic (actually algebraic) as it coincides with the map obtained from the divisor of support of the cokernel of the Higgs family on $\Sigma \times H^0(K)$ defining $\hat{E}$. Such a divisor gives a unique morphism $H^0(K) \to S^{2r(g-1)}(\Sigma)$ with the property that the divisor in $\Sigma \times H^0(K)$ is the pullback of the universal divisor in $\Sigma \times S^{2r(g-1)}(\Sigma)$, see [65, Theorem 16.4].

Consider now the fibre product W' of Ψ and the $2r(g-1)$: 1-branched covering $q : \Sigma \times S^{2r(g-1)-1}(\Sigma) \to \Sigma$. By the universality of the fibre product, W' is isomorphic to its image W in $\Sigma \times H^0(K)$.

A point $(z,\alpha) \in \Sigma \times H^0(K)$ is in W if and only if $\det(\Phi+\alpha)(z) = 0$, i.e. that $-\alpha(z)$ is an eigenvalue of Φ_z. The image of W under the evaluation map $\Sigma \times H^0(K) \to T^*\Sigma$ is therefore set theoretically the spectral curve. As the spectral curve is smooth the induced scheme structure on the image of W recovers the spectral curve as an algebraic curve. □

Remark 4.5.11. Theorem 4.5.10 gives a procedure of extracting data of a transformed Higgs bundle $\hat{E}$ to produce a spectral curve. It does so by exploiting that $\hat{E}$ is a family of homogeneous bundles which gives a morphism $\mathbb{P}^g \to S^{2r(g-1)}(\Sigma)$ this map has image in a fibre of $S^{2r(g-1)}(\Sigma) \to J$ which is a projective space. From this morphism we can recover the spectral curve. If the Nahm transform is an isometry between moduli spaces there ought to be an analogue of the Hitchin fibration in the 'moduli space' of transformed

Higgs bundles (if such a thing exists). Based on the above, the base of the fibration might be rational maps $\mathbb{P}^g \to \mathbb{P}^{2r(g-1)-g}$. Theorem 4.5.10 and Corollary 4.5.7 collectively proves that the generic fibre is the same Jacobian of line bundles as in the original Hitchin fibration.

5. Limiting holomorphic structure

In this chapter we will discuss the holomorphic structure at infinity in $J \times \mathbb{P}^g$ of a Nahm transformed Higgs bundle.

Firstly, we consider the big stratum $J \times W \subset J \times \mathbb{P}^{g-1}$ with $W \subset \mathbb{P}^{g-1} = \mathbb{P}(H^0(K))$ consisting of holomorphic differentials without multiple zeros. In Section 5.1 we see that all transformed Higgs bundles are isomorphic on $J \times W$.

In Section 5.2 we make use of the product structure of the divisor at infinity $J \times \mathbb{P}^{g-1}$ to describe the isomorphism class of $\hat{E}_\infty = \hat{E}|_{J\times\mathbb{P}^{g-1}}$ as a family of holomorphic bundles on $\mathbb{P}^{g-1}$ parametrised by J. We see that $\hat{E}_\infty$ does depend on the original Higgs bundle even though on $J \times W$ it is fixed. We consider the jump locus of the holomorphic structure of $\hat{E}_\infty$ considered as a family of holomorphic vector bundles, and see that it includes a subset depending on the holomorphic bundle in the original Higgs bundle. We shall see that in the special case of a stable bundle the family is constant, and we shall give a complete description in the case where the genus is two and the rank is two. Lastly, we study the special case of the rank three canonical Higgs bundle and see that quite surprisingly the holomorphic structure also depends on the curve on which the Higgs bundles live.

5.1 Stratification

In this section we consider the restriction of a transformed Higgs bundle to the big stratum $J \times W$ with $W \subset \mathbb{P}^{g-1} = \mathbb{P}(H^0(K))$ the complement of the discriminant locus of differentials with multiple zeros.

Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two. Recall that to define $\hat{E} \to J \times \mathbb{P}^g$ we extend $E \xrightarrow{\Phi} EK$ to a family

$$\mathcal{C}(E) = E \xrightarrow{\Theta} E \otimes K(1)$$

on $\Sigma \times \mathbb{P}^g$ where for $[a : \alpha] \in \mathbb{P}^g$ the Higgs bundle on $\Sigma \times [a : \alpha]$ is

$$E \xrightarrow{a\Phi + \alpha\,\mathrm{Id}} E \otimes K.$$

In this section we are interested in the extension to a family $\mathcal{C}(E)_W$ parametrised by $W \subset \mathbb{P}^{g-1} \subset \mathbb{P}^g$,

$$E \xrightarrow{\Theta_W} E \otimes K$$

on $\Sigma \times W$. As in the proof of [14, Proposition 3.1.10], the Fourier–Mukai transform $\hat{E}$ restricted to $J \times W$ is the Fourier–Mukai transform defined by the family $\Sigma \times W$. If we denote by q_{ij} the canonical projection to the ij'th factor of $J \times \Sigma \times W$, then

$$\hat{E}|_{J\times W} \simeq \mathbb{R}^1 q_{13_*}(q_{23}^*(\mathcal{C}(E)_W) \otimes q_{12}^*(\tilde{\mathcal{P}})) \simeq q_{13_*}(q_{23}^*(\mathcal{Q}_W) \otimes q_{12}^*(\tilde{\mathcal{P}}))$$

where $\mathcal{Q}_W$ is the cokernel of the injective sheaf map Θ_W. The isomorphism follows from the first hypercohomology spectral sequence.

As W is the complement of the discriminant locus, $\mathcal{Q}_W$ is the rank $r = \operatorname{rk} E$ trivial bundle supported on the divisor $D^W \subset \Sigma \times W$ for which $(z, \alpha) \in D^W$ if $\alpha(z) = 0$. This follows as for $\alpha \in W$ the cokernel of $E \xrightarrow{\alpha \operatorname{Id}} EK$ on $\Sigma \times \{\alpha\}$ is supported on the zeros $z_1^\alpha, \ldots, z_{2g-2}^\alpha$ of α, and in a neighbourhood of these points

$$0 \to \mathcal{O}^{\oplus r} \xrightarrow{z \operatorname{Id}} \mathcal{O}^{\oplus r} \to (\mathcal{O}/z\mathcal{O})^{\oplus r} \to 0.$$

As $\mathcal{O}_{\mathbb{P}^{g-1}}(1)$ is trivial on W it follows that $\mathcal{Q}_W \simeq \mathcal{O}_{D^W}^{\oplus r}$ on $\Sigma \times W$.

We therefore get

$$\hat{E}|_{J \times W} \simeq q_{13_*}(\mathcal{O}_{J \times D^W} \otimes q_{12}^*(\mathcal{P}))^{\oplus r}.$$

If we let $\iota : J \times D^W \hookrightarrow J \times \Sigma \times W$ denote the inclusion, the projection formula gives

$$\hat{E}|_{J \times W} \simeq q_{13_*}\iota_*(\iota^* q_{12}^* \tilde{\mathcal{P}})^{\oplus r} \simeq (q_{13} \circ \iota)_*(q_{12} \circ \iota)^*(\tilde{\mathcal{P}})^{\oplus r}.$$

Since D^W is a divisor with the property that for each $\alpha \in W$

$$D^W|_{\Sigma \times \{\alpha\}} = z_1^\alpha + \cdots + z_{2g-2}^\alpha$$

is a divisor of degree $2g-2$ we get a holomorphic map $f : W \to S^{2g-2}(\Sigma)$ mapping α to the collection of its zeros on Σ. As for fixed α the z_i^α's are distinct points, the image is in the complement of the branched locus of the quotient map $\Sigma \times \cdots \times \Sigma \to S^{2g-2}(\Sigma)$ defined by the symmetric group. The map

$$\rho : \Sigma \times S^{2g-3}(\Sigma) \to S^{2g-2}(\Sigma) \quad \text{defined by} \quad (z_1, \{z_2, \ldots, z_{2g-2}\}) \mapsto \{z_1, \ldots, z_{2g-2}\}$$

is holomorphic. Now D^W fits into a pullback diagram

$$\begin{tikzcd} D^W \arrow[r, "f'"] \arrow[d, "\rho'"] & \Sigma \times S^{2g-3}(\Sigma) \arrow[d, "\rho"] \\ W \arrow[r, "f"] & S^{2g-2}(\Sigma) \end{tikzcd}$$

Notice also that

$$\operatorname{Id} \times \rho' = q_{13} \circ \iota : J \times D^W \to J \times W$$
$$q_{12} \circ \iota = p_{12} \circ (\operatorname{Id} \times f') : J \times D^W \to J \times \Sigma$$

where $p_{12} : J \times \Sigma \times S^{2g-3}(\Sigma) \to J \times \Sigma$ is the canonical projection. Using these identities we have

$$\hat{E}|_{J \times W} \simeq (\operatorname{Id} \times \rho')_*((\operatorname{Id} \times f')^*(\hat{\mathcal{P}}))^{\oplus r} \quad \text{where} \quad \hat{\mathcal{P}} = p_{12}^* \tilde{\mathcal{P}}. \tag{5.1}$$

By noting that D^W is independent of (E, Φ) we see that the maps π' and f' derived from D^W are independent of (E, Φ), thus we have proved the following theorem and immediate corollary.

Theorem 5.1.1. *For a stable Higgs bundle (E, Φ) of degree zero and rank at least two, the transform $\hat{E}$ restricted to the big stratum $J \times W$ with $W \subset \mathbb{P}^{g-1} = \mathbb{P}(H^0(K))$ consisting of holomorphic differentials without multiple zeros, is independent of (E, Φ).*

Corollary 5.1.2. *Let (E, Φ) be a stable Higgs bundle of degree zero and rank at least two. Let $\alpha \in H^0(K)$ have simple zeros $z_1^\alpha, \ldots, z_{2g-2}^\alpha$, then*

$$\hat{E}|_{J \times [0:\alpha]} \simeq L_1^{\oplus r} \oplus \cdots \oplus L_{2g-2}^{\oplus r}$$

where L_i is the degree zero line bundle on J determined by the image of z_i^α by the Abel–Jacobi map.

Proof. As $\rho'^{-1}(\{\alpha\}) = \{(z_1^\alpha, \alpha), \ldots, (z_{2g-2}^\alpha, \alpha)\}$ it follows directly from (5.1) that

$$\hat{E}|_{J \times [0:\alpha]} \simeq (L_1 \oplus \cdots \oplus L_{2g-2})^{\oplus r}$$

completing the proof. □

Remark 5.1.3. Assume $\alpha \in H^0(K)$ has a multiple zero. It follows from Theorem 4.4.9 that the contribution to $\hat{E}|_{J \times [0:\alpha]}$ is determined by the local behaviour around the zeros of α. Assume $\alpha = z^k dz$ in a small neighbourhood over which E and EK trivialise, then the cokernel of $\mathcal{O} \xrightarrow{z^k} \mathcal{O}$ is $\mathcal{O}/\mathcal{I}^k$. Mukai [56, Theorem 4.12] shows that the transform of $\mathcal{O}/\mathcal{I}^k$ is a k-fold successive non-trivial extension of $\mathcal{O}$. The contribution of a multiplicity k zero to $\hat{E}|_{J \times [0:\alpha]}$ is r copies of a non-trivial holomorphic vector bundle of rank k.

5.2 Family of holomorphic bundles on $\mathbb{P}^{g-1}$

In this section we consider $\hat{E}_\infty = \hat{E}|_{J \times \mathbb{P}^{g-1}}$ as a family of holomorphic rank $2 \operatorname{rk} E(g-1)$ bundles on $\mathbb{P}^{g-1}$ parametrised by J with each member of the family denoted by $\hat{E}_{\infty,\xi}$.

Proposition 5.2.1. *The bundle $\hat{E}_{\infty,\xi}$ on $\mathbb{P}^{g-1}$ is for each $\xi \in J$ an extension of holomorphic vector bundles*

$$0 \to Q_\xi \to \hat{E}_{\infty,\xi} \to R_\xi \to 0$$

where Q_ξ and R_ξ are defined by

$$0 \to H^0(EL_\xi) \otimes \mathcal{O} \to H^0(EKL_\xi) \otimes \mathcal{O}(1) \to Q_\xi \to 0$$
$$0 \to R_\xi \to H^1(EL_\xi) \otimes \mathcal{O} \to H^1(EKL_\xi) \otimes \mathcal{O}(1) \to 0$$

Proof. Fix $\xi \in J$ and denote by i, i' the inclusions $i : \{\xi\} \times \mathbb{P}^{g-1} \to J \times \mathbb{P}^{g-1}$ and $i' : \{\xi\} \times \Sigma \times \mathbb{P}^{g-1} \to J \times \Sigma \times \mathbb{P}^{g-1}$. We denote by π_{13} and π'_{13} the canonical projections $\pi_{13} : J \times \Sigma \times \mathbb{P}^{g-1} \to J \times \mathbb{P}^{g-1}$ and $\pi'_{13} : \{\xi\} \times \Sigma \times \mathbb{P}^{g-1} \to \{\xi\} \times \mathbb{P}^{g-1}$. It follows from the first hypercohomology spectral sequence, that the transform of a Higgs bundle (E, Φ)

is $\hat{E} = \pi_{13*}(\pi_{23}^*\mathcal{Q}_\Theta \otimes \pi_{12}^*\mathcal{P})$, where $\mathcal{Q}_\Theta$ is the cokernel sheaf of the family (4.1) on $\Sigma \times \mathbb{P}^g$. If we restrict to the divisor $\Sigma \times \mathbb{P}^{g-1}$ for $\mathbb{P}^{g-1} = \mathbb{P}(H^0(K)) \subset \mathbb{P}^g$, then by an argument similar to [14, Proposition 3.1.10]

$$\hat{E}_\infty = \hat{E}|_{J\times\mathbb{P}^{g-1}} = \pi_{13*}(\pi_{23}^*\mathcal{Q}_\infty \otimes \pi_{12}^*\mathcal{P}),$$

where $\mathcal{Q}_\infty$ is cokernel of (4.1) restricted to $\Sigma \times \mathbb{P}^{g-1}$. We now use the general base change formula of [14, Corollary 2.1.4.(1)]. As $\hat{E}$ is locally free and $i' = \mathrm{Id} \times i$ only affects the locally free sheaf $\mathcal{P}$ we get

$$\hat{E}_{\infty,\xi} = i^*\hat{E} = \pi'_{13*}(\mathcal{Q}_\infty \otimes L_\xi).$$

As $Q_\infty L_\xi$ is the cokernel of $EL_\xi \xrightarrow{\Theta} EKL_\xi \otimes \mathcal{O}_{\mathbb{P}^{g-1}}(1)$ on $\Sigma \times \mathbb{P}^{g-1}$ we get a five term long exact sequence on $\mathbb{P}^{g-1}$ splitting into three short exact sequences:

$$\begin{aligned}
&0 \to \pi'_*(EL_\xi) \to \pi'_*(EL_\xi K)(1) \to Q_\xi \to 0\\
&0 \to Q_\xi \to \pi'_*(\mathcal{Q}_\infty L_\xi) \to R_\xi \to 0\\
&0 \to R_\xi \to R^1\pi_*(EL_\xi) \to R^1\pi'_*(EL_\xi K)(1) \to 0.
\end{aligned}$$

Notice that $R^i\pi_{13*}'(EL_\xi) \simeq H^i(EL_\xi) \otimes \mathcal{O}$ and likewise for direct image of EKL_ξ. As the map

$$H^0(EL_\xi) \otimes \mathcal{O} \to H^0(EL_\xi K) \otimes \mathcal{O}(1)$$

at a point $[\alpha] \in \mathbb{P}^{g-1}$ is multiplication by a representative α it is injective, and Q_ξ is therefore locally free. A similar argument shows that R_ξ is locally free, proving the result. $\square$

5.2.1 Steiner bundles

The holomorphic vector bundles Q_ξ and R_ξ in Proposition 5.2.1 are vector bundles on a projective space. Holomorphic vector bundles on projective spaces have not been classified, except in dimension one. However, many types of vector bundles are known. One particularly nice type are *Steiner bundles* of which $Q_\xi(-1)$ and R_ξ^* are examples.

Definition 5.2.2. A vector bundle E on $\mathbb{P}^n = \mathbb{P}(V)$ is a *Steiner bundle* if E admits a resolution of the form

$$0 \to I \otimes \mathcal{O}_{\mathbb{P}^n}(-1) \xrightarrow{\tau} W \otimes \mathcal{O}_{\mathbb{P}^n} \to E \to 0 \tag{5.2}$$

where I and W are finite dimensional vector spaces.

Bundles of this type have been studied by many authors, e.g. [21] who coined the term Steiner bundle. Steiner bundles are a generalisation of Schwarzenberger bundles [68] that are a special type of rank n bundles on $\mathbb{P}^n$.

The map τ between vector bundles $I \otimes \mathcal{O}_{\mathbb{P}^n}(-1)$ and $W \otimes \mathcal{O}_{\mathbb{P}^n}$ is uniquely determined by a tensor

$$t \in \mathrm{Hom}(V, \mathrm{Hom}(I, W)) = V^* \otimes I^* \otimes W.$$

The tensor should be such that for each $v \in V$, $\tau = t(v)$ must be fiberwise injective. When we in Section 5.2.5 consider the rank three canonical Higgs bundle, we will identify Steiner bundles by specifying the tensor t.

Proposition 5.2.3. *If F, G are Steiner bundles on $\mathbb{P}^n$, then $H^1(\mathbb{P}^n, F(1) \otimes G) = 0$*

Proof. It follows from the short exact sequence defining G and Bott's rule [64, p. 63] that $H^1(G(1)) = 0$ and $H^2(G) = 0$. If we tensor the sequence defining the Steiner bundle F by $G(1)$, then the long exact sequence of cohomology associated to this sequence gives the result. □

Corollary 5.2.4. *For each $\xi \in J$ the bundle $\hat{E}_{\infty,\xi}$ splits as*

$$\hat{E}_{\infty,\xi} \simeq Q_\xi \oplus R_\xi,$$

where Q_ξ and R_ξ are the bundles from Proposition 5.2.1.

Proof. This follows immediately from Proposition 5.2.3 as by definition $Q_\xi(-1)$ and R_ξ^* are Steiner bundles. □

5.2.2 Jump locus

Proposition 5.2.5. *The locus of J where the holomorphic structure of the family $\hat{E}_\infty$ changes contains the jump locus of*

$$\dim H^0(\Sigma, EL_\xi) \quad \textit{and} \quad \dim H^0(\Sigma, EKL_\xi)$$

considered as functions $J \to \mathbb{N}$.

Proof. Let $\xi \in J$ be given. It follows from Corollary 5.2.4 and the long exact cohomology sequences of the Steiner bundles $Q_\xi(-1)$ and R_ξ^* that

$$\begin{aligned}\dim H^0(\mathbb{P}^{g-1}, \hat{E}_{\infty,\xi}(-1)) &= \dim H^0(Q_\xi(-1)) + \dim H^0(R_\xi(-1)) \\ &= \dim H^0(\Sigma, EKL_\xi).\end{aligned}$$

This shows that the holomorphic structure of $\hat{E}_{\infty,\xi}(-1)$ and thereby of $\hat{E}_{\infty,\xi}$ change when a jump in $\dim H^0(\Sigma, EKL_\xi)$ occurs. Likewise, the global holomorphic sections of $\hat{E}^*_{\infty,\xi}$ are global holomorphic sections of R^*_ξ. By similar arguments as above,

$$\dim H^0(\mathbb{P}^{g-1}, \hat{E}^*_{\infty,\xi}) = \dim H^0(R^*_\xi) = \dim H^0(\Sigma, EL_\xi) - \chi(E)$$

proving that a change in $\dim H^0(\Sigma, EL_\xi)$ induces a change in holomorphic structure of $\hat{E}^*_{\infty,\xi}$ and thereby of $\hat{E}_{\infty,\xi}$. □

Remark 5.2.6. Proposition 5.2.5 is the first indication that despite the results of Section 5.1 the holomorphic structure of $\hat{E}_\infty$ is non-standard. In Section 5.2.4, the families are worked out completely when the genus is two and the rank is two as well.

5.2.3 Stable bundles

Proposition 5.2.7. *If (E, Φ) is a stable Higgs bundle of degree zero and rank at least two with E a stable bundle, then*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}^{\operatorname{rk} E(g-1)} \oplus \mathcal{O}(1)^{\operatorname{rk} E(g-1)}$$

on $\mathbb{P}^{g-1}$ for all $\xi \in J$.

Proof. The proof follows directly from Proposition 5.2.1 and Corollary 5.2.4 as

$$H^0(EL_\xi) = 0 \quad \text{and} \quad H^1(EKL_\xi) \simeq H^0(E^* L_{-\xi})^* = 0$$

for all ξ when E is stable of degree zero. □

5.2.4 Complete description for genus two and rank two

We consider the special case of genus two and rank two. In this case, we use Hitchin's classification of holomorphic vector bundles supporting a stabilising Higgs field, see Proposition 4.2.20. As the projective space $\mathbb{P}^{g-1}$ is $\mathbb{P}^1$ we give the splitting type of $\hat{E}_{\infty,\xi}$ for all ξ in each of the cases. The case where E itself is stable is covered by Proposition 5.2.7.

We can naturally define a divisor $\Theta \subset J^{g-1}(\Sigma)$ as the image of $S^{g-1}\Sigma \to J^{g-1}(\Sigma)$ given by associating to a collection of $g-1$ points of Σ its effective degree $g-1$-divisor. The divisor Θ coincides both set and scheme theoretically with the divisor of line bundles of degree $g-1$ with a non-zero holomorphic section. For any line bundle L of degree $g-1$ we get a divisor $\Theta_L \subset J^0(\Sigma) = J$. If L is a square root of K then by Riemann–Roch Θ_L is symmetric meaning that $L_\xi \in \Theta_L$ if and only if $L_{-\xi} \in \Theta_L$.

The following proposition has the splitting types in each of the cases from Proposition 4.2.20.

Proposition 5.2.8. *Let Σ have genus two and let (E, Φ) be a stable Higgs bundle of degree zero and rank two. Then generically*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O}.$$

Changes in splitting type in each of the cases are as follows:

- *$E \simeq U \otimes L$ where U is a non-trivial extension of $\mathcal{O}$ by itself and $\deg L = 0$. If ξ is such that $L_\xi \simeq L^{-1}$, then*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O}(-1).$$

- *$E \simeq L \oplus N$ with $\deg L = \deg N = 0$. If ξ is such that $L_\xi \simeq L^{-1}$ or $L_\xi \simeq N^{-1}$, then*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O}(-1).$$

- *$E \simeq L \oplus N$ with $\deg L = \deg N = 0$. If ξ is such that $L_\xi \simeq L^{-1}$ and $L_\xi \simeq N^{-1}$, then*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}(2) \oplus \mathcal{O}(2) \oplus \mathcal{O}(-1) \oplus \mathcal{O}(-1).$$

- *$E \simeq (L \oplus L^{-1}) \otimes N$ where $L^2 \simeq K$ and $N^2 \simeq \det E$. If ξ is such that $L_\xi N \in \Theta_L$, then*

$$\hat{E}_{\infty,\xi} \simeq \mathcal{O}(2) \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O}(-1).$$

Proof. The proof is based on the of dimensions of cohomology groups from the proof of Proposition 4.2.22. We repeat it here for convenience.

stable	$U \otimes L$	$L \oplus N$	$(K^{1/2} \oplus K^{-1/2}) \otimes N$
0 for all L_ξ	0 if $L_\xi \neq L^{-1}$	0 if $L_\xi \neq L^{-1}$ and $L_\xi \neq N^{-1}$	0 if $L_\xi N \notin \Theta_{K^{1/2}}$
	1 if $L_\xi = L^{-1}$	1 if $L_\xi = L^{-1}$ or $L_\xi = N^{-1}$	1 if $L_\xi N \in \Theta_{K^{1/2}}$
		2 if $L_\xi = L^{-1}$ and $L_\xi = N^{-1}$	

Dimensions of the cohomology group $H^0(EL_\xi)$ for the four different cases of Hitchin's classification in Proposition 4.2.20. The divisor $\Theta_{K^{1/2}}$ is the translate of the Θ-divisor in $J^1(\Sigma)$ to $J^0(\Sigma)$ using $K^{1/2}$.

It follows from Riemann–Roch that $h^1(EL_\xi) = 2 + h^0(EL_\xi)$ and an explicit case by case study shows that $h^1(EL_\xi) = h^0(E^* L_{-\xi} K) = h^0(EL_\xi K)$. It is therefore enough to understand how $h^0(\xi) = h^0(EL_\xi)$ depends on ξ to get a full understanding of variations in the splitting type of $\hat{E}_{\infty,\xi}$.

By the same argument as in Proposition 5.2.7 covering the stable case the generic splitting type in all four cases is

$$\mathcal{O}(1) \oplus \mathcal{O}(1) \oplus \mathcal{O} \oplus \mathcal{O},$$

which is furthermore the only splitting type if E is stable.

The different types of jumps correspond to the jumps in the function $h^0(\xi)$ which are apparent from the table. Assume $h^0(\xi) = 1$. Then by definition of Q_ξ and R_ξ they are cokernel and kernel of short exact sequences

$$0 \to \mathcal{O} \to \mathcal{O}(1)^{\oplus 3} \to Q_\xi \to 0$$
$$0 \to R_\xi \to \mathcal{O}^{\oplus 3} \to \mathcal{O}(1) \to 0$$

from which it follows that $Q_\xi \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a + b = 3$ and $R_\xi \simeq \mathcal{O}(c) \oplus \mathcal{O}(d)$ with $c + d = -1$. As $\mathcal{O}(1)^{\oplus 3} \to Q_\xi$ is surjective we must have $a, b \geq 1$, and likewise as $R_\xi \to \mathcal{O}^{\oplus 3}$ is injective $c, d \leq 0$. This gives

$$Q_\xi \simeq \mathcal{O}(2) \oplus \mathcal{O}(1) \quad \text{and} \quad R_\xi \simeq \mathcal{O} \oplus \mathcal{O}(-1).$$

If $h^0(\xi) = 2$, then Q_ξ and R_ξ are defined by

$$0 \to \mathcal{O}^{\oplus 2} \to \mathcal{O}(1)^{\oplus 4} \to Q_\xi \to 0$$
$$0 \to R_\xi \to \mathcal{O}^{\oplus 4} \to \mathcal{O}(1)^{\oplus 2} \to 0.$$

By similar arguments as above, $Q_\xi \simeq \mathcal{O}(a) \oplus \mathcal{O}(b)$ with $a + b = 4$ and $a, b \geq 1$. Since $E \simeq L \oplus N$ is a direct sum and as the map $H^0(E) \otimes \mathcal{O} \to H^0(EK) \otimes \mathcal{O}(1)$ respects the splitting the only option is

$$Q_\xi \simeq \mathcal{O}(2) \oplus \mathcal{O}(2).$$

By similar arguments using that E is a direct sum of degree zero line bundles it follows that

$$R_\xi \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1),$$

completing the proof. $\square$

5.2.5 The canonical Higgs bundle

In this section we observe a new interesting phenomena of the holomorphic structure of $\hat{E}_\infty$, namely that for the rank three canonical Higgs bundle

$$E \simeq K \oplus \mathcal{O} \oplus K^{-1} \quad \text{with} \quad \Phi = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

the holomorphic structure is affected by the type of curve. We do explicit calculations on the slice $\{0\} \times \mathbb{P}^{g-1}$ and denote by Q and R the cokernel and kernel from Proposition 5.2.1.

Since $E \simeq E^*$ we have $R = Q^*(1)$ and we just need to understand the cokernel Q

$$0 \to H^0(E) \otimes \mathcal{O} \xrightarrow{\Theta} H^0(E \otimes K) \otimes \mathcal{O}(1) \to Q \to 0,$$

where Θ at $[\alpha] \in \mathbb{P}^{g-1}$ is multiplication by $\alpha : H^0(E) \to H^0(EK)$. As the multiplication by α respects the splitting of E, the cokernel Q also splits as a direct sum of bundles $Q_1(1) \oplus Q_2 \oplus \mathcal{O}(1)$, where Q_1 and Q_2 are defined by the short exact sequences

$$0 \to H^0(K)(-1) \xrightarrow{\alpha} H^0(K^2) \otimes \mathcal{O} \to Q_1 \to 0 \tag{5.3}$$

$$0 \to \mathcal{O} \xrightarrow{\alpha} H^0(K)(1) \to Q_2 \to 0. \tag{5.4}$$

Notice that (5.4) is the Euler sequence defining the tangent space of $\mathbb{P}^{g-1}$ as an extension,

$$0 \to \mathcal{O}_{\mathbb{P}^{g-1}} \to \mathcal{O}_{\mathbb{P}^{g-1}}(1)^{\oplus g} \to T\mathbb{P}^{g-1} \to 0,$$

and hence we identify Q_2 with $T\mathbb{P}^{g-1}$. Thus we have a splitting

$$K \oplus \widehat{\mathcal{O} \oplus K^{-1}}_{\infty} = Q_1^* \oplus \mathcal{O} \oplus T^*\mathbb{P}^{g-1}(1) \oplus T\mathbb{P}^{g-1} \oplus \mathcal{O}(1) \oplus Q_1(1).$$

As $h^0(K^2) = 3g-3$ the rank of Q_1 is $2g-3$. We find explicit expressions for Q_1 in genus $2, 3$ and 4 below in terms of a tensor as described in Section 5.2.1.

Hyperelliptic curves

A compact curve Σ is *hyperelliptic* if it is realised as a double cover of $\mathbb{P}^1$, $p : \Sigma \to \mathbb{P}^1$. Hyperelliptic curves can be described as the zeros of a section of $\mathcal{O}_{\mathbb{P}^1}(2g+2)$ pulled back to the total space of $p : \mathcal{O}_{\mathbb{P}^1}(g+1) \to \mathbb{P}^1$. Let $\eta \in H^0(\mathcal{O}_{\mathbb{P}^1}(g+1), p^*\mathcal{O}_{\mathbb{P}^1}(g+1))$ be the tautological section of $\mathcal{O}_{\mathbb{P}^1}(g+1)$ pulled back along itself and $s \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g+2))$ a section. Then Σ given by the equation

$$\eta^2 - p^*s = 0$$

is hyperelliptic and all hyperelliptic curves can be given by such an equation.

The canonical bundle of Σ is given by the adjunction formula,

$$K_\Sigma \simeq (K_{\mathcal{O}(g+1)} \otimes p^*\mathcal{O}(2g+2))|_\Sigma.$$

For any vector bundle $p : V \to B$ over a base B, there is a short exact sequence on the total space of V relating the tangent space of V to the tangent space of B,

$$0 \to p^*V \to TV \xrightarrow{dp} p^*TB \to 0.$$

Taking determinants we have the following useful formula

$$K_V = p^*(\wedge^{\operatorname{rk} V} V^* \otimes K_B). \tag{5.5}$$

In this case, $V = \mathcal{O}(g+1)$ and (5.5) gives

$$K_{\mathcal{O}(g+1)} = p^*(\mathcal{O}(-g-3)).$$

Thus

$$K_\Sigma \simeq p^*(\mathcal{O}(g-1))|_\Sigma.$$

The space of holomorphic sections of K_Σ is g-dimensional and this is the same dimension as the space of holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(g-1)$. Therefore all holomorphic sections of $K_\Sigma \to \Sigma$ are pulled back from holomorphic sections of $\mathcal{O}_{\mathbb{P}^1}(g-1) \to \mathbb{P}^1$.

The space of holomorphic sections of K_Σ^2 has dimension $3g-3$, however $\mathcal{O}_{\mathbb{P}^1}(2g-2)$ only has $2g-1$ linearly independent global sections. Denote by $\{\eta_1, \ldots, \eta_{g-2}\}$ linearly independent global sections of K_Σ^2 which are not pulled back from $\mathbb{P}^1$, then

$$H^0(K_\Sigma^2) = H^0(\mathcal{O}_{\mathbb{P}^1}(2g-2)) \oplus \operatorname{span}_{\mathbb{C}}\{\eta_1, \ldots, \eta_{g-2}\}.$$

If we let $\mathbb{P}^1$ have coordinates $[z_0 : z_1]$ we choose as ordered bases for the vector spaces $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2g-2))$ and $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(g-1))$

$$\{z_0^{2g-2},\, z_0^{2g-3}z_1,\, \ldots,\, z_0 z_1^{2g-3},\, z_1^{2g-2}\} \quad \text{and} \quad \{z_0^{g-1},\, z_0^{g-2}z_1,\, \ldots,\, z_0 z_1^{g-2},\, z_1^{g-1}\}.$$

Let $a \in \mathbb{P}(H^0(K_\Sigma))$ be given by coordinates $a = [a_0 : \cdots : a_{g-1}]$, with the basis above this means $a = a_0 z_0^{g-1} + \cdots + a_{g-1} z_1^{g-1}$. Then by multiplying a with each of the basis elements we obtain the tensor

$$t(a) = \begin{pmatrix} a_0 & 0 & \dots & 0 \\ a_1 & a_0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ a_{g-1} & a_{g-2} & \dots & a_0 \\ 0 & a_{g-1} & \dots & a_1 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & a_{g-1} \\ 0 & \dots & \dots & 0 \\ \vdots & & & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix}$$

with $g-2$ zero rows at the bottom. This completely defines the Steiner bundle Q_1. Notice how this splits $Q_1 = S \oplus \mathcal{O}^{g-2}$ where S is a Schwarzenberger bundle of rank $g-1$ on $\mathbb{P}^{g-1}$.

Genus two

For genus $g = 2$ any curve is hyperelliptic and the splittings are determined by the results in the previous section. However, when $g = 2$ the family of bundles $\hat{E}_\infty$ lives on $\mathbb{P}^1$ and due to Grothendieck's classification we can determine $\hat{E}_{\infty,0}$ explicitly.

From the short exact sequence (5.3) we see that $Q_1 \simeq \mathcal{O}_{\mathbb{P}^1}(2)$ and thus

$$\widehat{K \oplus \mathcal{O} \oplus K^{-1}}_{\infty,0} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(2) \oplus \mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(3). \tag{5.6}$$

Genus three

When $g \geq 3$ there is no classification of vector bundles on $\mathbb{P}^{g-1}$. Below, we therefore describe the Steiner bundle Q_1 by its tensor t defined in Section 5.2.1.

When Σ is a curve of genus 3 it is either hyperelliptic or a smooth quartic in $\mathbb{P}^2$. The hyperelliptic case is treated above.

Quartic in $\mathbb{P}^2$ Let $i : \Sigma \hookrightarrow \mathbb{P}^2$ be a quartic curve, then by the adjunction formula

$$K_\Sigma = i^* K_{\mathbb{P}^2} \otimes i^* \mathcal{O}_{\mathbb{P}^2}(4) = i^*(\mathcal{O}_{\mathbb{P}^2}(-3) \otimes \mathcal{O}_{\mathbb{P}^2}(4)) = i^* \mathcal{O}_{\mathbb{P}^2}(1).$$

As the dimensions match up all holomorphic section of K_Σ and K_Σ^2 are pulled back from $\mathbb{P}^2$

$$H^0(\Sigma, K_\Sigma) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1)) \quad \text{and} \quad H^0(\Sigma, K_\Sigma^2) = H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2)).$$

As above, we let $[z_0 : z_1 : z_2] \in \mathbb{P}^2$ be coordinates and choose as ordered basis for $H^0(\Sigma, K_\Sigma)$ and $H^0(\Sigma, K_\Sigma^2)$

$$\{z_0,\, z_1,\, z_2\} \quad \text{and} \quad \{z_0^2,\, z_1 z_0,\, z_2 z_0,\, z_2 z_1,\, z_2^2,\, z_1^2 - z_2 z_0\}.$$

Let $a = [a_0 : a_1 : a_2] \in \mathbb{P}\big(H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(1))\big)$ be coordinates, that is $a = a_0 z_0 + a_1 z_1 + a_2 z_2$. The tensor defining the Steiner bundle is found by multiplying the basis elements of $H^0(\Sigma, K_\Sigma)$ with a

$$t(a) = \begin{pmatrix} a_0 & 0 & 0 \\ a_1 & a_0 & 0 \\ a_2 & a_1 & a_0 \\ 0 & a_2 & a_1 \\ 0 & 0 & a_2 \\ 0 & a_1 & 0 \end{pmatrix}.$$

Remark 5.2.9. Notice that the tensor differs from that of the hyperelliptic case in just one entry in the lowest row, telling us that the two cases are different - but only just. That is, the non-hyperellipticity of the curve is to some extend remembered by the transform at infinity.

Genus four

In genus 4, there are three classes of curves: hyperelliptic, intersections of a cubic and a singular or non-singular quadric polynomial in $\mathbb{P}^3$. Any non-singular quadric is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$ and the cubic polynomial is then a section of $\mathcal{O}(3,3) \to \mathbb{P}^1 \times \mathbb{P}^1$. The singular case is described as the zero locus in the total space of $p : \mathcal{O}_{\mathbb{P}^1}(2) \to \mathbb{P}^1$ of a degree three polynomial in η – the tautological section of $p^* \mathcal{O}_{\mathbb{P}^1}(2) \to \mathcal{O}_{\mathbb{P}^1}(2)$.

Non-singular quadric In this case, Σ is the zero locus of a section of $\mathcal{O}(3,3) \to \mathbb{P}^1 \times \mathbb{P}^1$, let $i : \Sigma \to \mathbb{P}^1 \times \mathbb{P}^1$ be the inclusion. By the adjunction formula the canonical bundle on Σ is related to the canonical bundle on $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathcal{O}(3,3)$,

$$K_\Sigma \simeq i^*(K_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}(3,3)).$$

Now the tangent bundle to $\mathbb{P}^1 \times \mathbb{P}^1$ is $\mathcal{O}(2,0) \oplus \mathcal{O}(0,2)$ and hence the canonical bundle is

$$K_{\mathbb{P}^1 \times \mathbb{P}^1} = \det(T^*(\mathbb{P}^1 \times \mathbb{P}^1)) = \mathcal{O}(-2,-2).$$

Thus,

$$K_{\mathbb{P}^1 \times \mathbb{P}^1} \otimes \mathcal{O}(3,3) = \mathcal{O}(-2,-2) \otimes \mathcal{O}(3,3) = \mathcal{O}(1,1)$$

and

$$K_\Sigma = i^*\mathcal{O}(1,1).$$

Therefore,

$$4 = \dim H^0(\Sigma, K_\Sigma) \geq \dim H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) \geq \dim H^0(\mathbb{P}^1, \mathcal{O}(1))^2 = 4,$$

and

$$H^0(\Sigma, K_\Sigma) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)).$$

As $K_\Sigma^2 = i^*\mathcal{O}(2,2)$ and $\dim H^0(\Sigma, K_\Sigma^2) = 9$ the same argument as above shows that $H^0(\Sigma, K_\Sigma^2)$ has a basis consisting of products of basis elements of $H^0(\mathbb{P}^1, \mathcal{O}(2))$. Choose the following ordered bases

$$\begin{aligned} H^0(\Sigma, K_\Sigma) = H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1,1)) &= \operatorname{span}_\mathbb{C}\{z_0w_0,\, z_0w_1,\, z_1w_0,\, z_1w_1\} \\ H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(2,2)) &= \operatorname{span}_\mathbb{C}\{z_0^2w_0^2,\, z_0^2w_0w_1,\, z_0z_1w_0^2,\, z_0z_1w_0w_1,\, z_0z_1w_1^2, \\ &\qquad z_1^2w_0w_1,\, z_1^2w_1^2,\, z_0^2w_1^2 - z_0z_1w_0^2,\, z_1^2w_0^2 - z_0z_1w_1^2\}, \end{aligned}$$

where $z = [z_0 : z_1] \in \mathbb{P}^1$ are coordinates on the first factor and $w = [w_0 : w_1] \in \mathbb{P}^1$ are coordinates on the second factor.

If we again let $a = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}(H^0(K_\Sigma))$ be a point, then in coordinates $a = a_0z_0w_0 + a_1z_0w_1 + a_2z_1w_0 + a_3z_1w_1$, and the tensor defining the quotient bundle is

$$t(a) = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & a_1 & 0 & 0 \\ 0 & 0 & a_2 & 0 \end{pmatrix}.$$

Singular quadric In this last case, the curve is given by the zero locus in the total space of $p : \mathcal{O}(2) \to \mathbb{P}^1$ of a degree three polynomial

$$\eta^3 + b_1\eta^2 + b_2\eta + b_3,$$

where $\eta \in H^0(\mathcal{O}(2), p^*\mathcal{O}(2))$ is the tautological section and the b_i's are pulled back sections of $\mathcal{O}(i) \to \mathbb{P}^1$.

By the adjunction formula and (5.5)

$$K_\Sigma \simeq p^*\mathcal{O}(2)|_\Sigma.$$

Since $\dim H^0(\Sigma, K_\Sigma) = 4$ and $\dim H^0(\mathbb{P}^1, \mathcal{O}(2)) = 3$ we have

$$H^0(\Sigma, K_\Sigma) = \mathbb{C}\eta \oplus H^0(\mathbb{P}^1, \mathcal{O}(2)) = \operatorname{span}_\mathbb{C}\{\eta,\, z_0^2,\, z_0z_1,\, z_1^2\},$$

with a choice of ordered basis. Now, $K_\Sigma^2 \simeq p^*\mathcal{O}(4)$ and the dimension of global holomorphic sections of K_Σ^2 is 9 whereas $\dim H^0(\mathbb{P}^1, \mathcal{O}(4)) = 5$, hence

$$\begin{aligned} H^0(\Sigma, K_\Sigma^2) &= H^0(\mathbb{P}^1, \mathcal{O}(4)) \oplus \mathbb{C}\eta^2 \oplus \operatorname{span}_\mathbb{C}\{\eta z_0^2,\, \eta z_0z_1,\, \eta z_1^2\} \\ &= \operatorname{span}_\mathbb{C}\{\eta^2,\, \eta z_0^2,\, \eta z_0z_1,\, \eta z_1^2,\, z_0^2z_1^2,\, z_0z_1^3,\, z_1^4,\, z_0^4 - \eta z_0z_1,\, z_0^3z_1 - \eta z_1^2\}, \end{aligned}$$

with a choice of ordered basis. Let $a = [a_0 : a_1 : a_2 : a_3] \in \mathbb{P}(H^0(K_\Sigma))$, then $a = a_0\eta + a_1z_0^2 + a_2z_0z_1 + a_3z_1^2$. In this basis, the tensor defining the quotient bundle is

$$t(a) = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ a_1 & a_0 & 0 & 0 \\ a_2 & a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 & a_0 \\ 0 & a_3 & a_2 & a_1 \\ 0 & 0 & a_3 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & a_1 & 0 & 0 \\ 0 & a_2 & a_1 & 0 \end{pmatrix}.$$

5.2.6 Summary

In the examples above we find different tensors defining the Steiner bundles Q_1 and therefore a different holomorphic structure of $\hat{E}_{\infty,0}$ depending on the type of curve. The dependencies are small but nevertheless present. The bundle $\hat{E}_{\infty,0}$ mainly consists of the same components for all genera

$$\widehat{K \oplus \mathcal{O} \oplus K^{-1}}_{\infty,0} = Q_1^* \oplus \mathcal{O} \oplus T^*\mathbb{P}^{g-1}(1) \oplus T\mathbb{P}^{g-1} \oplus \mathcal{O}(1) \oplus Q_1(1),$$

and as it is apparent from the explicit calculation that only a very small part of Q_1 differs in each of the cases.

Remark 5.2.10. Notice that the above calculations are independent of the Higgs field Φ and are therefore equally valid for any stable Higgs bundle (E, Φ) with $E \simeq K \oplus \mathcal{O} \oplus K^{-1}$.

6. Hodge theory for parabolic Higgs bundles and applications

In [41], Hitchin introduced a Dirac operator associated to a Higgs bundle and used Hodge theory to show that there are unique harmonic representatives for the first Dolbeault cohomology classes of a Higgs bundle, i.e. the first cohomology of the complex

$$\Omega^0(E) \xrightarrow{\bar{\partial}_E+\Phi} \Omega^1(E) \xrightarrow{\bar{\partial}_E+\Phi} \Omega^2(E).$$

In the first half of this chapter, we extend this result to parabolic Higgs bundles (Theorem 6.3.1) and compute the dimension of the cohomology (Theorem 6.3.3). We use the result to relate Dolbeault cohomology to hypercohomology of a certain two-term complex naturally obtained from a parabolic Higgs bundle, and to the de Rham comology of the associated flat connection. Due to the singularities introduced by the parabolic structure, the proofs of these theorems require Hodge theory which can handle singularities. The bulk of the proofs is setting up suitably weighted Sobolev spaces and proving Fredholmness of certain operators. We use the weighted Sobolev spaces introduced by Biquard [8] which were also used by Konno [50]. Section 6.1 introduces the setup, while the weighted Sobolev spaces are introduced in Section 6.2. The Hodge decomposition for parabolic Higgs bundles is Section 6.3, while Section 6.4 uses the Hodge theory to relate Dolbeault, de Rham, and hypercohomology of the parabolic Higgs bundle.

In the last half of the chapter, we apply the Hodge theory to a number of situations. Firstly, we prove in Section 6.5 that under mild assumptions on the parabolic data, the moduli space of parabolic Higgs bundles is a fine moduli space, i.e. it carries a universal parabolic Higgs bundle. Using this result and the Hodge theory, we construct in Section 6.6 a hyperholomorphic bundle on the moduli space.

Secondly, we consider a minimal non-trivial example of the hyperholomorphic bundle obtained in the case of parabolic Higgs bundles on $\mathbb{P}^1$ with four parabolic points. In this case, the moduli space is complex two-dimensional. If the weights are suitably chosen the hyperholomorphic vector bundle is a line bundle. We compute its topology and discuss whether the curvature is L^2. This is Section 6.7.

Thirdly, we discuss the construction of a special type of parabolic Higgs bundles called *limiting configurations.* Limiting configurations were introduced by Mazzeo, Swoboda, Weiss, and Witt [53] to discuss the asymptotics of the L^2-metric on the moduli space of ordinary Higgs bundles. This is Section 6.8. We also discuss the local shape of L^2-solutions to the Dirac–Higgs equations for limiting configurations. What seems like a technical condition to get Fredholmness of certain operators is recovered by considering the local solutions to the Dirac–Higgs equations for limiting configurations, giving the abstract theory a reality check.

Lastly, we use the Hodge theory of Section 6.3 to define a Nahm transform for parabolic Higgs bundles in Section 6.9. In the special case of parabolic Higgs bundles on elliptic curves, we show that the Nahm transform produces new finite energy doubly-periodic instantons.

6.1 Concepts and notation

We first set up notation which is fixed throughout. Let Σ be a compact Riemann surface with a choice of Kähler metric normalised to have area 1. Let ω be the associated Kähler form. We let $P_1, \dots, P_n \in \Sigma$ denote a collection of points called *parabolic points*, $D = P_1 + \cdots + P_n$ their divisor, and $\Sigma_0 = \Sigma \setminus \{P_1, \dots, P_n\}$ their complement.

Let E be a smooth complex vector bundle of rank l on Σ. A *parabolic structure* on E is a collection of flags at each of the parabolic points P_i together with an increasing sequence of real numbers called weights

$$\begin{gathered} E_{P_i} = F_1 E_{P_i} \supsetneq F_2 E_{P_i} \supsetneq \cdots \supsetneq F_{a_i} E_{P_i} \supsetneq F_{a_i+1} E_{P_i} = \{0\} \\ 0 \leq w_1^{(i)} < w_2^{(i)} < \cdots < w_{a_i}^{(i)} < 1. \end{gathered}$$

A collection of flags at each parabolic point without the weights is often called a *quasi-parabolic* structure. Each weight has multiplicity

$$m_{P_i}(w_j^{(i)}) = \dim(F_j E_{P_i} / F_{j+1} E_{P_i})$$

and to ease notation we often use an alternative set of weights $\alpha_k^{(i)}$ all of multiplicity one,

$$\alpha_k^{(i)} = w_j^{(i)} \quad \text{if} \quad l - \dim F_j E_{P_i} < k \leq l - \dim F_{j+1} E_{P_i}.$$

A holomorphic map between holomorphic vector bundles $\varphi : E \to F$ both with parabolic structures is called *parabolic* if $\alpha_j^{(i)}(E) > \alpha_k^{(i)}(F)$ implies $\varphi(F_j E_{P_i}) \subset F_{k+1} F_{P_i}$ for all $P_i \in D$ and *strongly parabolic* if $\alpha_j^{(i)}(E) \geq \alpha_k^{(i)}(F)$ implies $\varphi(F_j E_{P_i}) \subset F_{k+1} F_{P_i}$ for all $P_i \in D$. We denote by $\mathrm{ParHom}(E, F)$ and $\mathrm{SParHom}(E, F)$ the sheaves of parabolic and strongly parabolic homomorphisms, respectively.

Define the *parabolic degree* of a bundle E with parabolic structure by

$$\operatorname{pardeg}(E) = \deg(E) + \sum_{i=1}^{n} \sum_{k=1}^{l} \alpha_k^{(i)} = \deg(E) + \sum_{i=1}^{n} \sum_{k=1}^{a_i} m_{P_i}(w_k^{(i)}) w_k^{(i)},$$

where $\deg(E)$ is the degree of E in the usual sense. For a bundle with parabolic structure we define the *slope*

$$\operatorname{par}\mu(E) = \frac{\operatorname{pardeg}(E)}{\operatorname{rk} E}.$$

Let U_i be neighbourhood of a parabolic point P_i with z a local coordinate such that $z(P_i) = 0$. A smooth frame $\{e_k^{(i)}\}_{k=1}^l$ trivialising $E|_{U_i}$ such that

$$F_j E_{P_i} = \operatorname{span}\{e_k^{(i)} \mid \alpha_k^{(i)} \geq w_j^{(i)}\} \quad \text{for all} \quad 1 \leq j \leq a_i$$

is called *adapted*. Define a Hermitian metric on a bundle with parabolic structure by declaring $\left\{|z|^{-\alpha_k^{(i)}} e_k^{(i)}\right\}$ to be a unitary frame on $E|_{U_i}$ and extend smoothly to the rest of Σ. Such a metric is called an *adapted Hermitian metric*.

Notice that an adapted Hermitian metric vanishes at parabolic points like

$$\operatorname{diag}\left(|z|^{2\alpha_1^{(i)}}, \ldots, |z|^{2\alpha_l^{(i)}}\right)$$

with respect to the smooth frame $\{e_k^{(i)}\}$.

Let $\mathscr{C}'$ be the set of smooth holomorphic structures on a bundle E with parabolic structure. Fix a Hermitian metric adapted to the parabolic structure, then for $\bar{\partial}_A \in \mathscr{C}'$ denote by d_A the Chern connection of $\bar{\partial}_A$ with respect to the fixed adapted Hermitian metric on E, and by F_A its curvature.. Denote by d_A^Z and F_A^Z the induced connection on $\det E$ and its curvature. Fix $\bar{\partial}_{A_0} \in \mathscr{C}'$ such that

$$\frac{i}{2\pi} \Lambda F_{A_0}^Z = \operatorname{par}\mu(E)$$

where Λ is contraction by the Kähler form ω. Such a holomorphic structure always exists, see [8, Proposition 2.9]. Define the set of holomorphic structures with fixed induced holomorphic structure on $\det E$

$$\mathscr{C} = \{\bar{\partial}_A \in \mathscr{C}' \mid d_A^Z = d_{A_0}^Z\}.$$

At a parabolic point P_i, denote by

$$N_i = \{g \in \operatorname{End} E_{P_i} \mid g(F_j E_{P_i}) \subset F_{j+1} E_{P_i} \text{ for all } j\}$$

the nilpotent automorphisms with respect to the filtration at P_i.

Definition 6.1.1. Let E be a smooth vector bundle with parabolic structure. An operator $D'' = \bar{\partial}_A + \Phi$ is a *parabolic Higgs structure* if

1. $\bar{\partial}_A \in \mathscr{C}$
2. Φ is a section of $\operatorname{End}_0 E \otimes K$ which is $\bar{\partial}_A$-meromorphic on Σ and $\bar{\partial}_A$-holomorphic on Σ_0
3. Φ has at most simple poles with residues in N_i at P_i for all i.

Let $\mathscr{D}$ be the set of parabolic Higgs structures on E. For $D'' \in \mathscr{D}$ the pair $\mathbb{E} = (E, D'')$ is a *parabolic Higgs bundle*.

A subbundle V of E is a *Higgs subbundle* if V is a holomorphic subbundle of $(E, \bar{\partial}_A)$ and is preserved by Φ, i.e. $\Phi(V) \subset V \otimes K$. If V is a subbundle of E it has an induced parabolic structure

$$\begin{aligned} V_{P_i} = F_1 V_{P_i} \supsetneq F_2 V_{P_i} \supsetneq \cdots \supsetneq F_{b_i} V_{P_i} \supsetneq F_{b_i+1} V_{P_i} = \{0\} \\ 0 \leq x_1^{(i)} < x_2^{(i)} < \cdots < x_{b_i}^{(i)} < 1. \end{aligned}$$

by taking the greatest k such that $V_{P_i} \subset F_k E_{P_i}$ and setting $x_1^{(i)} = w_k^{(i)}$. Define $F_j V_{P_i}$ and $x_j^{(i)}$ inductively by assuming $x_{j-1}^{(i)} = w_k^{(i)}$ and $F_{j-1} V_{P_i} = V_{P_i} \cap F_k E_{P_i}$. Then define $F_j V_{P_i} = V_{P_i} \cap F_{k+1} E_{P_i}$ and the weight $x_j^{(i)} = w_m^{(i)}$, where m is the greatest integer such that $F_j V_{P_i} \subset F_m E_{P_i}$.

A parabolic Higgs bundle $\mathbb{E}$ is *stable* if for any Higgs subbundle $\mathbb{V}$ of $\mathbb{E}$

$$\mathrm{par}\mu(\mathbb{V}) = \frac{\mathrm{pardeg}(\mathbb{V})}{\mathrm{rk}\,\mathbb{V}} < \frac{\mathrm{pardeg}(\mathbb{E})}{\mathrm{rk}\,\mathbb{E}} = \mathrm{par}\mu(\mathbb{E}).$$

If $\mathbb{E} = (E, \bar{\partial}_A + \Phi)$ and $\mathbb{F} = (F, \bar{\partial}_B + \Psi)$ are two parabolic Higgs bundles, define the two-term complex $\mathbb{H}\mathrm{om}(\mathbb{E}, \mathbb{F})$:

$$\mathrm{ParHom}(E, F) \to \mathrm{SParHom}(E, F) \otimes K(D)$$

by $f \mapsto f\Phi - \Psi f$, and let $\mathbb{E}\mathrm{nd}(\mathbb{E}) = \mathbb{H}\mathrm{om}(\mathbb{E}, \mathbb{E})$.

From [74], we have the following useful properties.

Proposition 6.1.2 ([74]).

- *The endomorphisms and infinitesimal deformations of a parabolic Higgs bundle $\mathbb{E}$ are given by the hypercohomology groups $\mathbb{H}^0(\mathbb{E}\mathrm{nd}(\mathbb{E}))$ and $\mathbb{H}^1(\mathbb{E}\mathrm{nd}(\mathbb{E}))$, respectively.*
- *If $\mathbb{E}$, $\mathbb{F}$ are stable parabolic Higgs bundles with $\mathrm{par}\mu(\mathbb{E}) \geq \mathrm{par}\mu(\mathbb{F})$, then if $\mathbb{E}$ and $\mathbb{F}$ are isomorphic $\dim \mathbb{H}^0(\mathbb{H}\mathrm{om}(\mathbb{E}, \mathbb{F})) = 1$, otherwise it is 0.*

The moduli space of stable parabolic Higgs bundles was constructed by Konno [50] using appropriate Sobolev completions of $\mathscr{D}$ adapted to the parabolic structure before taking the quotient by the complex gauge group. Just as for ordinary Higgs bundles, the stability condition is equivalent to the existence of a Hermitian metric satisfying a PDE; this time the Hermitian metric must be adapted to the parabolic structure. The main theorem of [50] is the Hitchin–Simpson Theorem for parabolic Higgs bundles, stated here for the parabolic degree zero case which is relevant to us.

Theorem 6.1.3 ([50] Theorem 1.5). *Let (E, D'') be a parabolic Higgs bundle of parabolic degree zero. Then (E, D'') is stable if and only if D'' is irreducible and there exists an adapted Hermitian metric on E satisfying the* Higgs bundle equations

$$F_A + [\Phi, \Phi^*] = 0 \quad \textit{and} \quad \bar{\partial}_A \Phi = 0 \tag{6.1}$$

where $D'' = \bar{\partial}_A + \Phi$ with F_A and Φ^* defined with respect to the adapted Hermitian metric.

6.2 Sobolev spaces

In [8], Biquard defined the right notion of Sobolev completion of the space of holomorphic parabolic structures. Konno [50] followed Biquard's recipe and extended the completion to the space of parabolic Higgs structures. In this section we will take the same approach to sections of parabolic bundles. This is a straightforward adaptation of the Sobolev space constructions in [8, 50] but as the details are needed for later reference they are included here.

6.2.1 Weighted Sobolev spaces

Let U be the unit disk in $\mathbb{C}$ with Euclidean coordinates $z = x + iy$ or polar coordinates $z = re^{i\theta}$. Denote by L^p_k the standard Sobolev space of functions on U with k derivatives in L^p. We define a weighted Sobolev norm on $C^\infty(U)$ by

$$\|f\|_{W^p_{k,\delta}} = \left(\int_U \sum_{i+j\le k} \left| r^{i+j-\delta} \frac{d^i}{dx^i} \frac{d^j}{dy^j} f \right|^p \frac{dxdy}{r^2} \right)^{1/p}$$

where $\delta \in \mathbb{R}$. Notice that when changing to cylindrical coordinates $t = -\log r$ on $U \setminus \{0\}$ the weighted Sobolev norms are the ones defined by Lockhart and McOwen [51]. We will use a special weight suiting our purpose, namely $\delta = k - \frac{2}{p}$, and define

$$\|f\|_{W^p_k} = \|f\|_{W^p_{k,k-2/p}} = \left(\int_U \sum_{i+j\le k} \left| r^{i+j-k} \frac{d^i}{dx^i} \frac{d^j}{dy^j} f \right|^p dxdy \right)^{1/p}$$

for $f \in C^\infty(U)$. Define $W^p_k(U)$ to be $f \in L^p_{k;\mathrm{loc}}(U \setminus \{0\})$ with $\|f\|_{W^p_k}$ finite.

With this special choice of weight Biquard showed the following useful theorem demystifying the weighted Sobolev spaces.

Theorem 6.2.1 ([8] Theorem 1.3)**.** *If l is a nonnegative integer with $l - 1 < k - \frac{2}{p} < l$, then*

$$W^p_k(U) = \{ f \in L^p_k(U) \mid f(0) = 0, \dots, \nabla^{l-1} f(0) = 0 \}$$

and the W^p_k norm is equivalent to the L^p_k-norm on $W^p_k(U)$.

If $1 < p < 2$, Biquard's theorem gives the following useful identifications

$$W^p_0(U) = L^p(U), \quad W^p_1(U) = L^p_1(U), \quad W^p_2(U) = \{ f \in L^p_2(U) \mid f(0) = 0 \}.$$

6.2.2 Singular Chern connections

If $(E, \bar{\partial}_A + \Phi)$ is a parabolic Higgs bundle with an adapted Hermitian metric, then the parabolic structure induce singularities in the Chern connection d_A at the parabolic points. In this section we extend the Sobolev norms from above to sections of E such that we get an analytical setup capable of handling singular connections.

Fix a parabolic point P_i. For the remaining part of this section we ignore the index i when there is no room for confusion. Let U be a neighbourhood of P with local holomorphic coordinate $z = re^{i\theta}$ centered at P and $\{e_k\}$ a smooth adapted frame of E. With respect to the smooth frame $\{e_k\}$, define diagonal endomorphisms

$$S = \operatorname{diag}\big(|z|^{-\alpha_1}, \dots, |z|^{-\alpha_l}\big) \quad \text{and} \quad \alpha = \operatorname{diag}\big(\alpha_1, \dots, \alpha_l\big).$$

In the smooth frame $\{e_k\}$ we write the holomorphic structure $\bar{\partial}_A$ on $E|_U$ as $\bar{\partial}_A = \bar{\partial} + A$ where A is a $l \times l$-matrix of $(0,1)$-forms. In the unitary frame $\{|z|^{-\alpha_k} e_k\}$ it is

$$\bar{\partial}_A = \bar{\partial} - \frac{\alpha}{2}\frac{d\bar{z}}{\bar{z}} + S^{-1}AS.$$

In this frame, the Chern connection of $\bar{\partial}_A$ and the adapted Hermitian metric is

$$\begin{aligned} d_A &= d + \frac{\alpha}{2}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) + S^{-1}AS - (S^{-1}AS)^* \\ &= d + i\alpha d\theta + S^{-1}AS - (S^{-1}AS)^*. \end{aligned}$$

Notice that if α_k is non-zero, then d_A has a singularity at 0 in the direction of e_k. We decompose $E|_U$ smoothly as $E|_U = E_S \oplus E_R$ where

$$E_S = \operatorname{span}\{e_k \mid \alpha_k \neq 0\} \quad E_R = \operatorname{span}\{e_k \mid \alpha_k = 0\}$$

are called the *singular* and *regular* part of E around P, respectively. Write $u \in \Omega^0(E|_U)$ as $u = u_R + u_S$ according to the decomposition. If we write $d_0 = d + i\alpha d\theta$ with respect to the singular frame, then d_0 respects this decomposition:

$$(d_0 u)_R = d(u_R) \quad \text{and} \quad (d_0 u)_S = d(u_S) + i\alpha u_S d\theta.$$

We define a weighted Sobolev norm on $\Omega^0(E|_U)$ by

$$\|u\|_{D^p_k} = \|u_R\|_{L^p_k} + \|u_S\|_{W^p_k},$$

where the parabolic Hermitian metric is used to define the Sobolev norms. As d_0 respects the decomposition we get continuous maps

$$d_0 : D^p_k\Omega^0(E|_U) \to D^p_{k-1}\Omega^1(E|_U)$$

where $D^p_k\Omega^s(E|_U)$ is the Sobolev completion of $\Omega^s(E|_U)$ with respect to the norm $\|\cdot\|_{D^p_k}$.

6.2.3 Sobolev completions and Fredholm operators

Using the locally defined weighted Sobolev norm we can now define the function spaces we want to work with.

Definition 6.2.2. Let (E, D'') be a parabolic Higgs bundle with adapted Hermitian metric. The Sobolev space $D^p_k\Omega^s(E)$ is the Sobolev completion of $\Omega^s(E)$ with respect to the Sobolev norm $\|\cdot\|_{D^p_k}$.

Remark 6.2.3. The Sobolev norm $\|\cdot\|_{D^p_k}$ is the D^p_k-norm around the parabolic points patched with the standard L^p_k-norm on the complement.

Remark 6.2.4. Instead of using the decomposition of $E|_U$ into regular and singular parts we can use the W^p_k-norm on all of the sections. This gives Sobolev spaces $W^p_k\Omega^s(E)$.

Lemma 6.2.5. *If $1 < p < 2$, then the codimension of $W^p_2\Omega^0(E)$ in $D^p_2\Omega^0(E)$ is the total number of zero weights of E, i.e. $\sum_{P\in D} m_P(0)$.*

Proof. This follows directly from Theorem 6.2.1. $\square$

In [50], Konno extends the D^p_k-Sobolev norm to define a completion of holomorphic structures. That is define $\mathscr{C}^p_1 = \bar\partial_{A_0} + D^p_1\Omega^{0,1}(\operatorname{End}_0 E)$. If $\bar\partial_A \in \mathscr{C}^p_1$ we get continuous maps

$$D^p_2\Omega^0(V) \xrightarrow{\bar\partial_A} D^p_1\Omega^{0,1}(V) \quad \text{and} \quad D^p_1\Omega^{1,0}(V) \xrightarrow{\bar\partial_A} D^p_0\Omega^{1,1}(V) \tag{6.2}$$

where V is E or $\operatorname{End}_0 E$.

Using the theory of Lockhart and McOwen [51], Biquard [8] finds conditions for p under which the $\bar\partial_A$'s are Fredholm operators when $V = \operatorname{End}_0 E$:

Lemma 6.2.6. *If $p > 1$ satisfies the following conditions*

$$\begin{aligned} 1 < p &< \frac{2}{2 + \alpha^{(i)}_k - \alpha^{(i)}_j} \quad \text{if} \quad \alpha^{(i)}_j > \alpha^{(i)}_k \\ 1 < p &< \frac{2}{1 + \alpha^{(i)}_k - \alpha^{(i)}_j} \quad \text{if} \quad \alpha^{(i)}_j < \alpha^{(i)}_k \end{aligned}$$

for each parabolic point P_i, then the operators

$$\begin{aligned} \bar\partial_A &: D^p_2\Omega^0(\operatorname{End}_0 E) \to D^p_1\Omega^{0,1}(\operatorname{End}_0 E) \\ \bar\partial_A &: D^p_1\Omega^{1,0}(\operatorname{End}_0 E) \to D^p_0\Omega^{1,1}(\operatorname{End}_0 E) \end{aligned}$$

are Fredholm.

Using the same method, we find new conditions on p under which the $\bar\partial_A$'s are Fredholm operators when $V = E$ in (6.2).

Lemma 6.2.7. *If $p > 1$ satisfies the following conditions*

$$1 < p < \frac{2}{2 - \alpha_k^{(i)}} \quad \textit{and} \quad 1 < p < \frac{2}{1 + \alpha_k^{(i)}} \quad \textit{for} \quad \alpha_k^{(i)} \neq 0$$

for each parabolic point P_i, then

$$\begin{aligned} \bar\partial_A &: D_2^p\Omega^0(E) \to D_1^p\Omega^{0,1}(E) & \qquad \bar\partial_A &: D_1^p\Omega^{1,0}(E) \to D_0^p\Omega^{1,1}(E) \\ \partial_A &: D_2^p\Omega^0(E) \to D_1^p\Omega^{1,0}(E) & \qquad \partial_A &: D_1^p\Omega^{0,1}(E) \to D_0^p\Omega^{1,1}(E) \end{aligned}$$

are Fredholm operators. Here ∂_A is the $(1,0)$-part of the Chern connection of $\bar\partial_A$ with respect to an adapted Hermitian metric.

Proof. Using [51, Theorem 1.1] and the remark following it, it is enough to solve an eigenvalue problem locally around each parabolic point. In the singular frame constructed in Section 6.2.2, the operator is $\bar\partial_A = \bar\partial - \frac{\alpha}{2}\frac{d\bar z}{\bar z}$ and the equation (written in polar coordinates) to check for solutions is

$$0 = (\lambda + i\alpha_k)f + \frac{\partial f}{\partial\theta}, \tag{6.3}$$

where $\lambda \in \mathbb{C}$ and f is a function on the circle. A solution to this equation can only exist if $\mathrm{Re}(\lambda) = 0$ and $\mathrm{Im}(\lambda) + \alpha_k \in \mathbb{Z}$. From [51, Theorem 1.1] the imaginary part of λ is exactly the Soblev weight, which in the two cases we consider are $1 - \frac{2}{p}$ and $2 - \frac{2}{p}$. For the operators to be Fredholm there should be no solutions to equation (6.3), i.e. $\alpha_k - \frac{2}{p}$ is not an integer for any k. As $1 < p < 2$ the case $\alpha_k = 0$ does not give any restrictions on p. In the case of non-zero weights the condition on p is that

$$1 < p < \frac{2}{1 + \alpha_k} \quad \text{for all } \alpha_k \neq 0.$$

The same method is applied to show that ∂_A is a Fredholm operator in the two cases mentioned if p satisfies

$$1 < p < \frac{2}{2 - \alpha_k} \quad \text{for all } \alpha_k \neq 0.$$

Strictly speaking the results in [51] can only be used to prove Fredholmness of operators on the weighted Sobolev spaces W_k^p. By Theorem 6.2.1 this really only makes a difference when $k = 2$. But, as the codimension of $W_2^p\Omega^0(E)$ in $D_2^p\Omega^0(E)$ is finite the extended operators defined on the larger space are also Fredholm. □

Definition 6.2.8. A $p > 1$ is *compatible with the parabolic structure* of E if the assumptions of Lemmas 6.2.6 and 6.2.7 are satisfied.

Lemma 6.2.9. *If $(E, \bar{\partial}_A + \Phi)$ is a parabolic Higgs bundle and p is compatible with the parabolic structure, then $\Phi \in D_1^p\Omega^{1,0}(\operatorname{End}_0 E)$ and defines compact operators*

$$D_2^p\Omega^0(E) \xrightarrow{\Phi} D_1^p\Omega^{1,0}(E) \quad \text{and} \quad D_1^p\Omega^{0,1}(E) \xrightarrow{\Phi} D_0^p\Omega^{1,1}(E).$$

Proof. From [50, Lemma 2.6.(3)], Φ is in $D_1^p\Omega^{1,0}(\operatorname{End}_0 E)$. By Theorem 6.2.1, $D_1^p = L_1^p$ and $D_2^p \subset L_2^p$. The lemma now follows from standard Sobolev theory.

Let $\{u_j\}$ be a bounded sequence in $D_2^p\Omega^0(E)$. As the L_2^p-norm is equivalent to the D_2^p-norm the sequence is also bounded when considered in $L_2^p\Omega^0(E)$. The Sobolev embedding theorem embeds L_2^p compactly into C^0 when $1 < p < 2$. The embedded sequence has a Cauchy subsequence $\{u_{j_k}\}$, by compactness. As multiplication $C^0 \times L_1^p \to L_1^p$ is continuous, the sequence $\{\Phi u_{j_k}\}$ is Cauchy, and thus the operator $\Phi : D_2^p\Omega^0(E) \to D_1^p\Omega^1(E)$ is compact.

That also $\Phi : D_1^p\Omega^1(E) \to \Omega^2(E)$ is compact, follows from similar arguments when we notice that $D_1^p\Omega^1(E)$ embeds compactly in $L^2\Omega^1(E)$ and that the multiplication map $L^2 \times L_1^p \to L^p$ is continuous. □

Definition 6.2.10. For (E, D'') a parabolic Higgs bundle with adapted Hermitian metric and $D'' = \bar{\partial}_A + \Phi$ define $D' = \partial_A + \Phi^*$, using the adapted Hermitian metric to give both Φ^* and the Chern connection.

Corollary 6.2.11. *If $p > 1$ is compatible with the parabolic structure of E, then the operators*

$$D'', D' : D_2^p\Omega^0(E) \to D_1^p\Omega^1(E)$$

have finite dimensional kernels and closed images.

In the coming sections we will need the following lemma embedding our Sobolev spaces in L^2.

Lemma 6.2.12. *If $1 < p < 2$, the space $D_1^p\Omega^1(E)$ is embedded in $L^2\Omega^1(E)$, and therefore carries an inner product.*

Proof. As $1 < p < 2$, Theorem 6.2.1 embeds $D_1^P\Omega^1(E)$ as a subspace of $L_1^p\Omega^1(E)$. As Σ is a compact Riemann surface, the Sobolev embedding theorem embeds $L_1^p\Omega^1(E)$ in $L^q\Omega^1(E)$ where $q = \frac{2p}{1-p}$. Since $1 < p < 2$ we have $q > 2$, giving the final inclusion $L^q\Omega^1(E) \subset L^2\Omega^1(E)$. □

When $p > 1$ is compatible with the parabolic structure we have the following useful properties of sections in D_2^p and D_1^p.

Lemma 6.2.13. *Let $p > 1$ be compatible with the parabolic structure of E, and assume that u is a section of E and $\bar{\partial}_A u = 0$ on Σ.*

1. *If $u \in D_2^p\Omega^0(E)$, then u is $\bar{\partial}_A$-holomorphic on Σ.*
2. *If $u \in D_1^p\Omega^0(E)$, then u is $\bar{\partial}_A$-holomorphic on $\Sigma \setminus D$ and has at most a simple pole on D with residue in $E_{S,P}$ at P.*

Proof. The proof is the same as [50, Lemma 2.6] and follows from local considerations around parabolic points using the definition of D_k^p and the fact that p is adapted to the parabolic structure. □

6.3 Hodge theory

In the previous section we showed that if p is compatible with the parabolic structure on E, then the maps D'', D' have closed images. In this section we prove the following Hodge decomposition for stable parabolic Higgs bundles of parabolic degree zero. Throughout this section, we assume that the parabolic degree is zero.

Theorem 6.3.1. *Let (E, D'') be a stable parabolic Higgs bundle of parabolic degree zero. If $p > 1$ is compatible with the parabolic structure on E, then we have the following decompositions of $D_1^p\Omega^1(E)$.*

$$\begin{aligned} D_1^p\Omega^1(E) &\simeq \mathcal{H} \oplus \mathrm{im}(D') \oplus \mathrm{im}(D'') \\ \ker(D') &\simeq \mathrm{im}(D') \oplus \mathcal{H} \\ \ker(D'') &\simeq \mathrm{im}(D'') \oplus \mathcal{H} \end{aligned}$$

where

$$\mathcal{H} = \{\varphi \in D_1^p\Omega^1(E) \,:\, D'\varphi = 0 \quad \mathit{and} \quad D''\varphi = 0\}. \tag{6.4}$$

and the direct sum decomposition is with respect to the L^2-inner product.

Corollary 6.3.2. *If $p > 1$ is compatible with the parabolic structure on a stable parabolic Higgs bundle (E, D'') of parabolic degree zero, then each first cohomology class of*

$$D_2^p\Omega^0(E) \xrightarrow{D''} D_1^p\Omega^1(E) \xrightarrow{D''} D_0^p\Omega^2(E)$$

is represented by a unique harmonic element.

Theorem 6.3.3. *If $p > 1$ is compatible with the parabolic structure on a stable parabolic Higgs bundle (E, D'') of parabolic degree zero, then the space of harmonic sections $\mathcal{H}$ from* (6.4) *has dimension*

$$\dim \mathcal{H} = 2 \operatorname{rk} E(g-1) + \sum_{P \in D} \operatorname{rk} E_{S,P}$$

where $\operatorname{rk} E_{S,P}$ is the rank of the singular part of E at P. If $w_1(P) = 0$, then $\operatorname{rk} E_{S,P} = \operatorname{rk} E - m_P(w_1)$ otherwise $\operatorname{rk} E_{S,P} = \operatorname{rk} E$.

Remark 6.3.4. Notice the similarity in the dimension of $\mathcal{H}$ to the dimension of the moduli space of stable parabolic bundles with fixed determinant connection

$$2(\operatorname{rk} E^2 - 1)(g-1) + 2 \sum_{P \in D} \dim N_P.$$

The term $2 \dim N_P = \operatorname{rk} E^2 - \sum_k m_P(w_k)^2$ is the rank of $\operatorname{End} E$ minus the parts weighted by zero at $P \in D$.

6.3.1 Technical lemmas

The Kähler identities extend to parabolic Higgs bundles in the following way.

Lemma 6.3.5. *Let (E, D'') be a parabolic Higgs bundle with $p > 1$ compatible with the parabolic structure. For $D'' = \bar{\partial}_A + \Phi$ and $D' = \partial_A + \Phi^*$ we have the following Kähler identities*

1. $(D'')^\vee = -i[\Lambda, D']$
2. $(D')^\vee = i[\Lambda, D'']$

where $(D')^\vee, (D'')^\vee : D^p_k \Omega^s(E) \to D^p_{k-1}\Omega^{s-1}(E)$ for $k > 0$ and Λ is contraction with the fixed Kähler form on Σ. The adjoint is with respect to the L^2-inner product induced on $D^p_k \Omega^s(E)$ by $k > 0$.

Proof. The lemma is proved in a similar fashion as the usual Kähler identities using the compatibility of p with the parabolic structure to control the behaviour of sections around the parabolic points. □

The rest of this section is devoted to proving the above theorems. Let us therefore assume that (E, D'') is a fixed stable parabolic Higgs bundle of parabolic degree zero and $p > 1$ is compatible with the parabolic structure on E.

Lemma 6.3.6. *If D'' is irreducible and solves the Higgs bundle equations* (6.1)*, then*

$$D'' : D^p_2 \Omega^0(E) \to D^p_1 \Omega^1(E)$$

has trivial kernel.

Proof. The proof is similar to Lemma 2.1.4 using the irreducibility and that $D'D''$ is a real operator by the Higgs bundle equations. □

Corollary 6.3.7. *If D' is irreducible and satisfies the Higgs bundle equations* (6.1)*, then*

$$D' : D_2^p\Omega^0(E) \to D_1^p\Omega^1(E)$$

has trivial kernel.

Proof. The proof is equivalent to the proof of Lemma 6.3.6 as a non-zero solution to $D's = 0$ is equivalent to the existence of a Higgs subline bundle of the dual parabolic Higgs bundle $(E^*, -\Phi^*)$ which is also of parabolic degree zero. □

Remark 6.3.8. If the parabolic degree is non-zero, then the right handside of the Higgs bundle equations is the parabolic slope. The proof of Lemma 6.3.6 can be generalised to show that the lemma is valid only when the parabolic degree is at most zero. Likewise, Corollary 6.3.7 is only valid when the parabolic degree is at least zero. As we need both results to hold, we have to assume that the parabolic degree is zero.

Lemma 6.3.9. *If D'' and D' are irreducible and satisfies the Higgs bundle equations* (6.1)*, then the operators*

$$D_2^p\Omega^0(E) \xrightarrow{D'D''} D_0^p\Omega^2(E) \quad \text{and} \quad D_2^p\Omega^0(E) \xrightarrow{D''D'} D_0^p\Omega^2(E)$$

are isomorphisms.

Proof. We use results of Lockhart and McOwen [51] to prove that $D'D''$ is a Fredholm operator and to show that the index is zero. The result follows by applying Lemma 6.3.6. We focus on $D'D''$ as the other case is parallel to this.

As $D'D''(u) = \partial_A\bar{\partial}_A u + \Phi^*\Phi u$ and $\Phi^*\Phi : D_2^p\Omega^0(E) \to D_0^p\Omega^2(E)$ is compact, it is enough to prove that

$$\partial_A\bar{\partial}_A : D_2^p\Omega^0(E) \to D_0^p\Omega^2(E) \tag{6.5}$$

is Fredholm of index zero.

Following [51], we must consider solutions to $\partial_A\bar{\partial}_A u = 0$. Choose the singular frame constructed in Section 6.2.2 and use polar coordinates with the parabolic point as zero. We consider the Fourier transform of the $\partial_A\bar{\partial}_A u = 0$ locally around a parabolic point and look for solutions to an eigenvalue problem. To transform $\partial_A\bar{\partial}_A u = 0$ we replace $r\partial_r$ with $i\lambda$ and the equation becomes,

$$0 = -\lambda^2\hat{u} + (\partial_\theta + i\alpha_k)^2\hat{u},$$

where $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda = 1 - \frac{2}{p}$. A solution of this equation exists if $\operatorname{Re}\lambda = 0$ and if $\alpha_k \pm \operatorname{Im}\lambda$ is an integer. But as $\operatorname{Im}\lambda = 1 - \frac{2}{p}$ and p is compatible with the parabolic structure, no solutions exists, and the operator (6.5) is Fredholm. Again, we use the fact that $W^p_2\Omega^0(E)$ has finite codimension in $D^p_2\Omega^0(E)$.

As $-i\Lambda\partial_A\bar{\partial}_A = \bar{\partial}^*_A\bar{\partial}_A : \Omega^0(E) \to \Omega^0(E)$ is self-adjoint we get from [51, Theorem 7.4] that the index of

$$\partial_A\bar{\partial}_A : W^p_2\Omega^0(E) \to W^p_0\Omega^2(E) = L^p\Omega^2(E) = D^p_0\Omega^2(E) \tag{6.6}$$

is

$$-\frac{1}{2}\sum_{\operatorname{Im}\lambda=0} d(\lambda)$$

where $d(\lambda)$ is the total dimension of all local solutions to

$$0 = (r\partial_r)^2 u + (\partial_\theta + i\alpha_k)^2 u$$

of the form $r^{-i\lambda}P(\theta, r)$ where P is a polynomial in $-\log(r)$ with coefficients being functions defined on a circle.

When imposing $\operatorname{Im}(\lambda) = 0$ a direct computation shows that if $\alpha_k \neq 0$ there are no solutions to the equation. However, if $\alpha_k = 0$ there is a two-dimensional space of solutions. This shows that the index of (6.6) is

$$\operatorname{ind}\bigl(\partial_A\bar{\partial}_A : W^p_2\Omega^0(E) \to D^p_0\Omega^2(E)\bigr) = -\bigl|\{\alpha^{(i)}_k \ : \ \alpha^{(i)}_k = 0\}\bigr|.$$

But, by Lemma 6.2.5 this is exactly the opposite of the codimension of $W^p_2\Omega^0(E)$ in $D^p_2\Omega^0(E)$. Let the codimension be N. Then for some $n \leq N$

$$\begin{aligned} \dim\ker\bigl(\partial_A\bar{\partial}_A : D^p_2\Omega^0(E) \to D^p_0\Omega^2(E)\bigr) &= \dim\ker\bigl(\partial_A\bar{\partial}_A|_{W^p_2\Omega^0(E)}\bigr) + n \\ \dim\operatorname{coker}\bigl(\partial_A\bar{\partial}_A : D^p_2\Omega^0(E) \to D^p_0\Omega^2(E)\bigr) + N - n &= \dim\operatorname{coker}\bigl(\partial_A\bar{\partial}_A|_{W^p_2\Omega^0(E)}\bigr) \end{aligned}$$

giving that the index of (6.5) is zero. The index of

$$D'D'' : D^p_2\Omega^0(E) \to D^p_0\Omega^2(E)$$

is therefore also zero. □

Lemma 6.3.10. *If $\varphi \in D^p_1\Omega^1(E)$, then there are $\beta, \gamma \in D^p_2\Omega^0(E)$ such that*

$$D'(\varphi - D''\beta) = 0 \quad \textit{and} \quad D''(\varphi - D'\gamma) = 0.$$

Proof. If $\varphi \in D^p_1\Omega^1(E)$, then $D'\varphi \in D^p_0\Omega^2(E)$ and by Lemma 6.3.9 there is a $\beta \in D^p_0\Omega^0(E)$ such that $D'D''\beta = D'\varphi$. The existence of γ is equivalent. □

6.3.2 Proof of theorems

In this section we use the lemmas from Section 6.3.1 to prove Theorems 6.3.1 and 6.3.3.

Proof of Theorem 6.3.1. For $\varphi \in D_1^p\Omega^1(E)$ Lemma 6.3.10 gives $\beta, \gamma \in D_2^p\Omega^0(E)$ such that

$$\varphi - D''\beta - D'\gamma$$

is in $\mathcal{H}$, proving that

$$D_1^p\Omega^1(E) = \mathcal{H} + D''D_2^p\Omega^0(E) + D'D_2^p\Omega^0(E).$$

The trivial intersection of the spaces follows from Lemma 6.3.9, e.g. if $\varphi \in \mathcal{H} \cap \operatorname{im}(D')$, then $0 = D''\varphi = D''D'\beta$ giving $\beta = 0$. The decomposition of $\ker(D'')$ and $\ker(D')$ also follow from Lemmas 6.3.9 and 6.3.10.

The L^2-orthogonality of the decompositions follow directly from the Kähler identities.

□

Lemma 6.3.11. *The operator*

$$T = D' \oplus D'' : D_1^p\Omega^1(E) \to D_0^p\Omega^2(E)^{\oplus 2}$$

is Fredholm, and the index of T is $\dim \mathcal{H}$.

Proof. By Lemma 6.2.9 it is enough to prove that

$$\partial_A \oplus \bar\partial_A : W_1^p\Omega^1(E) = D_1^p\Omega^1(E) \to W_0^p\Omega^2(E)^{\oplus 2}$$

is Fredholm. As usual we use Lockhart and McOwen [51]. Assume a local section of $D_1^p\Omega^1(E)$ has the form $u = a_k \frac{e_k}{|z|^{\alpha_k}} dz + b_k \frac{e_k}{|z|^{\alpha_k}} d\bar z$. Then locally around a parabolic point the equations for u to be in the kernel of T are

$$\begin{aligned} 0 &= -ir\partial_r b_k - \partial_\theta b_k - i\alpha_k b_k \\ 0 &= ir\partial_r a_k - \partial_\theta a_k - i\alpha_k a_k. \end{aligned}$$

Fourier transforming these equations by replacing $-ir\partial_r$ by λ in the equations above, shows that they have no solutions if p is adapted to the parabolic structure. Therefore, $\partial_A \oplus \bar\partial_A$ is Fredholm.

By Lemma 6.3.9, $D' : D''(D_2^p\Omega^0(E)) \to D_0^p\Omega^2(E)$ and $D'' : D'(D_2^p\Omega^0(E)) \to D_0^p\Omega^2(E)$ are surjective, and thus $\operatorname{ind}(T) = \dim\ker(T) = \dim \mathcal{H}$. □

Proof of Theorem 6.3.3. By Lemma 6.3.11 we must compute the index of $T = D' \oplus D''$. As $D'D''$ is an isomorphism

$$\begin{aligned}
0 &= \operatorname{ind}(D'D'' : D_2^p\Omega^0(E) \to D_0^p\Omega^2(E)) \\
&= \operatorname{ind}(\partial_A\bar\partial_A : D_2^p\Omega^0(E) \to D_0^p\Omega^2(E)) \\
&= \operatorname{ind}(\bar\partial_A : D_2^p\Omega^0(E) \to D_1^p\Omega^{0,1}(E)) + \operatorname{ind}(\partial_A : D_1^p\Omega^{0,1}(E) \to D_0^p\Omega^2(E)),
\end{aligned}$$

giving that the index of T is

$$\begin{aligned}
\operatorname{ind}(T) &= \operatorname{ind}(\bar\partial_A : D_1^p\Omega^{1,0}(E) \to D_0^p\Omega^2(E)) + \operatorname{ind}(\partial_A : D_1^p\Omega^{0,1}(E) \to D_0^p\Omega^2(E)) \\
&= \operatorname{ind}(\bar\partial_A : D_1^p\Omega^{1,0}(E) \to D_0^p\Omega^2(E)) - \operatorname{ind}(\bar\partial_A : D_2^p\Omega^0(E) \to D_1^p\Omega^{0,1}(E)).
\end{aligned}$$

Now, Lemma 6.2.13 gives that

$$\begin{aligned}
\operatorname{ind}(\bar\partial_A : D_2^p\Omega^0(E) \to D_1^p\Omega^{0,1}(E)) &= \operatorname{ind}(\bar\partial_A : \Omega^0(E) \to \Omega^{0,1}(E)) = \chi(E) \\
\operatorname{ind}(\bar\partial_A : D_1^p\Omega^0(EK) \to D_0^p\Omega^{0,1}(EK)) &= \operatorname{ind}(\bar\partial_A : \Omega^0(SK) \to \Omega^{0,1}(SK)) = \chi(\mathcal{S}K(D))
\end{aligned}$$

where S is the sheaf of sections of E with at most a simple pole on D with residue in the *singular* part of E at each point of D and $\mathcal{S} = S(-D)$ is the sheaf of holomorphic sections taking values in the singular part of E. From the exact sheaf sequence

$$0 \to \mathcal{S} \otimes K(D) \to E \otimes K(D) \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is a sky-scraper sheaf on Σ supported on the parabolic points with a zero weight. The length of a stalk is exactly the multiplicity of the zero weight at that point. It follows that

$$\dim \mathcal{H} = \chi(\mathcal{S} \otimes K(D)) - \chi(E) = 2 \operatorname{rk} E(g-1) + \sum_{P \in D} \operatorname{rk} E_{S,P},$$

proving Theorem 6.3.3. □

Remark 6.3.12. The sheaf $\mathcal{S}$ can also be defined as the kernel of the projection map onto the regular part of E at each parabolic point. The fibre of $\mathcal{Q}$ at $P \in D$ can be identified with E_{R_P}. See the end of Section 6.4.1 for a more natural description of $\mathcal{S}$ as strictly parabolic homomorphism $\mathcal{O}$ to E.

6.4 Dolbeault, de Rham, and Hypercohomology

In this section we will use the Hodge theory from Section 6.3 to identify the Dolbeault and de Rham cohomology of a stable parabolic Higgs bundle of parabolic degree zero with the hypercohomology of $E \xrightarrow{\Phi} \mathcal{S} \otimes K(D)$.

6.4.1 Hypercohomology

In this section we prove the following theorem relating Dolbeault cohomology and hypercohomology.

Theorem 6.4.1. *If $(E, \bar{\partial}_A + \Phi)$ is a stable parabolic Higgs bundle of parabolic degree zero, then*

$$\mathbb{H}^1(E \xrightarrow{\Phi} \mathcal{S} \otimes K(D)) \simeq \mathcal{H}$$

where $\mathcal{H}$ is defined using the adapted Hermitian metric solving the Higgs bundle equations.

Throughout this section we let $(E, \bar{\partial}_A + \Phi)$ be a fixed stable parabolic Higgs bundle of parabolic degree zero with metric solving the Higgs bundle equations, and fix p compatible with the parabolic structure on E.

In this section we use the same notation for the spaces $D^p_k\Omega^s(E)$ and their sheaves of local sections.

To prove the theorem we need to find a resolution of $\mathcal{E}$ and $\mathcal{S} \otimes K(D)$ using the sheaves $D^p_k\Omega^s(E)$ of D^p_k-sections of $\Lambda^s T^*\Sigma \otimes E$.

Lemma 6.4.2. *The sequence*

$$0 \to \mathcal{E} \to D^p_2\Omega^0(E) \xrightarrow{\bar{\partial}_A} D^p_1\Omega^{0,1}(E)$$

is a resolution of $\mathcal{E}$.

Proof. Let $U \subset \Sigma$ be an open set. From Lemma 6.2.13 it follows that $\mathcal{E}(U)$ is the kernel of $\bar{\partial}_A$.

What remains is to prove surjectivity of $\bar{\partial}_A$. If there are no parabolic points in U surjectivity follows from the Dolbeault Lemma for Sobolev spaces. Assume U contains a parabolic point P.

Choose a local frame $\{e_k\}_{k=1}^l$ of holomorphic sections of E adapted to the parabolic structure. Let $ud\bar{z}$ be an element of $D^p_1\Omega^{0,1}(U, E)$ with $u = \sum u_k e_k$. As $\{e_k\}$ is a holomorphic frame $\bar{\partial}_A = \bar{\partial}$ and for each k we are seeking a solution to

$$\frac{\partial s_k}{\partial \bar{z}} = u_k.$$

It follows from Theorem 6.2.1 that the Dolbeault Lemma for Sobolev spaces gives a solution $s \in L^p_2(U)$. If the parabolic weight $\alpha_k = 0$, then we can take $s_k = s$. If $\alpha_k \neq 0$, then the solution we seek must vanish at 0. As $1 < p < 2$ the solution $s \in L^p_2(U)$ is continuous on U so $s_k = s - s(0) \in W^p_2(U)$ solves the equation. □

Lemma 6.4.3. *The sequence*

$$0 \to \mathcal{S} \otimes K(D) \to D^p_1\Omega^{1,0}(E) \xrightarrow{\bar{\partial}_A} D^p_0\Omega^{1,1}(E)$$

is a resolution of $\mathcal{S} \otimes K(D)$.

Proof. The proof is similar to Lemma 6.4.2 using Lemma 6.2.13 to identify $\mathcal{S} \otimes K(D)$ as the kernel of $\bar{\partial}_A$. $\square$

Proof of Theorem 6.4.1. By Lemma 6.2.9 multiplication by the Higgs field extends to a morphism of the resolutions in Lemmas 6.4.2 and 6.4.3.

$$\begin{tikzcd} \mathcal{E} \arrow[r, hook] \arrow[d, "\Phi"] & D_2^p\Omega^0(E) \arrow[r, "\bar{\partial}_A"] \arrow[d, "\Phi"] & D_1^p\Omega^{0,1}(E) \arrow[d, "\Phi"] \\ \mathcal{S} \otimes K(D) \arrow[r, hook] & D_1^p\Omega^{1,0}(E) \arrow[r, "\bar{\partial}_A"] & D_0^p\Omega^{1,1}(E) \end{tikzcd}$$

The hypercohomology of $\mathcal{E} \xrightarrow{\Phi} \mathcal{S} \otimes K(D)$ can be computed by this double complex by a spectral sequence argument. It follows that the first hypercohomology group is isomorphic to the first cohomology group of

$$D_2^p\Omega^0(E) \xrightarrow{\Phi+\bar{\partial}_A} D_1^p\Omega^{1,0}(E) \oplus D_1^p\Omega^{0,1}(E) \xrightarrow{\bar{\partial}_A+\Phi} D_0^p\Omega^{1,1}(E).$$

By Corollary 6.3.2 this cohomology is exactly $\mathcal{H}$. $\square$

If $\mathbb{E}_1, \mathbb{E}_2$ are two parabolic Higgs bundles, then $\mathbb{H}^0(\mathbb{H}\mathrm{om}(\mathbb{E}_1, \mathbb{E}_2))$ are the homomorphisms from $\mathbb{E}_1$ to $\mathbb{E}_2$.

Hausel's arguments in [33, Theorem 4.3] works equally well for the parabolic situation, giving the following result.

Corollary 6.4.4. *For any stable parabolic Higgs bundle $\mathbb{E}$ with* $\mathrm{pardeg}(\mathbb{E}) = 0$ *and $\mathbb{E}$ non-trivial, then*

$$\mathbb{H}^0(\mathbb{H}\mathrm{om}(\mathcal{O}, \mathbb{E})) = \mathbb{H}^2(\mathrm{Hom}(\mathcal{O}, \mathbb{E})) = 0,$$

where $\mathcal{O}$ is given parabolic weight zero and zero Higgs field.

The complex $\mathbb{H}\mathrm{om}(\mathcal{O}, \mathbb{E})$ is

$$\mathrm{ParHom}(\mathcal{O}, E) \to \mathrm{SParHom}(\mathcal{O}, E) \quad = \quad E \to \mathcal{S} \otimes K(D)$$

and thus the corollary is the holomorphic analogue of Lemma 6.3.6 and Corollary 6.3.7. This also justifies why hypercohomology of the complex $E \to \mathcal{S} \otimes K(D)$ is the right object to study as opposed to the more obvious choice $E \to E \otimes K(D)$. Notice that $\mathcal{S} = E$ if there are no zero-weights.

6.4.2 de Rham cohomology

Let (E, D'') be a stable parabolic Higgs bundle with parabolic degree zero and $D'' = \bar{\partial}_A + \Phi$. If we use the adapted Hermitian metric solving the Higgs bundle equations to define $D' = \partial_A + \Phi^*$, then

$$D = D' + D'' = d_A + \Phi + \Phi^*$$

is a flat connection on E with singularities at the parabolic points.

Conversely, if D is a flat connection, then a Hermitian metric h induce a splitting $D = d_h + \Psi$ into a unitary and a self-adjoint part. Splitting further $d_h = \partial_E + \bar{\partial}_E$ and $\Psi = \Phi + \Phi^*$ into types we can define the operator $D'' = \bar{\partial}_E + \Phi$ and $D' = \partial_E + \Phi^*$. Simpson [70] showed there is a natural one-to-one correspondence between stable parabolic Higgs bundles of parabolic degree zero and irreducible flat connections on a vector bundle with parabolic structure which behaves sufficiently nice around the parabolic points (tameness). The equivalence goes through the existence of a so-called harmonic metric: the operator D'' defined via the metric satisfies $(D'')^2 = \bar{\partial}_E(\Phi) = 0$, i.e that Φ is holomorphic with respect to $\bar{\partial}_E$. This is an extension of Donaldson [22] and Corlette [19] to the parabolic situation.

Using the harmonic metric to define weighted Sobolev spaces as above we have operators

$$D_2^p\Omega^0(E) \xrightarrow{D} D_1^p\Omega^1(E) \quad \text{and} \quad D_1^p\Omega^1(E) \xrightarrow{D} D_0^p\Omega^2(E).$$

Theorem 6.4.5. *If (E, D) is a bundle with a parabolic structure and an irreducible flat connection, then the first cohomology group of*

$$D_2^p\Omega^0(E) \xrightarrow{D} D_1^p\Omega^1(E) \xrightarrow{D} D_0^p\Omega^2(E)$$

is isomorphic to $\mathcal{H}$ defined using the harmonic metric from [70].

Proof. We use the harmonic metric to split D to operators D' and D''. We know from Theorem 6.3.1 that every $\varphi \in D_1^p\Omega^1(E)$ has a unique decomposition $\varphi = \eta + D'\beta + D''\gamma$ for $\beta, \gamma \in D_2^p\Omega^0(E)$. If $\beta = \gamma$, then $D\varphi = 0$ gives $\ker D = \mathcal{H} \oplus \operatorname{im}(D)$. This proves the theorem. The connection D is flat $0 = D^2 = D'D'' + D''D'$, and therefore

$$0 = D^2\varphi = D''D'\beta + D'D''\gamma = D''D'(\beta - \gamma).$$

Since $D''D'$ is an isomorphism (Lemma 6.3.9) $\beta = \gamma$. $\square$

6.5 Universal bundle

In this section we will discuss the existence of a holomorphic universal parabolic Higgs bundle on $\mathscr{M} \times \Sigma$ where $\mathscr{M}$ is the moduli space of stable parabolic Higgs bundles. This was first constructed algebraically by Yokogawa [78] showing that $\mathscr{M}$ is a coarse moduli space. Thaddeus [74] mentions that a universal parabolic Higgs bundle can be constructed using standard arguments similar to Newstead [62, §5.5]. In this section we make the construction explicit to determine conditions on the parabolic data to determine when the universal parabolic Higgs bundle exists. We follow Hausel [33, Section 5] who also use Newstead's approach. All ingredients used in this section appear elsewhere but not collectively. Before we embark on this, we briefly review Konno's construction [50] of the moduli space $\mathscr{M}$.

Let us first stress which objects we fix from the beginning:

- Σ a compact Riemann surface with Kähler form ω normalised for Σ to have area 1.
- E a smooth vector bundle on Σ of rank l and topological degree d.
- $D = P_1 + \cdots + P_n$ a divisor of n distinct points.
- A quasi-parabolic structure, i.e. a flag in the fibre E_{P_i} at each P_i.
- A collection of weights $\alpha_k^{(i)}$ at each P_i.
- A Hermitian metric adapted to the parabolic structure.
- A $\bar{\partial}$-operator, $\bar{\partial}_{A_0}$, inducing a holomorphic structure on $\det E$.

A set of parabolic weights is called *generic* if all semi-stable parabolic Higgs bundles are stable for this choice of weights. Throughout this section we will assume the fixed set of weights is generic.

Recall the definition of $\mathscr{D}$ from Section 6.1 as the space parametrising parabolic Higgs structures $D'' = \bar{\partial}_A + \Phi$ with Φ being $\bar{\partial}_A$-meromorphic on Σ with at most simple poles at D with strictly parabolic residues. Since the flags at each parabolic point is fixed we consider only gauge transformations preserving the flags

$$\mathscr{G}^c = \{g \in \Omega^0(\mathrm{ParEnd}(E)) \,|\, \det g_x = 1 \quad \text{for any} \quad x \in \Sigma\}.$$

If $p > 1$ is compatible with the fixed parabolic structure we define completions of $\mathscr{D}$ and $\mathscr{G}^c$

$$\mathscr{D}_1^p = \{D'' = \bar{\partial}_A + \Phi \in \mathscr{C}_1^p \times D_1^p\Omega^{1,0}(\mathrm{End}_0 E) \,|\, \bar{\partial}_A \Phi = 0\}$$
$$\mathscr{G}^c{}_2^p = \{g \in D_2^p\Omega^0(\mathrm{End}\, E) \,|\, \det g_x = 1 \quad \text{for any} \quad x \in \Sigma\}.$$

From Biquard's Sobolev embedding theorem (Theorem 6.2.1) we see that ${\mathscr{G}^c}^p_2$ is a group with a right action on $\mathscr{D}^p_1$. It furthermore follows that $\mathscr{D}$ is dense in $\mathscr{D}^p_1$ and $\mathscr{G}^c$ is dense in ${\mathscr{G}^c}^p_2$. Define $\mathscr{D}^{st}$ as the subspace of $\mathscr{D}$ consisting of *stable* parabolic structures and extend this to ${\mathscr{D}^{st}}^p_1$. Konno then defines

$$\mathscr{M} := {\mathscr{D}^{st}}^p_1 / {\mathscr{G}^c}^p_2$$

to be the moduli space of stable parabolic Higgs bundles and shows that it is a finite dimensional hyperkähler manifold. Konno furthermore shows that the metric on $\mathscr{M}$ is *complete*.

Definition 6.5.1. A parabolic structure is *good* if the collection of weights is generic and one of the following three conditions hold for the quasi-parabolic structure:

- The rank l and topological degree d are coprime.
- There is a parabolic point $P_i \in D$ such that $\dim F^k E_{P_i}$ is coprime to the rank for some $1 \leq k \leq a_i$.
- There is a parabolic point $P_i \in D$ such that $\dim F^k E_{P_i}$ and $l + d$ are coprime for some $1 \leq k \leq a_i$.

Remark 6.5.2. Having a good parabolic structure is a rather mild condition on the quasi-parabolic data, which is often satisfied in examples, e.g. if at one parabolic point the flag is full, the parabolic structure is good.

Lemma 6.5.3. *If $\mathscr{C}^p_1$ is the completion of the space of $\bar{\partial}$-operators with respect to the weighted Sobolev norm defined from a fixed good parabolic structure, then there exists a ${\mathscr{G}^c}^p_2$-equivariant holomorphic line bundle $L_{\mathscr{C}}$ on $\mathscr{C}^p_1$ on which $\mathbb{C}^* \subset {\mathscr{G}^c}^p_2$ acts by scalar multiplication with trivial character.*

Proof. This is [12, Proposition 1.7]. We reiterate the argument here for completeness.

Define bundles $E_{\mathscr{C}} = \mathscr{C}^p_1 \times E$ and $E^{i,k}_{\mathscr{C}} = \mathscr{C}^p_1 \times F^k E_{P_i}$ for $P_i \in D$ on $\mathscr{C}^p_1 \times \Sigma$. Here E is the fixed smooth complex vector bundle with fixed good parabolic structure. If ${\mathscr{G}^c}^p_2$ acts trivially on Σ, then $E_{\mathscr{C}}$ and $E^{i,k}_{\mathscr{C}}$ are naturally ${\mathscr{G}^c}^p_2$-equivariant bundles on which $\mathbb{C}^*$ acts by scalar multiplication. Fix a line bundle N of degree 1 on Σ and for each $m \in \mathbb{Z}$ let $E_{\mathscr{C}}(m) = E_{\mathscr{C}} \otimes q^* N^m$ where $q : \mathscr{C}^p_1 \times \Sigma \to \Sigma$ is the projection.

As $\mathscr{C}^p_1$ is the space of $\bar{\partial}$-operators on E we get a holomorphic line bundle $\operatorname{Det} E_{\mathscr{C}}(m)$ on $\mathscr{C}^p_1$ via Quillen's determinant construction [66]. The action of $\lambda \in \mathbb{C}^*$ on $\operatorname{Det} E_{\mathscr{C}}(m)$ is by $\lambda^{d+l(1-g)+ml}$ with g the genus of Σ. If l and d are coprime, then there exists integers a, b such that $al + b(d + l(g-1)) = 1$, and $\mathbb{C}^*$ acts by scalar multiplication with trivial character on

$$L_{\mathscr{C}} = (\operatorname{Det} E_{\mathscr{C}}(1))^a \otimes (\operatorname{Det} E_{\mathscr{C}})^{b-a}.$$

If $\dim F^k E_{P_i}$ and l are coprime for some i and k and if a and b are integers such that $a \dim F^k E_{P_i} + bl = 1$, then the action of $\mathbb{C}^*$ on

$$L_{\mathscr{C}} = (\det E_{\mathscr{C}}^{i,k})^a \otimes (\operatorname{Det} E_{\mathscr{C}})^{-b} \otimes (\operatorname{Det} E_{\mathscr{C}}(1))^b$$

is by scalar multiplication with a trivial character.

If $\dim F^k E_{P_i}$ and $l + d$ are coprime for some i and k let a, b be integers such that $a \dim F^k E_{P_i} + b(d + l) = 1$, then the action of $\mathbb{C}^*$ is scalar multiplication with a trivial character on

$$L_{\mathscr{C}} = (\det E_{\mathscr{C}}^{i,k})^a \otimes (\operatorname{Det} E_{\mathscr{C}}(1))^b \otimes (\det E_{\mathscr{C}}^{i})^{b(g-1)}$$

where $E_{\mathscr{C}}^i$ is the restriction of $E_{\mathscr{C}}$ to $\mathscr{C}_1^p \times \{P_i\}$. □

Proposition 6.5.4. *Let $\mathscr{M}$ be the moduli space of stable parabolic Higgs bundles with parabolic structure fixed as above. If the parabolic structure is good, then there is a holomorphic universal parabolic Higgs bundle on $\mathscr{M} \times \Sigma$.*

Proof. The proof relies on the construction of a bundle $E_{\mathscr{M}}$ on $\mathscr{M} \times \Sigma$ parametrising the holomorphic structure of (E, D'') for every $D'' \in \mathscr{M}$. To do this we follow a similar construction by Atiyah and Bott [3, p. 579–580]. Let $E_{\mathscr{C}}$ be the tautological bundle on $\mathscr{C}_1^p \times \Sigma$, and let $\mathscr{G}^c{}_2^p$ act trivially on Σ, then $\mathbb{C}^* \subset \mathscr{G}^c{}_2^p$ acts by scalar multiplication on $E_{\mathscr{C}}$. As the parabolic structure is good there is a holomorphic line bundle $L_{\mathscr{C}}$ on $\mathscr{C}_1^p \times \Sigma$ on which $\mathbb{C}^*$ acts by scalar multiplication (Lemma 6.5.3).

Denote by $E_{\mathscr{D}}$ the pullback of $E_{\mathscr{C}} \otimes L_{\mathscr{C}}^{-1}$ to $\mathscr{D}^{st}{}_1^p \times \Sigma$. Now $\mathbb{C}^*$ acts trivially on $E_{\mathscr{D}}$ and so $E_{\mathscr{D}}$ is a $\mathscr{G}^c{}_2^p/\mathbb{C}^*$-equivariant bundle. The only automorphisms of a stable parabolic Higgs bundle are scalar multiples of the identity (Proposition 6.1.2), so $\mathscr{G}^c{}_2^p/\mathbb{C}^*$ acts freely on $\mathscr{D}^{st}{}_1^p$. It follows that $E_{\mathscr{C}} \otimes L_{\mathscr{C}}^{-1}$ reduces to a holomorphic vector bundle $E_{\mathscr{M}}$ on $\mathscr{M} \times \Sigma$ with the property that $E_{\mathscr{M}}|_{\bar\partial_A + \Phi} \simeq (E, \bar\partial_A)$.

To conclude the proof, we need a universal Higgs field. This follows exactly as in [33]. The fibre of the projection $\mathscr{D}_1^p \to \mathscr{C}_1^p$ is canonically $H^0(\Sigma, \operatorname{SParEnd}_0(E) \otimes K(D))$, as shown in [50, Lemma 2.6]. This gives a tautological section

$$\Phi_{\mathscr{D}} \in H^0(\mathscr{D}_1^p, \operatorname{SParEnd}_0(E_{\mathscr{D}}) \otimes K(D)).$$

The section $\Phi_{\mathscr{D}}$ defines a Higgs bundle $E_{\mathscr{D}} \xrightarrow{\Phi_{\mathscr{D}}} E_{\mathscr{D}} \otimes K(D)$ on $\mathscr{D}^{st}{}_1^p \times \Sigma$ which is $\mathscr{G}^c{}_2^p/\mathbb{C}^*$-equivariant and thus gives a holomorphic universal Higgs bundle $E_{\mathscr{M}} \xrightarrow{\Phi_{\mathscr{M}}} E_{\mathscr{M}} \otimes K(D)$ on $\mathscr{M} \times \Sigma$. □

Konno's construction of the moduli space of stable parabolic Higgs bundles shows that it is a *coarse* moduli space. Using Proposition 6.5.4 as the main ingredient one can show that under mild assumptions the moduli space is actually *fine*.

Definition 6.5.5. Two families $\mathbb{E}_T$, $\mathbb{F}_T$ of stable parabolic Higgs bundles on $T \times \Sigma$ are *equivalent* if there exists a line bundle L on T such that $\mathbb{F}_T \simeq \mathbb{E}_T \otimes \pi_T^*(L)$ where $\pi : T \times \Sigma \to T$ is the projection.

The following lemma from [74] shows that two families are equivalent exactly if they give the same map to the coarse moduli space $\mathscr{M}$.

Lemma 6.5.6. *If $\mathbb{E}_T$ and $\mathbb{F}_T$ are families of stable parabolic Higgs bundles over Σ parametrised by a variety T, and suppose $\mathbb{E}_t \simeq \mathbb{F}_t$ for all $t \in T$. Then $\mathbb{E}_T$ and $\mathbb{F}_T$ are equivalent.*

Corollary 6.5.7. *If $\mathscr{M}$ is the moduli space of stable parabolic Higgs bundles with fixed good parabolic structure, then $\mathscr{M}$ is fine.*

Proof. The corollary follows directly by the arguments of [62, Theorem 5.12] using Proposition 6.5.4 and Lemma 6.5.6. □

Remark 6.5.8. In Konno's construction [50], a flag at each parabolic point is fixed throughout and the gauge transforms used preserve the flags. Another way to construct $\mathscr{M}$ is given by Yokogawa [78] where one does not need to fix the flags and therefore can use all gauge transforms. The two moduli spaces are isomorphic and we find the latter description more useful for explicitly describing various submanifolds of $\mathscr{M}$ as we will do in Section 6.7.

6.6 A hyperholomorphic bundle

In this section we use the Hodge theory for parabolic Higgs bundles developed in Section 6.3 to construct a hyperholomorphic vector bundle on the moduli space of parabolic Higgs bundles. Throughout this section we consider stable parabolic Higgs bundles of parabolic degree zero with fixed good parabolic structure. We denote by $\mathscr{M}$ the moduli space of these.

Theorem 6.6.1. *There is a hyperholomorphic vector bundle $(\mathbb{D}, \nabla)$ on $\mathscr{M}$ of rank*

$$2l(g-1) + nl - m_0 \quad \textit{with} \quad m_0 = \sum_{P \in \tilde{D}} m_P(w_1)$$

where $\tilde{D} \subset D$ are the parabolic points where the lowest weight is zero. This bundle is called the Dirac–Higgs *bundle.*

Proof. From Proposition 6.5.4 there is a universal Higgs bundle $E_{\mathscr{M}} \xrightarrow{\Phi_{\mathscr{M}}} E_{\mathscr{M}} \otimes K(D)$ on $\mathscr{M} \times \Sigma$. This Higgs bundle gives two families of operators $D''_{A,\Phi} = \bar{\partial}_A + \Phi$ and

$D'_{A,\Phi} = \partial_A + \Phi^*$ acting on $D^p_1\Omega^1(E)$. Here we use the fixed adapted Hermitian metric. Combining these we get an elliptic differential operator

$$\mathcal{D}^*_{A,\Phi} = \begin{pmatrix} \bar{\partial}_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : D^p_1(\Omega^{1,0}(E)) \oplus D^p_1(\Omega^{0,1}(E)) \to D^p_0\Omega^{1,1}(E)^{\oplus 2}.$$

From Theorem 6.3.1 it follows that $\ker \mathcal{D}^*_{A,\Phi}$ is finite-dimensional and by Theorem 6.3.3 the dimension is independent of $D''_{A,\Phi} \in \mathscr{M}$. Together, this produces a vector bundle $\mathbb{D}$ on $\mathscr{M}$ of the right dimension. As

$$D^p_1\Omega^{1,0}(E) \oplus D^p_1\Omega^{0,1}(E) \subset L^2\Omega^{1,0}(E) \oplus L^2\Omega^{0,1}(E) =: L^2$$

there is an inner product on the infinite dimensional trivial bundle $\mathscr{M} \times L^2$ on $\mathscr{M}$. The inclusion $i : \ker \mathcal{D}^*_{A,\Phi} \hookrightarrow L^2$ of fibres embeds $\mathbb{D}$ as a subbundle of $\mathscr{M} \times L^2$. We define a connection on $\mathbb{D}$ by projecting the trivial connection to $\mathbb{D}$

$$\nabla = P\underline{d}i.$$

The bundle with connection $(\mathbb{D}, \nabla)$ is the *Dirac–Higgs* bundle.

What remains is to prove that the curvature of ∇ is of type $(1,1)$ with respect to all complex structures on $\mathscr{M}$. To do this we need the Hodge theory developed in Section 6.3.

The tangent space to $\mathscr{C}^p_1 \times D^p_1\Omega^{1,0}\operatorname{End}_0(E)$ at $(\bar{\partial}_A, \Phi)$ is naturally isomorphic to

$$\mathscr{T}^p_1 = D^p_1\Omega^{0,1}\operatorname{End}_0(E) \times D^p_1\Omega^{1,0}\operatorname{End}_0(E),$$

and for $(\dot{\beta}, \dot{\Phi}) \in \mathscr{T}^p_1$ we have (as in the non-parabolic case) three inequivalent complex structures

$$I(\dot{\beta}, \dot{\Phi}) = (i\dot{\beta}, i\dot{\Phi}) \quad J(\dot{\beta}, \dot{\Phi}) = (i\dot{\Phi}^*, -i\dot{\beta}^*) \quad K(\dot{\beta}, \dot{\Phi}) = (-\dot{\Phi}^*, \dot{\beta}^*).$$

For the complex structure I, consider the family of complexes

$$D^p_2\Omega^0(E) \xrightarrow{\bar{\partial}_A + \Phi} D^p_1\Omega^1(E) \xrightarrow{\bar{\partial}_A + \Phi} D^p_0\Omega^2(E).$$

An infinitesimal deformation of $\bar{\partial}_A + \Phi$ is $\dot{\beta} + \dot{\Phi}$ where $(\dot{\beta}, \dot{\Phi}) \in \mathscr{T}^p_1$ and thus I acts as multiplication by i on the derivative of $\bar{\partial}_A + \Phi$. In other words, this complex varies holomorphically with respect to I. From Lemmas 6.3.6 and 6.3.9 the complex is exact with cohomology concentrated in degree one. Furthermore, it is split by the inverse of $D''_{A,\Phi}D'_{A,\Phi}$. The complex is a so-called infinite dimensional monad and from general theory [23, Section 3.1.3] the cohomology defines a Hermitian holomorphic bundle on $\mathscr{M}$. By Theorem 6.3.1 it follows that $\mathbb{D}$ has a holomorphic structure with respect to I with which ∇ is compatible.

For the complex structure J, consider the de Rham complex for the family of flat connections $d_A + \Phi + \Phi^*$

$$D_2^p\Omega^0(E) \xrightarrow{d_{dR}} D_1^p\Omega^1(E) \xrightarrow{d_{dR}} D_0^p\Omega^2(E),$$

where $d_{dR} = d_A + \Phi + \Phi^*$. This complex varies holomorphically with respect ot J as J is multiplication by i on $\dot\beta - \dot\beta^* + \dot\Phi + \dot\Phi^*$. Again from Lemmas 6.3.6 and 6.3.9 the zeroth and second cohomology vanish and the family is an infinite dimensional monad defining a Hermitian bundle on $\mathscr{M}$ holomorphic with respect to J. From Theorem 6.4.5 it follows that $\mathbb{D}$ has a holomorphic structure with respect to J with which ∇ is compatible.

The argument for the complex structure K is equivalent to that for J, but we instead consider the family of complexes with differentials $d_A - i\Phi + i\Phi^*$. The complex structure K is multiplication by i on $\dot\beta - \dot\beta^* - i\dot\Phi + i\dot\Phi^*$. It follows from arguments similar to Theorem 6.4.5 that the cohomology is concentrated in degree one and that $\mathbb{D}$ has a holomorphic structure with respect to K with which ∇ is compatible. $\square$

6.6.1 Dirac–Higgs and Green's operator

In the proof of Theorem 6.6.1 we introduced an operator $\mathcal{D}^*_{A,\Phi}$ associated to any parabolic Higgs bundle. In this section we will explore this operator further and discuss some of its properties.

Let (E, D'') with $D'' = \bar\partial_A + \Phi$ be a parabolic Higgs bundle, and assume p is compatible with the parabolic structure. We follow Section 2.1 and define a Dirac-operator coupled to a parabolic Higgs bundle by

$$\mathcal{D}_{A,\Phi} = \begin{pmatrix} \partial_A & -\Phi \\ \Phi^* & -\bar\partial_A \end{pmatrix} : D_2^p\Omega^0(E)^{\oplus 2} \to D_1^p\Omega^{1,0}(E) \oplus D_1^p\Omega^{0,1}(E).$$

We call $\mathcal{D}_{A,\Phi}$ the Dirac–Higgs operator. The operator $\mathcal{D}^*_{A,\Phi}$ defined in the proof of Theorem 6.6.1

$$\mathcal{D}^*_{A,\Phi} = \begin{pmatrix} \bar\partial_A & \Phi \\ \Phi^* & \partial_A \end{pmatrix} : D_1^p\Omega^{1,0}(E) \oplus D_1^p\Omega^{0,1}(E) \to D_0^p\Omega^{1,1}(E)^{\oplus 2}$$

is the adjoint of $\mathcal{D}_{A,\Phi}$ with respect to the L^2-inner product on $D_1^p\Omega^1(E)$ in the following sense:

Lemma 6.6.2. *Let $s \in D_2^p\Omega^0(E)^{\oplus 2}$ and $u \in D_1^p\Omega^{1,0}(E) \oplus D_1^p\Omega^{0,1}(E)$, then with respect to the L^2-inner product induced on $D_1^p\Omega^1(E)$*

$$\left\langle u, \mathcal{D}_{A,\Phi} s \right\rangle = \left\langle i\Lambda \mathcal{D}^*_{A,\Phi} u, s \right\rangle.$$

Proof. This follows directly from the Kähler identities (Lemma 6.3.5). $\square$

The connection on the Dirac–Higgs bundle is defined by a projector for $\ker \mathcal{D}^*_{A,\Phi}$. The projection operator can be defined using a Green's operator for the Laplacian $\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}$.

Lemma 6.6.3. *The projection* $P_{A,\Phi} : D_1^p\Omega^{1,0}(E) \oplus D_1^p\Omega^{0,1}(E) \to \ker \mathcal{D}^*_{A,\Phi}$ *is given by*

$$P_{A,\Phi} = \mathrm{Id} - \mathcal{D}_{A,\Phi} G_{A,\Phi} \mathcal{D}^*_{A,\Phi}$$

where $G_{A,\Phi}$ *is the Green's operator for the Dirac–Higgs Laplacian*

$$\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi} : D_2^p\Omega^0(E)^{\oplus 2} \to D_0^p\Omega^{1,1}(E)^{\oplus 2}.$$

When $(E, D''_{A,\Phi})$ *is stable*

$$G_{A,\Phi} = \begin{pmatrix} (D''_{A,\Phi}D'_{A,\Phi})^{-1} & 0 \\ 0 & (D''_{A,\Phi}D'_{A,\Phi})^{-1} \end{pmatrix}.$$

Proof. If $G_{A,\Phi}$ is the Green's operator for $\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}$, i.e the right inverse, then clearly $\mathrm{Id} - \mathcal{D}_{A,\Phi}G_{A,\Phi}\mathcal{D}^*_{A,\Phi}$ projects to $\ker \mathcal{D}^*_{A,\Phi}$ and $\mathcal{D}_{A,\Phi}G_{A,\Phi}\mathcal{D}^*_{A,\Phi}$ to the orthogonal complement. Let $s = s_1 + s_2 \in D_2^p\Omega^0(E)^{\oplus 2}$, then

$$\mathcal{D}^*_{A,\Phi}\mathcal{D}_{A,\Phi}s = \begin{pmatrix} D''_{A,\Phi}D'_{A,\Phi}s_1 \\ -D'_{A,\Phi}D''_{A,\Phi}s_2 \end{pmatrix} = D''_{A,\Phi}D'_{A,\Phi}s$$

where the last equality is the Higgs bundle equations. As $(E, D''_{A,\Phi})$ is stable, Lemma 6.3.9 shows that $D''_{A,\Phi}D'_{A,\Phi}$ are isomorphisms. □

6.7 PARABOLIC HIGGS BUNDLES ON $\mathbb{P}^1$ WITH 4 PARABOLIC POINTS

In this section we discuss the Dirac–Higgs bundle from Theorem 6.6.1 in a special case where it is a line bundle. We consider parabolic Higgs bundles on $\mathbb{P}^1$ with four parabolic points. Let $D = z_1 + z_2 + z_3 + z_4$ be a collection of four distinct points on $\mathbb{P}^1$, and denote by $\mathbb{P}^1_4$ the curve with a fixed choice of D. The bundles under consideration must be of rank two and of parabolic degree zero. The parabolic structure will be a full flag at each point of D with weights distributed according to the following table, with $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Points	z_1	z_2	z_3	z_4
Weights	0	0	0	α_4
	α_1	α_2	α_3	α_5

Notice that such a choice of weights is *generic* in the sense that all semi-stable parabolic Higgs bundles are stable.

In this case, the holomorphic description of a parabolic Higgs bundle is $E \xrightarrow{\Phi} E(2)$ with Φ nilpotent at each point of D.

In this section we only fix the type of the quasi-parabolic structure but let the actual flag at each parabolic point be part of the moduli problem. This is Yokogawa's approach [78]. Yokogawa's description is better suited for the explicit calculations in this section.

Lemma 6.7.1. *Let $\mathbb{E} = (E, \Phi)$ be a stable parabolic Higgs bundle on $\mathbb{P}^1_4$ with the above specified weights. Then $E \simeq \mathcal{O} \oplus \mathcal{O}(-1)$.*

Proof. As $\mathrm{pardeg}(\mathbb{E}) = 0$ and $\sum_i \alpha_i = 1$ the topological degree of E is -1, giving that $E \simeq \mathcal{O}(a) \oplus \mathcal{O}(-a-1)$.

If $a \geq 0$, then by stability, the part of Φ mapping $\mathcal{O}(a) \to \mathcal{O}(-a-1)$ cannot be non-zero. Therefore $a \leq \frac{1}{2}$, that is $a = 0$.

If $a < 0$, then similarly, the part of Φ mapping $\mathcal{O}(-a-1) \to \mathcal{O}(a+2)$ must be non-zero, giving that $a = -1$. Either way, $E \simeq \mathcal{O} \oplus \mathcal{O}(-1)$. □

Denote by $\mathscr{M}$ the moduli space of stable rank two parabolic Higgs bundles of parabolic degree zero on $\mathbb{P}^1_4$ with the weights specified as above.

Proposition 6.7.2. *The moduli space $\mathscr{M}$ is non-empty and is a complex elliptic fibration over $\mathbb{C}$ with one singular fibre of type $\tilde{D}_4$.*

Proof. Define a parabolic structure on $\mathcal{O} \oplus \mathcal{O}(-1)$ by having the subspace aligning with $\mathcal{O}(-1)$ at every z_i. If $\Phi = \begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$, then the only Φ-invariant subbundle is $\mathcal{O}(-1)$ which has parabolic degree $-\alpha_4 < 0$, proving that $\mathscr{M}$ is non-empty.

The other properties follow from more general theory. By [50], the moduli space has complex dimension two. In this case, the Hitchin fibration is $\mathscr{M} \xrightarrow{\chi} \mathbb{C}$ where the base is $H^0(K^2(D)) = \mathbb{C}$ as we require Φ to be nilpotent with respect to the flag at each point of D. The general fibre of the fibration is the Jacobian of a double cover of $\mathbb{P}^1_4$ branched at D, i.e. an elliptic curve. The only singular fibre is the nilpotent cone $\chi^{-1}(0)$. From Hausel's description of the nilpotent cone [32, Theorem 4.4.5] and Kodaira's classification of singular fibres of elliptic fibrations [5, p. 201], the nilpotent cone is a union of five spheres sitting in a $\tilde{D}_4$-configuration, see Figure 6.1. □

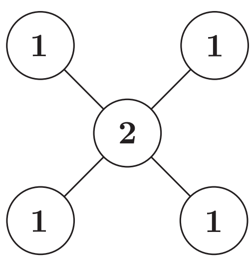

Figure 6.1: The $\tilde{D}_4$ Dynkin diagram represents a configuration of five spheres with edges indicating a single point of transverse intersection and the number is the algebraic multiplicity of the sphere.

Remark 6.7.3. The moduli space $\mathscr{M}$ is similar to Hausel's Toy Model [32, p. 176f], except that stable parabolic bundles with vanishing Higgs field constitute a component of the nilpotent cone in Hausel's Toy Model. In the present case, $\mathcal{O} \oplus \mathcal{O}(-1)$ is not stable as a parabolic Higgs bundle.

Definition 6.7.4. Denote by $(\mathbb{L}, \nabla) \to \mathscr{M}$ the Dirac–Higgs bundle for stable parabolic Higgs bundles of parabolic degree zero and rank two on $\mathbb{P}^1_4$ with parabolic weights as specified in the beginning of this section.

6.7.1 The nilpotent cone

In this section we give an explicit description of each component of the nilpotent cone.

The flag at a parabolic point is a choice of a line in $\mathbb{C}^2$ and thus $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ is the configuration space for the flags, where first factor corresponds to the line at z_1 and so on. The line at a point z corresponding to $\beta \in \mathbb{P}^1$ is $\operatorname{span}\{\beta e_1 + e_2\}$ where $\operatorname{span}\{e_1\} = \mathcal{O}_z$ and $\operatorname{span}\{e_2\} = \mathcal{O}(-1)_z$, so $\beta = 0$ is $\mathcal{O}(-1)_z$ and $\beta = \infty$ is $\mathcal{O}_z$.

The automorphism group of $\mathcal{O} \oplus \mathcal{O}(-1)$

$$\operatorname{Aut}(\mathcal{O} \oplus \mathcal{O}(-1)) = \left\{ \begin{pmatrix} \varepsilon & \varepsilon\gamma \\ 0 & \varepsilon\delta \end{pmatrix} : \varepsilon, \delta \in \mathbb{C}^* \text{ and } \gamma \in H^0(\mathcal{O}(1)) \right\}$$

acts on the configuration space by $\beta_i \mapsto \frac{\beta_i + \gamma(z_i)}{\delta}$, where $\beta_i \in \mathbb{P}^1$ is the line at z_i. Thus the parameter ε acts trivially and it is really the quotient $\operatorname{Aut}(\mathcal{O} \oplus \mathcal{O}(-1))/\mathbb{C}^*$ which acts on the configuration space. In terms of $(\delta, \gamma) \in \mathbb{C}^* \times H^0(\mathcal{O}(1))$ the action is

$$(\delta, \gamma) \cdot (\beta_1, \beta_2, \beta_3, \beta_4) = \left(\frac{\beta_1 + \gamma(z_1)}{\delta}, \frac{\beta_2 + \gamma(z_2)}{\delta}, \frac{\beta_3 + \gamma(z_3)}{\delta}, \frac{\beta_4 + \gamma(z_4)}{\delta} \right).$$

Notice that any parameter $\beta_i = \infty$ is preserved by the action.

Lemma 6.7.5. *If $(\beta_1, \beta_2, \beta_3, \beta_4)$ is a configuration of flags at the parabolic points and there is a stabilising Higgs field, then at most one of the β_i's is ∞.*

Proof. Assume a Higgs field $\Phi = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}$ is written with respect to the decomposition $\mathcal{O} \oplus \mathcal{O}(-1)$. If $\beta_i = \infty$ for some i, then a and c must vanish at z_i as Φ is strictly parabolic. Due to stability, c must be a non-zero section of $\mathcal{O}(1)$. Therefore stability is violated if there is more than one i with $\beta_i = \infty$. □

One of the benefits of working with strictly parabolic Higgs fields is that we have the usual $\mathbb{C}^*$-action on $\mathscr{M}$. From [70, Theorem 8] we know that the Higgs bundles fixed by $U(1) \subset \mathbb{C}^*$ split as a direct sum $\mathbb{E}_1 \oplus \mathbb{E}_2$ of parabolic bundles. In terms of configurations, the flags must align with either $\mathcal{O}$ or $\mathcal{O}(-1)$ at each parabolic point. From Lemma 6.7.5 we have at most one flag aligning with $\mathcal{O}$.

Proposition 6.7.6. *The fixed point set of the* $\mathrm{U}(1)$*-action on* $\mathscr{M}$ *has five irreducible components, which are*

- $\mathbb{P}^1$ *parametrised by Higgs field* $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$, $\varphi \in H^0(\mathcal{O}(1))$ *(up to scale) and flag configuration* $(0,0,0,0)$.
- 4 *isolated points given by Higgs fields* $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$ *(up to scale) with* φ *vanishing at a* z_i *where the corresponding* $\beta_i = \infty$*, and* $\beta_j = 0$ *for* $j \neq i$.

The Morse indices are 0 *and* 2*, respectively.*

Proof. This follows directly from considering [70, Theorem 8] in this specific case. The Morse indices are from [29, Proposition 3.11] specified to this case. $\square$

Corollary 6.7.7. *The Poincaré polynomial of* $\mathscr{M}$ *is* $1 + 5t^2$.

From [32, Theorem 4.4.5] the nilpotent cone consists of five $\mathbb{P}^1$'s in a $\tilde{D}_4$-configuration with the central one consisting of fixed points of the $\mathrm{U}(1)$-action.

Proposition 6.7.8. *The nilpotent cone consists of five* $\mathbb{P}^1$*'s: the central* X_0*, and four non-intersecting* $X_1, \ldots, X_4$ *intersecting* X_0 *once transversally.*

- X_0 *is parametrised by Higgs fields* $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$ *(up to scale),* $\varphi \in H^0(\mathcal{O}(1))$ *and flag configuration* $(0,0,0,0)$.
- X_i*,* $i = 1, \ldots, 4$*, is parametrised by the flag configuration* $\beta_i \in \mathbb{P}^1$ *and* $\beta_j = 0$ *for* $j \neq i$ *and Higgs field* $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$ *(up to scale) with* $\varphi(z_i) = 0$.

Proof. From Proposition 6.7.6 the central component is given. Consider the configurations $(\beta, 0, 0, 0)$ and Higgs field $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$. This parabolic Higgs bundle is certainly stable as the only invariant subbundle is $\mathcal{O}(-1)$ which has negative parabolic degree. This bundle is inequivalent to an element of the central sphere as only the scaling automorphism $\begin{pmatrix} 1 & 0 \\ 0 & \delta \end{pmatrix}$ will keep the three zeros fixed. Likewise the configurations $(\beta, 0, 0, 0)$ and $(0, \beta, 0, 0)$ are inequivalent. Consider the scaling automorphism acting on Higgs fields by conjugation. Then $\delta \in \mathbb{C}^*$ acts on φ as $\delta^{-1}\varphi$. Thus δ acts on φ and β in the same way. By normalising φ using δ, the component X_1 is parametrised by $\beta_1 \in \mathbb{P}^1$. $\square$

Notice how the non-central components are connecting the central sphere to the isolated fixed points of the $\mathrm{U}(1)$-action.

6.7.2 Topology of the hyperholomorphic line bundle

Theorem 6.7.9. *The degrees of the hyperholomorphic line bundle $\mathbb{L} \to \mathscr{M}$ restricted to the components of the nilpotent cone are as given in the following table.*

Component	X_0	X_1	X_2	X_3	X_4
Degree	2	-1	-1	-1	0

When we fix a complex structure on $\mathscr{M}$ it induces a holomorphic structure on $\mathbb{L}$. The holomorphic line bundle is denoted by $\mathcal{L}$. On $\mathscr{M} \times \mathbb{P}^1_4$ let

$$\mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\Theta} S_{\mathscr{M}} \tag{6.7}$$

be a universal parabolic Higgs bundle. Here Θ is a universal parabolic Higgs field and $S_{\mathscr{M}}$ is a universal bundle of strictly parabolic homomorphisms from $\mathcal{O}$ to the parabolic vector bundle in question, i.e for all $(E, \Phi) \in \mathscr{M}$

$$S_{\mathscr{M}}|_{(E,\Phi)} = \mathrm{SParHom}(\mathcal{O}, E) \otimes \mathcal{O}(2).$$

The holomorphic line bundle $\mathcal{L}$ can be obtained by the direct image of (6.7) along the projection to $\mathscr{M}$.

Notice that we have here included $K(D)$ in the definition of $S_{\mathscr{M}}$ and that S are holomorphic sections. This is different to Section 6.3, but we've done this to ease notation. Notice furthermore that in Yokogawa's construction of $\mathscr{M}$, the universal bundle $\mathcal{O} \oplus \mathcal{O}(-1)|_{\mathscr{M}\times D}$ comes equipped with a subsheaf defining the quasi-parabolic structure.

The proof of Theorem 6.7.9 amounts to identifying $S_{\mathscr{M}}$ in all of the five cases above. Before we do that we need the following lemma.

Lemma 6.7.10. $R^1\pi_*(S_{\mathscr{M}}) = 0$*, where $\pi : \mathscr{M} \times \mathbb{P}^1_4 \to \mathscr{M}$ is the projection.*

Proof. It is enough to prove that for any $(E, \Phi) \in \mathscr{M}$, $S_{\mathscr{M}}|_{\{(E,\Phi)\}\times\mathbb{P}^1_4}$ is either $\mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Let (E, Φ) be a fixed parabolic Higgs bundle, and denote by S the restriction of $S_{\mathscr{M}}$. By definition of $S = \mathrm{SParHom}(\mathcal{O}, E(2))$, there is an exact sequence

$$0 \to \mathrm{SParHom}(\mathcal{O}, E(2)) \to \mathrm{ParHom}(\mathcal{O}, E(2)) \to \mathcal{Q} \to 0$$

where $\mathcal{Q}$ is a sky-scraper sheaf supported on z_1, z_2, z_3 and the stalk is of length one at each point. Consequently, S has degree zero. From the action of the automorphism group on the configuration space of flags we can assume that the flags at two of the points z_1, z_2, z_3 align with $\mathcal{O}(-1)$, i.e. that the corresponding β_i is 0. Assume this is z_2, z_3. Let (f, g) be a homomorphism $\mathcal{O} \to \mathcal{O}(2) \oplus \mathcal{O}(1)$; this is strictly parabolic if f vanish at z_2, z_3 and $f(z_1)/g(z_1) = \beta_1$. There is no condition at z_4 as there is no zero-weight. If f vanish completely g is unconstrained, so $S \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$, while if f is non-zero we consider it a section of $\mathcal{O}$ and the constraint on g makes it a section of $\mathcal{O}$, and $S \simeq \mathcal{O} \oplus \mathcal{O}$. $\square$

Proof of Theorem 6.7.9. Consider the restriction of $\mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\Theta} S_{\mathscr{M}}$ to $X_i \times \mathbb{P}^1_4$, and denote by $\pi : X_i \times \mathbb{P}^1_4 \to X_i$ the projection. Then

$$\mathcal{L}|_{X_i} \simeq \mathbb{R}^1\pi_*(\mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\Theta} S_{X_i}).$$

By the five term exact sequence for hypercohomology (1.6), the line bundle $\mathcal{L}|_{X_i}$ sits in an exact sequence

$$0 \to \pi_*(\mathcal{O} \oplus \mathcal{O}(-1)) \to \pi_* S_{X_i} \to \mathcal{L}|_{X_i} \to 0$$

where surjectivity follows as $H^1(\mathbb{P}^1_4, \mathcal{O} \oplus \mathcal{O}(-1)) = 0$. We calculate the Chern character of $\mathcal{L}|_{X_i}$ by the Grothendieck–Riemann–Roch formula

$$\mathrm{ch}(\mathcal{L}_{X_i}) = \mathrm{ch}(\pi_* S_{X_i}) - \mathrm{ch}(\pi_*(\mathcal{O} \oplus \mathcal{O}(-1))) = \pi_*(\mathrm{ch}(S_{X_i})\,\mathrm{Td}(\mathbb{P}^1_4)) - 1$$

where we used Lemma 6.7.10 and that $\pi_*(\mathcal{O} \oplus \mathcal{O}(-1))$ is the trivial line bundle.

The Chern character of S_{X_i} varies from case to case, and the rest of the proof is a study of each individual case.

Consider first the central component X_0, which by Proposition 6.7.8 is parametrised by the Higgs field $\begin{pmatrix} 0 & 0 \\ \varphi & 0 \end{pmatrix}$, $\varphi \in H^0(\mathcal{O}(1))$ (up to scale) and fixed parabolic structure $(0,0,0,0)$. Using the same arguments as in Lemma 6.7.10, S_{X_0} restricts to $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on every slice of $X_0 \times \mathbb{P}^1_4$. As the Higgs field is the coordinate on X_0, we have on $X_0 \times \mathbb{P}^1_4$ that $S_{X_0} \simeq (\mathcal{O}(-1) \oplus \mathcal{O}(1)) \otimes \mathcal{O}_{X_0}(1)$. Therefore $\mathrm{ch}(S_{X_0}) = 2 + 2\hat{h}$ where $\hat{h}$ is a generator of $H^2(X_0, \mathbb{Z})$. Combining this with the above, we see that $\mathrm{ch}(\mathcal{L}|_{X_0}) = 1 + 2\hat{h}$

On the non-central components X_i, $i \neq 0$, the gauge-equivalence class of the Higgs field is fixed. Using similar arguments as in Lemma 6.7.10, we have a short exact sequence of sheaves on $X_i \times \mathbb{P}^1_4$

$$0 \to S_{X_i} \to \mathcal{O}(2) \oplus \mathcal{O}(1) \to \iota_* \mathcal{Q}_{X_i} \to 0 \tag{6.8}$$

where $\iota_* \mathcal{Q}_{X_i}$ is a sheaf supported on $X_i \times \{z_1, z_2, z_3\}$ with ι being the inclusion. The quotient $\mathcal{Q}_{X_i}$ is a line bundle on each $X_i \times \{z_j\}$ whose degree depends on the parabolic structure at z_j. Let h denote a generator of $H^2(\mathbb{P}^1_4, \mathbb{Z})$. It follows from the short exact sequence (6.8) that $c_1(\iota_* \mathcal{Q}_{X_i}) = 3h$. Furthermore, $c_2(\iota_* \mathcal{Q}_{X_i}) = -\iota_* c_1(\mathcal{Q}_{X_i})$ by the action of push forward on Chern classes. The Chern character of S_{X_i} is therefore

$$\mathrm{ch}(S_{X_i}) = 2 + 3h - \mathrm{ch}(\iota_* \mathcal{Q}_{X_i}) = 2 - \iota_* c_1(\mathcal{Q}_{X_i}).$$

If $i \in \{1,2,3\}$, then $\mathcal{Q}_{X_i}$ on $X_i \times \{z_i\}$ is the quotient

$$0 \to \mathcal{O}_{X_i}(-1) \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{Q}_{X_i} \to 0$$

as the parabolic structure at z_i is defined by the coordinate on X_i. On the other two connected components, the parabolic structure does not change and the sequence defining $\mathcal{Q}_{X_i}$ is

$$0 \to \mathcal{O} \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{Q}_{X_i} \to 0.$$

Therefore $\mathrm{ch}(S_{X_i}) = 2 - h\hat{h}$ where $\hat{h} \in H^2(X_i, \mathbb{Z})$ is a generator, and

$$\mathrm{ch}(\mathcal{L}|_{X_i}) = \pi_*((2 - h\hat{h})(1 + h)) - 1 = 1 - \hat{h}.$$

If $i = 4$, then $\mathcal{Q}_{X_i}$ is the trivial line bundle on $X_4 \times \{z_1, z_2, z_3\}$ as the parabolic structure at z_1, z_2, z_3 is independent of the point in X_i. Therefore, $c_1(\mathcal{Q}_{X_i}) = 0$, $\mathrm{ch}(S_{X_i}) = 2$, and

$$\mathrm{ch}(\mathcal{L}|_{X_4}) = \pi_*(2(1 + h)) - 1 = 1.$$

This concludes the proof. □

Corollary 6.7.11. *The hyperholomorphic line bundle $\mathbb{L}$ is not isomorphic to the hyperholomorphic line bundle in [43].*

Proof. The hyperholomorphic line bundle constructed in [43] has curvature $\omega_1 + dd_1^c\mu$ where μ is the moment map for the U(1)-action on $\mathscr{M}$ and ω_1 is the Kähler form for the metric on $\mathscr{M}$ with complex structure I, see [43, Section 2.1]. As the irreducible components of the fibres of the Hitchin fibration are complex submanifolds of $\mathscr{M}$ with respect to complex structure I, ω_1 restricts to a Kähler form on each irreducible component of the nilpotent cone. As the components are curves, evaluating ω_1 gives the volume of the component with respect to ω_1 which is always non-zero. It follows from Theorem 6.7.9 that the degree of $\mathbb{L}$ is 0 on X_4, showing that the hyperholomorphic line bundle from [43] is topologically different to $\mathbb{L}$. □

Remark 6.7.12. Comparing the curvatures in the toy model of Section 2.3 and the $\mathbb{C}^2$-model of [43, Example 2, Section 2]

$$d\bar{u} \wedge dv - du \wedge d\bar{v} \qquad \text{and} \qquad du \wedge d\bar{u} - dv \wedge d\bar{v},$$

the two hyperholomorphic line bundles were most likely different.

Corollary 6.7.13. *The degree of $\mathbb{L}$ is 1 on a generic fibre of the Hitchin fibration.*

Proof. This follows immediately as a generic fibre is homologous to $2X_0+X_1+X_2+X_3+X_4$, see Figure 6.1. □

6.7.3 Analytical properties

Not much is known about the hyperkähler metric on $\mathscr{M}$. All we essentially know is that it is hyperkähler and complete. There are however several indications that the metric should be of type ALG, i.e. that near infinity $\mathscr{M}$ is diffeomorphic, up to a finite covering, to the total space of a 2-torus fibration over $\mathbb{R}^2$ minus a ball, and has a metric that is asymptotically adapted to the fibration in the sense that it is asymptotically flat.

Cherkis and Kapustin [17] were the first suggest the metric to be of type ALG. They used a Nahm transform between periodic monopoles with singularities and singular Higgs bundles on a cylinder. The moduli space of periodic monopoles is hyperkähler of type ALG and as Nahm transforms are usually isometries it is believed that also the moduli space of parabolic Higgs bundles on $\mathbb{P}^1$ is of type ALG.

Recently, Hein [36] has constructed complete hyperkähler ALG-metrics on the complex manifold underlying $\mathscr{M}$ with complex structure I. He did this by compactifying it to an elliptic fibration over $\mathbb{P}^1$ by adding a singular fibre, solving a complex Monge–Ampère equation, and removing the added singular fibre again. It is not known whether all hyperkähler metrics are obtained this way.

The asymptotics of the hyperkähler metric on Higgs bundle moduli spaces are a part of a large conjectural framework set up by Gaiotto, Moore, and Nietzke [26, 27], and indeed they conjecture that in general the asymptotics should be semi-flat with respect to the Hitchin fibration.

It is believed that part of the asymptotic properties of the hyperkähler metrics on Higgs bundle moduli spaces can be extracted from the recent work of Mazzeo, Swoboda, Weiss, and Witt [53, 54]. There are good reasons to believe that this also extend to parabolic Higgs bundles. If so, this would be further support for the conjecture by Cherkis and Kapustin.

There is in other words, a good amount of evidence leaning in favour of the conjecture. If the metric indeed is of type ALG we can use the topological information from Theorem 6.7.9 to draw analytical conclusions about the connection on the hyperholomorphic line bundle $\mathbb{L} \to \mathscr{M}$.

Theorem 6.7.14. *If the hyperkähler metric on $\mathscr{M}$ is of type* ALG*, then the instanton connection of the hyperholomorphic line bundle* $(\mathbb{L}, \nabla)$*, does not have finite energy.*

Proof. As $\mathbb{L}$ is a line bundle, the curvature two-form is closed. As being hyperholomorphic in complex two dimensions is the same as having anti-self-dual curvature two-form, the curvature two-form is in fact harmonic.

It follows from [34, Corollary 10] that the L^2-harmonic two-forms is the image of the compactly supported degree two-cohomology under the natural inclusion map into the normal degree two-cohomology

$$L^2\mathcal{H}^2(\mathscr{M}, g) \simeq \mathrm{im}(H^2_c(\mathscr{M}) \xrightarrow{j} H^2(\mathscr{M})).$$

If $\beta \in L^2\mathcal{H}^2(\mathscr{M}, g)$, then $\beta = j(\gamma)$ for some $\gamma \in H^2_c(\mathscr{M})$ where $\gamma = \sum_{i=0}^4 c_i \,\mathrm{PD}[X_i]$ for some $c_i \in \mathbb{Z}$ and $\mathrm{PD}[X_i]$ are the Poincaré duals of the generators of $H_2(\mathscr{M})$. As a generic fibre intersects $[X_i]$ trivially for all i evaluating β on a generic fibre is zero.

As $\mathbb{L}$ is a line bundle, the curvature two-form of ∇ represents up to a constant the first Chern class of $\mathbb{L}$. From Corollary 6.7.13, we know that $\mathbb{L}$ has degree one on a generic fibre of the Hitchin fibration, and the arguments above now show that ∇ does not have finite energy. □

6.7.4 A generic fibre of the Hitchin fibration

In this section we investigate the generic fibre of the Hitchin fibration in detail and compute the Chern character of the hyperholomorphic line bundle restricted to a generic fibre without using Theorem 6.7.9.

Proposition 6.7.15. *Let $c \in \mathbb{C}^*$. Then $\chi^{-1}(c)$ is isomorphic to an abelian variety: the Jacobian of degree one line bundles on C, a double cover of $\mathbb{P}^1_4$ branched at the parabolic points.*

Proof. Let $c \in \mathbb{C}^*$. Define the spectral curve $C \subset \mathcal{O}(2) \to \mathbb{P}^1_4$ as the solutions of the equation

$$\eta^2 + c(z - z_1)(z - z_2)(z - z_3)(z - z_4) = 0$$

where η is the tautological section of $\mathcal{O}(2)$ pulled back to its total space. The curve $p : C \to \mathbb{P}^1_4$ is a smooth double cover of $\mathbb{P}^1_4$ and is clearly branched at the parabolic points z_i. The curve C has genus one by the Riemann–Hurwitz formula.

If L is a degree one line bundle on C, then $p_*L \simeq \mathcal{O} \oplus \mathcal{O}(-1)$ and carries a parabolic structure at z_i given by the kernel of the evaluation map at $p^{-1}(z_i)$. That is, on C there is an exact sequence

$$0 \to N \to p^*p_*L \xrightarrow{ev} L \to O$$

where N is a line bundle given as the kernel of the evaluation map. The parabolic structure of p_*L at z_i is given by N_{z_i} in $p^*p_*L_{z_i}$.

As C is smooth, general theory [7] proves that $\chi^{-1}(c)$ is the Jacobian of degree one line bundles on C. □

Using this description we have the following Chern character for the hyperholomorphic line bundle restricted to a generic fibre of the Hitchin fibration.

Proposition 6.7.16. *Let $c \in \mathbb{C}^*$, then $\operatorname{ch}(\mathbb{L}|_{\chi^{-1}(c)}) = 1 + t$ where t is a generator of $H^2(\chi^{-1}(c), \mathbb{Z})$, with $\chi^{-1}(c) = J(C)$ considered as the Jacobian of degree one line bundles on the elliptic curve C defined by c.*

Proof. Let $c \in \mathbb{C}^*$ be fixed. Denote by $p : C \to \mathbb{P}^1_4$ the spectral curve defined by c, and $J = J^1(C) = \chi^{-1}(c)$ it's degree one Jacobian. We can either describe the holomorphic line bundle $\mathcal{L}|_J$ as the direct image of a complex on $J \times \mathbb{P}^1_4$ or on $J \times C$. As $J \times C$ has a universal bundle, the latter description is more convenient.

As in the proof of Theorem 6.7.9, it is essential to get a description of $\operatorname{SParHom}(\mathcal{O}, p_*L)$ for all degree one line bundles on C. As C is an elliptic curve, all sections of a degree one line bundle vanish at the same point. Using similar arguments as in the proof of Lemma 6.7.10, we see that $\operatorname{SParHom}(\mathcal{O}, p_*L)$ is $\mathcal{O} \oplus \mathcal{O}$ for all L, except if L is the line bundle L_0 whose sections vanish at $p^{-1}(z_4)$, in which case it is $\mathcal{O}(-1) \oplus \mathcal{O}(1)$.

Let $\mathcal{P}$ be the Poincaré line bundle on $J \times C$ normalised such that $\mathcal{P}|_{\{L_0\}\times C} \simeq L_0$. The following argument shows that the direct image of $\mathcal{P}L_0$ on $J \times C$ to $J \times \mathbb{P}^1_4$ is S_J, the universal bundle of strictly parabolic homomorphisms from $\mathcal{O}$ to E, restricted to J.

As LL_0 has degree two, $p_*(LL_0)$ is a rank two bundle of degree zero. As C is an elliptic curve $h^0(p_*(LL_0)) = h^0(LL_0) = 2$, and thus $p_*(LL_0) \simeq \mathcal{O}(a) \oplus \mathcal{O}(-a)$ with $a \in \{0, 1\}$. Furthermore, $h^0(p_*(LL_0) \otimes \mathcal{O}(-1)) = h^0(LL_0 \otimes p^*\mathcal{O}(-1))$ is either 1 or 0 depending on whether $L \simeq L_0$ or not. If $L \simeq L_0$, $a = 1$ and $p_*(L_0^2) \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$, and if $L \not\simeq L_0$, $a = 0$ and $p_*(LL_0) \simeq \mathcal{O} \oplus \mathcal{O}$.

The above shows that the two term complex on $J \times C$

$$\mathcal{P} \xrightarrow{\eta} \mathcal{P}L_0$$

pushed to $J \times \mathbb{P}^1_4$ is a universal parabolic Higgs bundle on $J \times \mathbb{P}^1_4$. Here we abuse notation by identifying $\mathcal{P}L_0$ by its image in $\mathcal{P} \otimes p^*\mathcal{O}(2)$ under a map vanishing on $J \times \{p^{-1}(z_1), p^{-1}(z_2), p^{-1}(z_3)\}$.

As $h^1(C, L) = 0$ when $\deg(L) > 0$, it follows that $R^1\pi_*(\mathcal{P}) = 0$ and $R^1\pi_*(\mathcal{P}L_0) = 0$ where $\pi : J \times \mathbb{C} \to J$ is the canonical projection. Following the procedure of Theorem 6.7.9,

$$\begin{aligned}\operatorname{ch}(\mathbb{L}|_{\chi^{-1}(c)}) &= \operatorname{ch}(\pi_*(\mathcal{P}L_0)) - \operatorname{ch}(\pi_*(\mathcal{P})) = \pi_*(\operatorname{ch}(\mathcal{P})(\operatorname{ch}(L_0) - 1)\operatorname{Td}(S)) \\ &= \pi_*((1 + t + c + s - c_2(\mathcal{P}))(1 + c - 1)) = 1 + t,\end{aligned}$$

where c and t are generators of $H^2(C, \mathbb{Z})$ and $H^2(J, \mathbb{Z})$, respectively, and s is the $(1,1)$-part of $c_1(\mathcal{P})$ in the Künneth decomposition of $H^2(J \times C, \mathbb{Z})$ and therefore $cs = 0$. □

6.8 Limiting configurations for Higgs bundles

In this section we shift focus to recent developments linking the asymptotics of the L^2-metric on the moduli space of *ordinary* Higgs bundles with certain parabolic Higgs bundles known as *limiting configurations.* If the rank and degree are coprime the moduli space of stable Higgs bundle is smooth and the L^2-metric described in Section 1.1.3 is complete. As mentioned in Section 6.7.3, Gaiotto, Moore, and Nietzke [26, 27] conjectured that the metric is asymptotically semi-flat with respect to the Hitchin fibration. Recently, Mazzeo, Swoboda, Weiss, and Witt [53] have suggested using limiting configurations to verify the semi-flatness conjecture for rank two. In this section we give a construction of limiting configurations suggested by Hitchin and discuss the local shape of solutions to the Dirac–Higgs equations for a limiting configuration. When considering the local L^2-solutions, we recover the requirements of Lemma 6.2.7 on $p > 1$ for the general parabolic theory to work using the Sobolev space L^p_1. This provides a nice reality check for our general theory.

Before we give the definition of a limiting configuration, we need the following local singular solution to the Higgs bundle equations. Let U be an open disk (or $\mathbb{C}$) centered at 0 and denote by $U^\times = U \setminus \{0\}$. Let (E, h) be a smooth Hermitian vector bundle of rank two on U. Choose a unitary frame trivialising E on $U^\times$. In this frame define

$$A^{\mathrm{fid}}_\infty = \frac{1}{8}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\left(\frac{dz}{z} - \frac{d\bar{z}}{\bar{z}}\right) \quad \text{and} \quad \Phi^{\mathrm{fid}}_\infty = \begin{pmatrix}0 & r^{1/2}\\ r^{1/2}e^{i\theta} & 0\end{pmatrix} dz \tag{6.9}$$

where $\Phi^{\mathrm{fid}}_\infty$ is specified in polar coordinates. Notice that the connection A^{fid}_∞ is singular at 0, $\Phi^{\mathrm{fid}}_\infty$ is continuous at 0 and otherwise smooth, $\Phi^{\mathrm{fid}}_\infty$ is normal, and $\det \Phi^{\mathrm{fid}}_\infty = -z dz^2$ has a simple zero. The pair $(A^{\mathrm{fid}}_\infty, \Phi^{\mathrm{fid}}_\infty)$ is called the *singular fiducial Higgs pair.* It is not difficult to see that $(A^{\mathrm{fid}}_\infty, \Phi^{\mathrm{fid}}_\infty)$ satisfies the Higgs bundle equations on $U^\times$.

Definition 6.8.1. Let (E, h) be a rank two Hermitian vector bundle on a Riemann surface Σ of genus $g \geq 2$ and $D = P_1 + \cdots + P_{4g-4}$. A *limiting configuration* (A_∞, Φ_∞) is a Higgs pair on $\Sigma^\times = \Sigma \setminus D$ satisfying the decoupled Higgs bundle equations

$$F(A_\infty) = 0 \quad [\Phi_\infty, \Phi_\infty^*] = 0 \quad \bar{\partial}_{A_\infty}\Phi_\infty = 0$$

and which agrees with $(A^{\mathrm{fid}}_\infty, \Phi^{\mathrm{fid}}_\infty)$ near each point of D with respect to some unitary frame for (E, h).

Let $q \in H^0(K^2)$ have simple zeros and $\pi : C \to \Sigma$ be the $2:1$-covering of Σ branched at the $4g-4$ zeros of q defined by the square root of q (i.e. C is the curve in the total space of the canonical bundle defined by the equation $\eta^2 = q$ where η is the tautological section). Finally, let $\sigma : C \to C$ be the involution permuting the sheets.

Proposition 6.8.2. *Let L be a line bundle on C with the property that $\sigma^*L \otimes L \simeq \pi^*K$, then there is a Hermitian metric on the holomorphic vector bundle*

$$E = \pi_*(L \oplus \sigma^*L)^\sigma$$

such that the Chern connection and the Higgs field

$$\pi_* \begin{pmatrix} \sqrt{q} & 0 \\ 0 & -\sqrt{q} \end{pmatrix}$$

is a limiting configuration on Σ with singularities at the zeros of q.

Proof. Let (N, h) be a *flat* line bundle on C which when considered as a holomorphic line bundle is in the Prym-variety of C, i.e. $\sigma^*N \simeq N^*$. Let $V_C = N \oplus \sigma^*N$ be a rank two holomorphic bundle on C. Define a Hermitian metric $H_C = h \oplus \sigma^*h$ on V_C and Higgs field $\Phi_C = \operatorname{diag}(\sqrt{q}, -\sqrt{q})$. Then (V_C, Φ_C, H_C) is a Higgs bundle with a Hermitian metric with Φ_C normal on C and H_C flat. As (V_C, Φ_C, H_C) is σ-invariant it descends to an orbifold Hermitian Higgs bundle in the sense of Nasatyr and Steer [61]. To such an orbifold bundle exists a parabolic bundle with rational weights. The general construction is a bit involved, but in this case, we can obtain the parabolic bundle directly by considering V_C as a locally free sheaf and only push down the σ-invariant sections, that is

$$V = \pi_*(N \oplus \sigma^*N)^\sigma \quad \text{and} \quad \Phi = \pi_*\Phi_C = \pi_*(\operatorname{diag}(\sqrt{q}, -\sqrt{q})).$$

Then V is a holomorphic vector bundle of rank two with $\det V \simeq K^{-1}$. The parabolic structure at the branch points p_i is given as in the proof of Proposition 6.7.15 by the kernel of the evaluation map. The weight of this subspace is $\frac{1}{2}$ while the whole fibre has weight 0.

In a holomorphic frame adapted to the parabolic structure, the parabolic metric and Higgs field are

$$H \sim \begin{pmatrix} 1 & 0 \\ 0 & r \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} dz.$$

As $N \oplus \sigma^*N$ is flat, the Chern connection on V is flat away from the branch points. However, with respect to this metric Φ is not normal. Furthermore, notice that the behaviour of the metric in a limiting configuration at a zero of $\det \Phi$ is $\operatorname{diag}(r^{-1/2}, r^{1/2})$.

This can be rectified by a simple twist of N and its Hermitian metric. Let $K^{1/2}$ be a square root of K and consider the line bundle $N \otimes \pi^*K^{1/2}$, then

$$\pi_*(N \otimes \pi^*K^{1/2} \oplus \sigma^*(N \otimes \pi^*K^{1/2}))^\sigma = \pi_*((N \oplus N^*) \otimes \pi^*K^{1/2})^\sigma = V \otimes K^{1/2}.$$

Firstly, $\det(V \otimes K^{1/2}) \simeq \mathcal{O}$. Secondly, to get a Hermitian metric on $V \otimes K^{1/2}$ we multiply the parabolic metric H on V by a section of $K^{-1/2} \otimes \bar{K}^{-1/2}$. As q is a quadratic differential

$(q\bar{q})^{-1/4}$ is exactly such a section, and as q has only simple zeros the behaviour at the branch points of $H(q\bar{q})^{-1/4}$ is

$$H(q\bar{q})^{-1/4} \sim \begin{pmatrix} r^{-1/2} & 0 \\ 0 & r^{1/2} \end{pmatrix}.$$

Notice that with respect to this new metric Φ is indeed normal, and as q is a holomorphic section of K^2, the Hermitian metric $(q\bar{q})^{-1/4}$ on $K^{1/2}$ is flat away from the zeros of q. Hence the metric $H(q\bar{q})^{-1/4}$ on $V \otimes K^{1/2}$ over $\Sigma^\times$ is flat and thus the Higgs bundle equations are satisfied. □

Let (E, Φ) be a $\mathrm{SL}(2,\mathbb{C})$-Higgs bundle, i.e. $\det E \simeq \mathcal{O}$ and $\mathrm{Tr}\,\Phi = 0$. Assume furthermore that $q = -\det \Phi \in H^0(K^2)$ has simple zeros $D = P_1 + \cdots + P_{4g-4}$. The spectral curve C is smooth, and thus there is a line bundle N in the Prym-variety of C such that $E = \pi_* N$. A limiting configuration associated to a Higgs bundle (E, Φ), is a limiting configuration which is complex gauge equivalent to (E, Φ) on $\Sigma^\times$. Using the line bundle N and following the proof of of Proposition 6.8.2 we get a limiting configuration (V, Φ') with the required properties. A gauge transform between (E, Φ) and (V, Φ') on $\Sigma^\times$ exists as on C, $\pm\sqrt{q}$ are the eigenvalues of $\pi^*\Phi$ on $C \setminus \pi^* D$ and the flat bundle $N \oplus N^*$ on $C \setminus \pi^* D$ is the eigenspace decomposition of E.

The asymptotics of the singular Hermitian metric of a limiting configuration is $r^{\pm 1/2}$, and thus the weights of the parabolic structure are $\pm\frac{1}{4}$. Notice how tensoring with $K^{1/2}$ and its metric changed the weights from 0 and $\frac{1}{2}$. Previously in this chapter, we required the weights of the parabolic structure to be in $[0,1)$. By doing an elementary modification at each of the zeros of q, the topological degree of the vector bundle underlying the limiting configuration goes down by $\deg D = 4g-4$ while the weight $-\frac{1}{4}$ becomes $\frac{3}{4}$ and $\frac{1}{4}$ is left unchanged. Notice that the parabolic degree is unchanged and is still zero. In the remaning part of this section, we will consider a limiting configuration as a parabolic Higgs bundle of parabolic degree zero with $D = P_1 + \cdots + P_{4g-4}$ as parabolic points and weights $\frac{1}{4}, \frac{3}{4}$.

6.8.1 LOCAL SHAPE OF SOLUTIONS TO THE DIRAC–HIGGS EQUATIONS FOR LIMITING CONFIGURATIONS

In this section we consider the singular fiducial Higgs pair $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$ defined in (6.9) solving the Higgs bundle equations on $\mathbb{C}^\times$, the complex plane with the origin removed. The Dirac–Higgs equations for $(A_\infty^{\mathrm{fid}}, \Phi_\infty^{\mathrm{fid}})$ are

$$0 = \begin{pmatrix} \bar{\partial} - \frac{1}{8}\frac{d\bar{z}}{\bar{z}} & 0 & 0 & r^{1/2} dz \\ 0 & \bar{\partial} + \frac{1}{8}\frac{d\bar{z}}{\bar{z}} & r^{1/2} e^{i\theta} dz & 0 \\ 0 & r^{1/2} e^{-i\theta} d\bar{z} & \partial + \frac{1}{8}\frac{dz}{z} & 0 \\ r^{1/2} d\bar{z} & 0 & 0 & \partial - \frac{1}{8}\frac{dz}{z} \end{pmatrix} \begin{pmatrix} a_{\mathrm{out}} dz \\ a_{\mathrm{inn}} dz \\ b_{\mathrm{inn}} d\bar{z} \\ b_{\mathrm{out}} d\bar{z} \end{pmatrix}.$$

The equations clearly split into two sets of coupled equations which we call the *outer* and *inner* equations. We will focus on each set of equations separately and use same procedure as in Section 2.2 to find all L^2-solutions.

L^2-solutions to outer equations The outer equations in polar coordinates are

$$\begin{aligned} 0 &= \frac{1}{2}e^{i\theta}\partial_r a_{\text{out}} + \frac{i}{2r}e^{i\theta}\partial_\theta a_{\text{out}} - \frac{1}{8r}e^{i\theta}a_{\text{out}} - r^{1/2}b_{\text{out}} \\ 0 &= \frac{1}{2}e^{-i\theta}\partial_r b_{\text{out}} - \frac{i}{2r}e^{-i\theta}\partial_\theta b_{\text{out}} - \frac{1}{8r}e^{-i\theta}b_{\text{out}} - r^{1/2}a_{\text{out}} \end{aligned}$$

Expanding a_{out} and b_{out} in Fourier series $a_{\text{out}} = \sum_{n\in\mathbb{Z}} a_n(r)e^{in\theta}$ and $b_{\text{out}} = \sum_{n\in\mathbb{Z}} b_n(r)e^{in\theta}$, the equations for the Fourier coefficients are

$$0 = 16r^2 a_n''(r) + (3+8n-16n^2-64r^3)a_n(r) \quad \text{and} \quad b_{n+1}(r) = \tfrac{1}{2}r^{-1/2}a_n'(r) - \tfrac{4n+1}{8}r^{-3/2}a_n(r).$$

Assuming that $a_n(r) = r^{1/2}f_n(\frac{4}{3}r^{3/2})$ the equation for f_n is the modified Bessel differential equation (2.4) with $\nu = \frac{1-4n}{6}$ and thus a_n is either $r^{1/2}I_{\pm\frac{1-4n}{6}}(\frac{4}{3}r^{3/2})$ or $r^{1/2}K_{\frac{1-4n}{6}}(\frac{4}{3}r^{3/2})$. From Lemma 2.2.1 we know that only $K_\nu(x)$ decays for large x and so we discard I_ν as a valid solution. Lemma 2.2.1 also shows that the only values for which $r^{1/2}K_{\frac{1-4n}{6}}(\frac{4}{3}r^{3/2})$ is L^2 around zero in $\mathbb{C}$ is for n either 0 or -1. The integrability condition also applies to b_{n+1} which is determined by a_n to be $-r^{1/2}K_{\frac{5+4n}{6}}(\frac{4}{3}r^{3/2})$. This radial function on $\mathbb{C}^\times$ is also only L^2 if n is either 0 or -1. That is, there is a two-dimensional space of L^2-solutions to the outer Dirac–Higgs equations of the singular fiducial Higgs pair. The solutions have the shape

$$\begin{aligned} a_{\text{out}} &= c_0 r^{1/2}K_{\frac{1}{6}}(\tfrac{4}{3}r^{3/2}) + c_{-1}r^{1/2}K_{\frac{5}{6}}(\tfrac{4}{3}r^{3/2})e^{-i\theta} \\ b_{\text{out}} &= -c_{-1}r^{1/2}K_{\frac{1}{6}}(\tfrac{4}{3}r^{3/2}) - c_0 r^{1/2}K_{\frac{5}{6}}(\tfrac{4}{3}r^{3/2})e^{i\theta}. \end{aligned} \tag{6.10}$$

Notice that these solutions are not just local L^2-solutions but actually *global* L^2-solutions on $\mathbb{C}^\times$.

L^2-solutions to inner equations The inner equations in polar coordinates are

$$\begin{aligned} 0 &= \frac{1}{2}e^{i\theta}\partial_r a_{\text{inn}} + \frac{i}{2r}e^{i\theta}\partial_\theta a_{\text{inn}} + \frac{1}{8r}e^{i\theta}a_{\text{inn}} - r^{1/2}e^{i\theta}b_{\text{inn}} \\ 0 &= \frac{1}{2}e^{-i\theta}\partial_r b_{\text{inn}} - \frac{i}{2r}e^{-i\theta}\partial_\theta b_{\text{inn}} + \frac{1}{8r}e^{-i\theta}b_{\text{inn}} - r^{1/2}e^{-i\theta}a_{\text{inn}}. \end{aligned}$$

We proceed as for the outer equations by expanding the equations in terms of Fourier series $a_{\text{inn}} = \sum_{n\in\mathbb{Z}} a_n(r)e^{in\theta}$ and $b_{\text{inn}} = \sum_{n\in\mathbb{Z}} b_n(r)e^{in\theta}$. The equations for the Fourier coefficients are

$$0 = 16r^2 a_n''(r) - (5-24n+16n^2+64r^3)a_n(r) \quad \text{and} \quad b_n(r) = \tfrac{1}{2}r^{-1/2}a_n'(r) - \tfrac{4n-1}{8}r^{-3/2}a_n(r).$$

Again, assuming that $a_n(r) = r^{1/2}f_n(\frac{4}{3}r^{3/2})$ the equation for f_n is the modified Bessel differential equation (2.4) with $\nu = \frac{3-4n}{6}$. By the asymptotics we discard the modified

Bessel functions $I_{\pm\nu}$, and we must have $a_n(r) = r^{1/2}K_{\frac{3-4n}{6}}(\frac{4}{3}\,r^{3/2})$ and thus $b_n(r) = -r^{1/2}K_{\frac{3+4n}{6}}(\frac{4}{3}\,r^{3/2})$. It follows from Lemma 2.2.1 that the only n for which a_n or b_n as functions on $\mathbb{C}^\times$ is L^2, is $n = 0$. By the special identity for $K_{\frac{1}{2}}$, we see that

$$r^{1/2}K_{\frac{1}{2}}(\tfrac{4}{3}\,r^{3/2}) = \sqrt{\frac{3}{2\pi}}\,r^{-1/4}e^{-4/3r^{3/2}}. \tag{6.11}$$

We conclude that there is only a one-dimensional space of L^2-solution to the inner Dirac–Higgs equations of the singular fiducial Higgs pair. The solutions have the shape

$$a_{\text{inn}} = c_0 r^{-1/4}e^{-4/3r^{3/2}} \quad \text{and} \quad b_{\text{inn}} = -c_0 r^{-1/4}e^{-4/3r^{3/2}}.$$

Again we notice that this is a global L^2-solution on $\mathbb{C}^\times$.

Combining the results above we get the following proposition.

Proposition 6.8.3. *The space of global L^2-solutions to the Dirac–Higgs equations for the singular fiducial Higgs pair defined on $\mathbb{C}^\times$ is three-dimensional.*

6.8.2 Comparing L^2 and L^p_1-solutions

In Proposition 6.8.3 we saw the singular fiducial Higgs pair on $\mathbb{C}$ has three linearly independent solutions to the Dirac–Higgs equations. As a parabolic Higgs bundle, a limiting configuration has $4g-4$ parabolic points with parabolic weights $\frac{1}{4}$ and $\frac{3}{4}$ at each point. From the dimension formula Theorem 6.3.3, we see that for a $p > 1$ compatible with the parabolic structure of the limiting configuration, the space of global solutions to the Dirac–Higgs equations has dimension $12g - 12$, i.e. a three-dimensional contribution from each parabolic point.

A $p > 1$ is by Definition 6.2.8 compatible with the parabolic structure if p satisfies the conditions of Lemmas 6.2.6 and 6.2.7. The condition in Lemma 6.2.6 is the original condition found by Biquard [8] and reused by Konno [50] in his construction of the moduli space of parabolic Higgs bundles. This condition requires $1 < p < \frac{4}{3}$. The condition in Lemma 6.2.7 is new, and in this case more restrictive as it requires both $1 < p < \frac{8}{7}$ and $1 < p < \frac{8}{5}$.

From the Sobolev embedding theorem we know that $L^p_1 \subset L^2$ when $1 < p < 2$. As a reality check on our theory, we consider the explicit L^2-solutions to the outer and inner equations, and investigate for which $p > 1$ they are also L^p_1-solutions.

Proposition 6.8.4. *The L^2-solutions to the inner Dirac–Higgs equations for the singular fiducial Higgs pair are in L^p_1 if $1 < p < \frac{8}{5}$ and the L^2-solutions to the outer Dirac–Higgs equations are in L^p_1 if $1 < p < \frac{8}{7}$, matching the conditions from Lemma 6.2.7.*

Proof. From (6.11), we know that the behaviour of the derivative of a L^2-solution to the inner equations on a neighbourhood U of 0 is $-\frac{1}{4}r^{-5/4}$. For this be in L^p_1 as a function on $U^\times$ we must have

$$-\frac{5}{4}p+1>-1 \quad \text{or equivalently} \quad p<\frac{8}{5}.$$

From (6.10), we know that the behaviour around 0 of a solution to the outer equations is determined by the behaviour of $r^{1/2}K_\nu(\frac{4}{3}r^{3/2})$ for ν being $\frac{1}{6}$ or $\frac{5}{6}$. By Lemma 2.2.1 the behaviour around 0 is therefore $c\,r^{\frac{1-3\nu}{2}}$ where the constant c depends on ν. As a radial function on $U^\times$ the derivative $c'\,r^{-\frac{1+3\nu}{2}}$ is in L^p if

$$-\frac{1+3\nu}{2}p+1>-1 \quad \text{or equivalently} \quad p<\frac{4}{3\nu+1}$$

which for $\nu=\frac{1}{6}$ is $p<\frac{8}{3}$ and for $\nu=\frac{5}{6}$ is $p<\frac{8}{7}$. Notice that a solution of the outer equations both have a Bessel function of index $\frac{1}{6}$ and one of index $\frac{5}{6}$ making $p<\frac{8}{3}$ obsolete. □

6.8.3 Limiting configurations as limits of Higgs bundles

The construction of limiting configurations in [53] is rather different to the construction given in Proposition 6.8.2. In [53] they see limiting configurations as limits of solutions to the Higgs bundle equations. Given a Higgs pair (A,Φ) on a Hermitian vector bundle satisfying the Higgs bundle equations, then by the Hitchin–Simpson Theorem there is a pair $(A_t, t\Phi_t)$ in the complex gauge orbit of $(A, t\Phi)$ satisfying the rescaled Higgs bundle equations

$$F(A_t)+t^2[\Phi_t,\Phi_t^*]=0 \quad \text{and} \quad \bar{\partial}_{A_t}\Phi_t=0. \tag{6.12}$$

The limit of (A_t,Φ_t) as $t\to\infty$ is how limiting configurations are perceived in [53]. Locally around the zeros of $\det\Phi$, the pairs (A_t,Φ_t) agree with a so-called fiducial Higgs pair $(A_t^{\mathrm{fid}},\Phi_t^{\mathrm{fid}})$.

The one-parameter family $(A_t^{\mathrm{fid}},\Phi_t^{\mathrm{fid}})$ is defined as

$$A_t^{\mathrm{fid}}=f_t(r)\begin{pmatrix}1&0\\0&-1\end{pmatrix}\left(\frac{dz}{z}-\frac{d\bar{z}}{\bar{z}}\right) \quad \text{and} \quad \Phi_t^{\mathrm{fid}}=\begin{pmatrix}0&r^{1/2}e^{h_t(r)}\\r^{1/2}e^{i\theta}e^{-h_t(r)}&0\end{pmatrix}dz$$

satisfying the rescaled Higgs bundle equations (6.12) which in this case are

$$f_t'(r)=2t^2r^2\sinh(2h_t(r)) \quad \text{and} \quad f_t(r)=\frac{1}{8}+\frac{1}{4}rh_t'(r).$$

These equations can further be reduced and rewritten such that h_t is a solution to the Painlevé III equation with certain decay properties, see [53, Section 3] for further details.

We could play the same game as in Section 6.8.1 and ask for the local solutions to the Higgs bundle equations for $(A_t, t\Phi_t)$. As $\det\Phi_t$ has a simple zero at the origin we expect from Lemma 2.4.1 there to be a one-dimensional space of solutions, and that is indeed exactly what we observe. The Dirac–Higgs equations again split into two sets of equations an outer set and an inner set. It turns out, that the outer set of equations do not have any L^2-solutions whereas the inner set of equations have a one-dimensional space which can be specified directly in terms of the function h_t above.

In this light, it is interesting that the limiting configuration which is supposed to be the limit of stable Higgs pairs has a three-dimensional space of solutions. We can understand this from the sheaf theoretic picture. The limiting Higgs field Φ_∞^{fid} is continuous at zero and so its residue vanish, leaving a two-dimensional cokernel. Furthermore, that the determinant vanish at the parabolic points yields yet another free parameter for the cokernel.

6.9 Nahm transform for parabolic Higgs bundles

In this section we define a Nahm transform for parabolic Higgs bundles of parabolic degree zero on a Riemann surface of genus $g \geq 1$. The main ingredient is the Hodge theory in Section 6.3.

We fix a Riemann surface Σ of genus $g \geq 1$. Let $\mathscr{M}$ denote the moduli space of parabolic Higgs bundles of rank l, parabolic degree zero, and with fixed weights. For each $\mathbb{E} \in \mathscr{M}$ there is a family of parabolic Higgs bundles parametrised by T^*J

$$E \otimes \tilde{\mathcal{P}} \xrightarrow{\Theta} E \otimes K(D) \otimes \tilde{\mathcal{P}} \tag{6.13}$$

on $T^*J \times \Sigma$ where J is the Jacobian of degree zero line bundles on Σ and $\tilde{\mathcal{P}} \to J \times \Sigma$ is a Poincaré line bundle pulled back by a choice of Abel–Jacobi map. Restricted to a slice defined by $(\xi, \alpha) \in T^*J \simeq J \times H^0(\Sigma, K)$ the family is

$$\mathbb{E}_{(\xi,\alpha)} = E \otimes L_\xi \xrightarrow{\Phi+\alpha\,\mathrm{Id}} E \otimes K(D) \otimes L_\xi.$$

Notice that we use an inclusion of the holomorphic sections of $E \otimes K$ into the meromorphic sections with simple poles at D. As α is holomorphic this does not change the residues of $\Phi + \alpha\,\mathrm{Id}$ which are still strictly parabolic.

To define $\tilde{\mathcal{P}}$ we need to chose a base point $z_0 \in \Sigma$ and thereby an Abel–Jacobi map $j : \Sigma \to J$ by mapping z to the divisor class of $z - z_0$.

If $\mathbb{E}$ is a stable parabolic Higgs bundle of degree zero, then so is the restriction of the above family to each slice. This is proved in the same way as [14, Lemma 3.1.7].

For each $\mathbb{E} \in \mathscr{M}$, the above defines a family of stable parabolic Higgs bundles of parabolic degree zero parametrised by the hyperkähler manifold T^*J of degree zero and rank one Higgs bundles. By equipping these rank one bundles with weight zero, we consider them as parabolic rank one Higgs bundles of parabolic degree zero. Since we furthermore have a universal parabolic Higgs bundle on $T^*J \times \Sigma$ we can repeat the construction and proof of Theorem 6.6.1 and to each $\mathbb{E} \in \mathscr{M}$ obtain a *hyperholomorphic bundle* $(\hat{\mathbb{E}}, \hat{\nabla})$ on T^*J of rank

$$2l(g-1) + nl - m_0 \quad \text{with} \quad m_0 = \sum_{P \in \tilde{D}} m_P(w_1)$$

where $\tilde{D} \subset D$ are the points P where $w_1(P) = 0$.

Definition 6.9.1. The pair $(\hat{\mathbb{E}}, \hat{\nabla})$ associated as in Theorem 6.6.1 to a stable parabolic Higgs bundle $\mathbb{E}$ of parabolic degree zero, is called the *Nahm transform* of $\mathbb{E}$.

Remark 6.9.2. If the fixed parabolic structure is good, then the Nahm transform of $\mathbb{E} \in \mathscr{M}$ could be defined as the pullback of the Dirac–Higgs bundle on $\mathscr{M}$ by the orbit map $\mathcal{N}_{\mathbb{E}} : T^*J \to \mathscr{M}$ defined by $(\xi, \alpha) \mapsto (E, D''_{\xi,\alpha})$ with $D''_{\xi,\alpha} = \bar{\partial}_{A,\xi} + \Phi + \alpha \operatorname{Id}$ where $\bar{\partial}_{A,\xi}$ is the holomorphic structure induced by $\bar{\partial}_A$ on $E \otimes L_\xi$.

Proposition 6.9.3. *Let $\hat{\mathbb{E}}$ be the Nahm transform of a parabolic Higgs bundle $\mathbb{E}$, then $\hat{\mathbb{E}}$ extends to a holomorphic bundle $\hat{\mathcal{E}}$ on $J \times \mathbb{P}^g$ when $\hat{\mathbb{E}}$ is considered as a holomorphic bundle on $T^*J \simeq J \times H^0(K)$.*

Proof. It follows from Theorem 6.4.1 that the Nahm transform $\hat{\mathbb{E}}$ of $\mathbb{E}$, when considered as a holomorphic bundle with respect to the complex structure I on $T^*J \simeq J \times H^0(K)$, can be defined as the hyperdirect image of the family

$$E \otimes \tilde{\mathcal{P}} \xrightarrow{\Theta} \mathcal{S} \otimes K(D) \otimes \tilde{\mathcal{P}}$$

on $J \times H^0(K) \times \Sigma$ along the projection to $J \times H^0(K)$. For each $(\xi, \alpha) \in J \times H^0(K)$ the above family is

$$E \otimes L_\xi \xrightarrow{\Phi + \alpha \operatorname{Id}} \mathcal{S} \otimes K(D) \otimes L_\xi.$$

The proof is completely analogous to Bonsdorff's [14, Theorem 3.1.12]. Extend the family of parabolic Higgs bundles to a family

$$E \otimes \tilde{\mathcal{P}} \xrightarrow{\Theta} \mathcal{S} \otimes K(D) \otimes \tilde{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{P}^g}(1) \tag{6.14}$$

on $J \times \mathbb{P}^g \times \Sigma$ with $\mathbb{P}^g = \mathbb{P}(H^0(K) \oplus \mathbb{C}\Phi)$ by defining

$$\Theta = t\Phi + \sum_{i=1}^{g} a_i \alpha_i \tag{6.15}$$

where $\{\alpha_1, \ldots, \alpha_g\}$ is a basis for $H^0(K)$ and $[t : a_1 : \cdots : a_g]$ are homogenous coordinates on $\mathbb{P}^g$. If the hyperdirect image of the two-term complex (6.14) along the projection to $J \times \mathbb{P}^g$ is locally free it will be an extension of $\hat{\mathbb{E}}$. Locally-freeness will follow if the hypercohomology of (6.14) restricted to each $(\xi, [t : \alpha])$-slice is concentrated in degree one.

This is indeed true when $t \neq 0$ as the corresponding parabolic Higgs bundle is stable. If $t = 0$, then the parabolic Higgs bundle is of the form

$$E \otimes L_\xi \xrightarrow{\alpha \, \mathrm{Id}} \mathcal{S} \otimes K(D) \otimes L_\xi.$$

where $\alpha \neq 0$. As $\alpha \, \mathrm{Id}$ is an injective sheaf map and its cokernel is a skyscraper sheaf it follows from the first hypercohomology spectral sequence [30, p. 443] that the only non-zero hypercohomology group is the first, and that it is isomorphic to the zero'th cohomology of the cokernel of $\alpha \, \mathrm{Id}$. $\square$

The holomorphic bundle $\hat{\mathcal{E}} \to J \times \mathbb{P}^g$ is called the *extended Nahm transform.*

Proposition 6.9.4. *The Chern character of the extended Nahm transform $\hat{\mathcal{E}}$ of $\mathbb{E} \in \mathscr{M}$ is*

$$\mathrm{ch}(\hat{\mathcal{E}}) = 2l(g-1) + nl - m_0 + (d + l(g-1) + nl - m_0)\hat{h} - lt\hat{h}$$

where l is the rank of $\mathbb{E}$, d is the topological degree of $\mathbb{E}$, n is the number of parabolic points, m_0 the total multiplicity of zero weights, h is a generator for $H^2(\mathbb{P}^g, \mathbb{Z})$, $t \in H^2(J, \mathbb{Z})$ is the Poincaré dual of the Θ-divisor on J, and $\mathrm{ch}(\mathcal{O}_{\mathbb{P}^g}(1)) = e^h = 1 + \hat{h}$.

Proof. From Proposition 6.9.3, the extended Nahm transform $\hat{\mathcal{E}}$ is

$$\hat{\mathcal{E}} = \mathbb{R}^1\pi_*(E \otimes \tilde{\mathcal{P}} \xrightarrow{\Theta} \mathcal{S} \otimes K(D) \otimes \tilde{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{P}^g}(1))$$

where $\pi : J \times \mathbb{P}^g \times \Sigma \to J \times \mathbb{P}^g$ is the projection. As the zeroth and second hypercohomology vanish, it follows from Grothendieck–Riemann–Roch that

$$\begin{aligned}\mathrm{ch}(\hat{\mathcal{E}}) &= \mathrm{ch}(\pi_!(\mathcal{S} \otimes K(D) \otimes \tilde{\mathcal{P}} \otimes \mathcal{O}_{\mathbb{P}^g}(1)) - \mathrm{ch}(\pi_!(E \otimes \tilde{\mathcal{P}}))) \\ &= \pi_*(\mathrm{Td}\,\Sigma \;\; \mathrm{ch}\,\tilde{\mathcal{P}} \; (\mathrm{ch}\,\mathcal{S} \;\; \mathrm{ch}\,K(D) \;\; \mathrm{ch}\,\mathcal{O}(1) \; - \; \mathrm{ch}\,E))\end{aligned}$$

The Chern character of $\tilde{\mathcal{P}}$ is

$$\mathrm{ch}(\tilde{\mathcal{P}}) = 1 + c - tx$$

where c is the component in $H^1(J, \mathbb{Z}) \otimes H^1(\Sigma, \mathbb{Z})$ under the Künneth decomposition, x is a generator of $H^2(\Sigma, \mathbb{Z})$, and $t \in H^2(J, \mathbb{Z})$ is the Poincaré dual of the Θ-divisor of J, see e.g. [2, Chapter 8.2]. Therefore,

$$x^2 = 0 \quad \text{and} \quad x \cdot c = 0$$

and thus

$$\mathrm{ch}(\tilde{\mathcal{P}})\,\mathrm{Td}(\Sigma) = 1 - (g-1)x + c - tx.$$

The sheaf $\mathcal{S}$ sits in an exact sequence

$$0 \to \mathcal{S} \otimes K(D) \to E \otimes K(D) \to \mathcal{Q} \to 0$$

on $J \times \mathbb{P}^g \times \Sigma$ with $\mathcal{Q}$ is supported on $J \times \mathbb{P}^g \times \{P_{i_1}, \dots, P_{i_m}\}$ where P_{i_k} are the parabolic points with zero weights. Therefore,

$$\begin{aligned}\operatorname{ch}(\mathcal{S} \otimes K(D)) &= \operatorname{ch}(E) \operatorname{ch}(K(D)) - \operatorname{ch}(\mathcal{Q}) \\ &= l + (d + 2l(g-1) + ln - m_0)x.\end{aligned}$$

Combining all of this, and using that π_* is integration along Σ, we get

$$\begin{aligned}\operatorname{ch}(\hat{\mathcal{E}}) =& 2l(g-1) + nl - m_0 \\ &+ (\operatorname{ch}(\mathcal{O}(1)) - 1)(d + l(g-1) + nl - m_0) \\ &+ lt(1 - \operatorname{ch}(\mathcal{O}(1)))\end{aligned}$$

which is the desired formula when using the shorthand notation $\operatorname{ch}(\mathcal{O}(1)) = 1 + \hat{h}$. □

Remark 6.9.5. When there are no parabolic points and the topological degree therefore is zero, the Chern character formula in Proposition 6.9.4 reduces to the Chern character in [14, Proposition 3.1.15].

Theorem 6.9.6. *Let $\mathbb{E}$, $\mathbb{F} \in \mathscr{M}$. If the extended Nahm transforms are isomorphic $\hat{\mathcal{E}} \simeq \hat{\mathcal{F}}$, then $\mathbb{E}$ and $\mathbb{F}$ are isomorphic as parabolic Higgs bundles.*

Proof. The proof is very similar to [14, Theorem 3.2.1]. The theorem will follow if we can recover the parabolic Higgs bundle from $\hat{\mathcal{E}}$. The proof will use two spectral sequence arguments. Given a parabolic Higgs bundle $E \xrightarrow{\Phi} EK(D)$ the first step in defining the holomorphic bundle $\hat{\mathcal{E}}$ on $J \times \mathbb{P}^g$ is to extend $E \xrightarrow{\Phi} \mathcal{S}K(D)$ on Σ to a family

$$E \xrightarrow{\Theta} \mathcal{S}K(D) \otimes \mathcal{O}_{\mathbb{P}^g}(1)$$

on $\Sigma \times \mathbb{P}^g$. As in [14, Lemma 3.2.1.1], the sheaf map Θ is injective. Denote by $\mathcal{Q}$ the cokernel sheaf

$$0 \to E \xrightarrow{\Theta} \mathcal{S}K(D) \to \mathcal{Q} \to 0.$$

As the hypercohomology of a stable parabolic Higgs bundle is concentrated in degree one, it follows from the first hypercohomology spectral sequence that

$$\hat{\mathcal{E}} \simeq \pi_*(q^* j_*(\mathcal{Q}) \otimes \mathcal{P})$$

where the maps are

$$\Sigma \times \mathbb{P}^g \xrightarrow{j} \hat{J} \times \mathbb{P}^g \xleftarrow{q} \hat{J} \times J \times \mathbb{P}^g \xrightarrow{\pi} J \times \mathbb{P}^g$$

with j the Abel–Jacobi map and q, π are the canonical projections.

Notice that there is a spectral sequence with second page

$$E_2^{s,t} = R^s q_*(\pi^*(R^t \pi_*(q^*(j_* \mathcal{Q}) \otimes \mathcal{P})) \otimes \mathcal{P}^{-1})$$

converging to $j_*\mathcal{Q}$ if $s+t=g$ and 0 otherwise [6, Cha. 3, Cor 3.4]. Since $R^t\pi_*(q^*(j_*\mathcal{Q})\otimes\mathcal{P})$ is only non-zero for $t=0$, it follows that

$$R^g q_*(\pi^*(\hat{\mathcal{E}})\otimes\mathcal{P}^{-1})\simeq j_*\mathcal{Q}.$$

From $\hat{\mathcal{E}}$ on $J\times\mathbb{P}^g$ we have obtained a sheaf on $\Sigma\times\mathbb{P}^g$ by pulling back $R^g q_*(\pi^*\hat{\mathcal{E}}\otimes\mathcal{P}^{-1})$ by the Abel–Jacobi map. To recover the vector bundle and Higgs field from $\mathcal{Q}$ we use the relative Beilinson spectral sequence [64, Theorem 4.1.11] on $\mathcal{Q}\otimes\mathcal{O}_{\mathbb{P}^g}(-1)$. The first page has terms

$$E_1^{s,t}=\mathcal{O}_{\mathbb{P}^g}(s)\boxtimes R^t\rho_*(\mathcal{Q}\otimes\Omega^{-s}_{\Sigma\times\mathbb{P}^g/\Sigma}(-s-1))$$

where $\rho:\Sigma\times\mathbb{P}^g\to\Sigma$ is the projection.

Using that $\mathcal{Q}$ is the cokernel, it follows that only $E_1^{-1,0}$ and $E_1^{0,0}$ are non-zero and are equal to $E\otimes\mathcal{O}_{\mathbb{P}^g}(-1)$ and $\mathcal{S}K(D)$, respectively. The convergence of the spectral sequence gives a short exact sequence on $\Sigma\times\mathbb{P}^g$

$$0\to E\otimes\mathcal{O}_{\mathbb{P}^g}(-1)\xrightarrow{\Theta}\mathcal{S}K(D)\to\mathcal{Q}\otimes\mathcal{O}_{\mathbb{P}^g}(-1)\to 0.$$

That the sheaf map is indeed Θ follows from the definition of the spectral sequence. By restricting the sequence to $\Sigma\times[1:0]$ we recover the parabolic Higgs bundle. □

Recall, that a holomorphic vector bundle V on an abelian variety J is *homogeneous* if $\tau_x^*V\simeq V$ for all $x\in J$, where $\tau_x:J\to J$ is translation by $x\in J$.

Proposition 6.9.7. *Let $\mathbb{E}$ be a stable parabolic Higgs bundle of parabolic degree zero. Let $[t:\alpha]\in\mathbb{P}^g$ be such that $\det(t\Phi+\alpha\,\mathrm{Id})$ is a non-trivial section of $K^l((l-1)D)$. Then $\hat{\mathcal{E}}|_{J\times[t:\alpha]}$ is a homogeneous bundle.*

Proof. The proof is exactly the same as the proof of Theorem 4.4.6 adapted to the parabolic setting.

□

When we in the next section consider parabolic Higgs bundles on elliptic curves, we can be more specific about the asymptotic holomorphic structure of a Nahm transformed Higgs bundle.

6.9.1 Doubly-periodic instantons

In this section we will consider the special case of $g=1$ where $T^*J=J\times\mathbb{C}$ is a complex two-dimensional surface and J is the dual to the Riemann surface $\hat{J}=\Sigma$ of genus 1 on which the parabolic Higgs bundles live.

Fix a parabolic structure and let $\mathscr{M}$ be the moduli space of parabolic Higgs bundles of parabolic degree zero. For $\mathbb{E} \in \mathscr{M}$ the Nahm transform $(\hat{\mathbb{E}}, \hat{\nabla})$ is a Hermitian vector bundle of rank $nl - m_0$ with a unitary connection satisfying the anti-self-duality equation

$$*F_{\hat{\nabla}} = -F_{\hat{\nabla}},$$

where $*$ is the Hodge-$*$ on $J \times \mathbb{C}$ equipped with the flat Euclidean metric.

Theorem 6.9.8. *If $\mathbb{E} \in \mathscr{M}$ is a stable parabolic Higgs bundle of parabolic degree zero, then $\hat{\nabla}$ has finite energy.*

Remark 6.9.9. In a recent paper, Mochizuki [55] gave an equivalence between L^2-instantons on $J \times \mathbb{C}$ and harmonic bundles with so-called wild singularities on $\hat{J}$. The equivalence is given by a Nahm transform. The parabolic Higgs bundles we study are covered by Mochizuki's more general argument.

We will use a similar approach to that used by Jardim [46, Theorem 3] to prove Theorem 6.9.8. First, we establish some notation. Let $\mathbb{E} = (E, D'')$ be the fixed parabolic Higgs bundle. The family (6.13) defines two families of differential operators $D''_{z,w}$ and $D'_{z,w}$ by

$$\begin{aligned} D''_{z,w}s &= \bar{\partial}_{A,z}s + \Phi_w s = \bar{\partial}_A s + z d\bar{\xi} s + \Phi s + w d\xi s = D''s + z d\bar{\xi} s + w d\xi s \\ D'_{z,w}s &= \partial_{A,z}s + \Phi^*_w s = \partial_A s - \bar{z} d\xi s + \Phi^* s + \bar{w} d\bar{\xi} s = D's - \bar{z} d\xi s + \bar{w} d\bar{\xi} s \end{aligned}$$

where $z, w \in \mathbb{C}$ and ξ is a coordinate on $\hat{J}$ with $d\xi$ trivialising the canonical bundle. The $\bar{\partial}$-operator $\bar{\partial}_{A,z}$ is the $\bar{\partial}$-operator on $E \otimes L_z$ induced by $\bar{\partial}_A$ on E. As the canonical bundle of $\hat{J}$ is trivial, holomorphic sections have the form $w d\xi$ where w is a constant. Likewise, denote by $\mathcal{D}_{z,w}$ and $\mathcal{D}^*_{z,w}$ the Dirac–Higgs operators for $(E, D''_{z,w})$, see Section 6.6.1.

Let u be a local section of $\hat{\mathbb{E}}$. Then consider it as a local section of $E \otimes (K_{\hat{J}} \oplus \bar{K}_{\hat{J}})$ on $J \times \mathbb{C} \times \hat{J}$ with coordinates (z, w, ξ). If we consider the Dirac–Higgs operator and the trivial connection $\underline{d}$ as operators acting on bundles on $J \times \mathbb{C} \times \hat{J}$, then $\underline{d}\mathcal{D}^*_{z,w}$ and $\mathcal{D}^*_{z,w}\underline{d}$ are operators

$$\Gamma(\pi_2^*(E \otimes (K_{\hat{J}} \oplus \bar{K}_{\hat{J}}))) \to \Gamma(\pi_2^*(E \otimes (K_{\hat{J}} \wedge \bar{K}_{\hat{J}})^{\oplus 2}) \otimes \pi_1^*(T^*(J \times \mathbb{C})))$$

where π_1, π_2 are projections to $J \times \mathbb{C}$ and $\hat{J}$, respectively.

Lemma 6.9.10. *Let $u(z,w,\xi) = u_1(z,w,\xi)+u_2(z,w,\xi)$ be a section of $\pi_2^*(E\otimes(K_{\hat{J}}\oplus\bar{K}_{\hat{J}}))$, then*

$$[\underline{d}, \mathcal{D}^*_{z,w}]u = \Omega u = \begin{pmatrix} dz & dw \\ d\bar{w} & -d\bar{z} \end{pmatrix} \begin{pmatrix} d\bar{\xi}\wedge u_1 \\ d\xi \wedge u_2 \end{pmatrix}.$$

Proof. The lemma follows as $\mathcal{D}^*_{z,w}$ depends linearly on (z,w). Expanding the commutator and using the following identities

$$\begin{aligned} &[\underline{d}, \bar{\partial}_A]u_1 = [\underline{d}, \partial_A]u_2 = 0 && [\underline{d}, \Phi]u_2 = [\underline{d}, \Phi^*]u_1 = 0 \\ &[\underline{d}, zd\bar{\xi}]u_1 = dz(d\bar{\xi}\wedge u_1) && [\underline{d}, \bar{z}d\xi]u_2 = d\bar{z}(d\xi\wedge u_2) \\ &[\underline{d}, \bar{w}d\bar{\xi}]u_1 = d\bar{w}(d\bar{\xi}\wedge u_1) && [\underline{d}, wd\xi]u_2 = dw(d\xi\wedge u_2) \end{aligned}$$

defines the operator Ω. □

Corollary 6.9.11. *If u is a section of $\hat{\mathbb{E}}$ considered as a section of $E\otimes(K_{\hat{J}}\oplus\bar{K}_J)$ on $J\times\mathbb{C}\times\hat{J}$, then*

$$\mathcal{D}^*_{z,w}\underline{d}u = \Omega u.$$

Proof. This follows directly from Lemma 6.9.10 as $\mathcal{D}^*_{z,w}u = 0$ for all (z,w). □

Lemma 6.9.12. *There exists $R>0$ and $C>0$ depending on Φ and Φ^*, such that for when $|w|>R$*

$$C|w|^2\|s\|^2_{L^2} \leq \left|\langle \omega s, D''_{z,w}D'_{z,w}s\rangle\right|$$

for all $s\in D^p_2\Omega^0(E)$. Here ω is the Kähler form on $\hat{J}$.

Proof. First, notice that if $s\in D^p_2\Omega^0(E)$, then $D'_{z,w}s\in D^p_1\Omega^1(E)$ and by Lemmas 6.2.12 and 6.3.5

$$\left|\langle \omega s, D''_{z,w}D'_{z,w}s\rangle\right| = \|D'_{z,w}s\|^2_{L^2} < \infty.$$

Let $s\in D^p_2\Omega^0(E)$ have unit length with respect to the L^2-norm. As

$$D''_{z,w}D'_{z,w}s = D''_{z,0}D'_{z,0} + |w|^2 d\xi\wedge d\bar{\xi}s + wd\xi\wedge\Phi^*s - \bar{w}d\bar{\xi}\wedge\Phi s,$$

it follows that

$$\left|\langle \omega s, D''_{z,w}D'_{z,w}s\rangle\right| \geq \left|\|D'_{z,0}s\|^2_{L^2} + |w|^2 - \left|\langle \omega s, wd\xi\wedge\Phi^*s - \bar{w}d\bar{\xi}\wedge\Phi s\rangle\right|\right|.$$

To determine the size of the last term we use that $\Phi\in D^p_1\Omega^1(\operatorname{End}_0 E)$ so the L^2-norm is finite,

$$\begin{aligned} \left|\langle \omega s, wd\xi\wedge\Phi^*s\rangle - \langle \omega s, \bar{w}d\bar{\xi}\wedge\Phi s\rangle\right| &\leq |w|(\|\Phi^*s\|_{L^2} + \|\Phi s\|_{L^2}) \\ &\leq |w|(\|\Phi^*\|_{L^2} + \|\Phi\|_{L^2}) = |w|R \end{aligned}$$

with R depending only on Φ and Φ^*.

When $|w| > R$ we have $|w|^2 - R|w| > 0$ and there is a C such that $|w|^2 - R|w| \geq C|w|^2$ for $|w| > R$, therefore

$$\left|\langle \omega s, D''_{z,w} D'_{z,w} s\rangle\right| \geq \|D'_{z,0}s\|^2_{L^2} + |w|^2 - R|w| \geq C|w|^2,$$

as required. $\square$

Proof of Theorem 6.9.8. Let $\{u_i\}$ be a local frame for $\hat{\mathbb{E}}$. With respect to the L^2-inner product, the (i,j)-coefficient of $F_{\hat{\nabla}}$ is

$$\begin{aligned}(F_{\hat{\nabla}})_{ij} &= \left\langle u_i, \hat{\nabla}^2 u_j \right\rangle \\ &= \langle u_i, \underline{d} P \underline{d} u_j\rangle \\ &= -\left\langle u_i, \underline{d}\mathcal{D}_{z,w} G_{z,w} \mathcal{D}^*_{z,w} \underline{d} u_j\right\rangle \\ &= \left\langle \underline{d} u_i, \mathcal{D}_{z,w} G_{z,w} \mathcal{D}^*_{z,w} \underline{d} u_j\right\rangle \\ &= \left\langle i\Lambda \mathcal{D}^*_{z,w} \underline{d} u_i, G_{z,w} \mathcal{D}^*_{z,w} \underline{d} u_j\right\rangle \\ &= \langle i\Lambda\Omega u_i, G_{z,w}\Omega u_j\rangle\end{aligned}$$

where the fourth equality is from $\mathcal{D}_{z,w} G_{z,w} \mathcal{D}^*_{z,w}$ being projection onto the orthogonal complement of $\ker \mathcal{D}^*_{z,w}$ and the second to last equality is Lemma 6.6.2.

Using Lemmas 6.6.3 and 6.9.10 and splitting $u_i = u_i^1 + u_i^2$ into types, we expand the above equation. The anti-self-duality of $F_{\hat{\nabla}}$ and that the metric on $J \times \mathbb{C}$ is Euclidean gives that it is enough to consider the asymptotics of the coefficients

$$\left\langle i\Lambda(d\bar{\xi} \wedge u_i^1), \hat{G}_{z,w}(d\bar{\xi} \wedge u_j^1)\right\rangle \quad \text{and} \quad \left\langle i\Lambda(d\bar{\xi} \wedge u_i^1), \hat{G}_{z,w}(d\xi \wedge u_j^2)\right\rangle$$

where $\hat{G}_{z,w}$ is the inverse of the isomorphism

$$D''_{z,w} D'_{z,w} : D_2^p \Omega^0(E) \to D_0^p \Omega^{1,1}(E).$$

If $s \in D_2^p\Omega^0(E)$ is such that $D''_{z,w}D'_{z,w}s$ is of the form $d\bar{\xi} \wedge u$ for $u \in D_1^p\Omega^{1,0}$, then $\|D''_{z,w}D'_{z,w}s\|_{L^2} < \infty$. It follows from Lemma 6.9.12 and Cauchy–Schwarz that for such s

$$C|w|^2 \|s\|_{L^2} \leq \|D''_{z,w} D'_{z,w} s\|_{L^2} \qquad \text{for} \qquad |w| > R. \tag{6.16}$$

Using the estimate (6.16) we get that for $|w| > R$

$$\begin{aligned}\|d\bar{\xi} \wedge u\|_{L^2} &= \|D''_{z,w} D'_{z,w} \hat{G}_{z,w}(d\bar{\xi} \wedge u)\|_{L^2} \\ &= \|D''_{z,w} D'_{z,w} s_{z,w}\|_{L^2} \\ &\geq C|w|^2 \|s_{z,w}\|_{L^2} \\ &= C|w|^2 \|\hat{G}_{z,w}(d\bar{\xi} \wedge u)\|_{L^2}\end{aligned}$$

and thus

$$\begin{aligned}\left|\left\langle i\Lambda(d\bar{\xi}\wedge u_i^1), \hat{G}_{z,w} d\bar{\xi}\wedge u_j^1\right\rangle\right| &\le \|d\bar{\xi}\wedge u_i^1\|_{L^2}\|\hat{G}_{z,w}(d\bar{\xi}\wedge u_j^1)\|_{L^2}\\ &\le \frac{1}{C}|w|^{-2}\|d\bar{\xi}\wedge u_i^1\|_{L^2}\|d\bar{\xi}\wedge u_j^1\|_{L^2}\end{aligned}$$

when $|w| > R$. A similar estimate for the other coefficient follows by a parallel argument. Together, this proves $|F_{\hat{\nabla}}| = O(|w|^{-2})$ and therefore that $\hat{\nabla}$ has finite energy. □

Remark 6.9.13. What is really shown in the proof of Theorem 6.9.8 is that the curvature decays quadratically $|F_{\hat{\nabla}}| \sim |w|^{-2}$ for $|w| \to \infty$. This is a stronger statement than being L^2.

Remark 6.9.14. From the description of the curvature $F_{\hat{\nabla}}$ in the proof of Theorem 6.9.8 and Lemma 6.9.10, it follows directly that $F_{\hat{\nabla}}$ is of type $(1,1)$ and orthogonal to the Kähler form on $J \times \mathbb{C}$ associated to the flat Euclidean metric. This gives a different proof that $F_{\hat{\nabla}}$ is an anti-self-dual two-form.

Asymptotic holomorphic structure

In this section we consider the holomorphic structure of $\hat{\mathbb{E}}|_{J_w}$ as $|w|$ tends to ∞.

Proposition 6.9.15. *Let $\mathbb{E}$ be a stable parabolic Higgs bundle of parabolic degree zero on an elliptic curve. Then $\hat{\mathcal{E}}|_{J_\infty}$ is a homogenous bundle determined by the parabolic structure of $\mathbb{E}$.*

Proof. As $d\xi$ does not vanish on $\hat{J}$ it is an injective sheaf map $E \to EK$. It then follows from Proposition 6.9.7 that $\hat{\mathcal{E}}|_{J_\infty}$ is a homogeneous bundle. The cokernel of

$$0 \to E \xrightarrow{d\xi} \mathcal{S}K(D) \to \mathcal{Q} \to 0$$

is a skyscraper sheaf supported on D of lengths $l - m(w_1(P))$ at each $P \in D$ with $w_1(P) = 0$ otherwise of length l. The sheaf $\mathcal{Q}$ is therefore completely determined by the parabolic structure. As $\hat{\mathcal{E}}|_{J_\infty} \simeq \pi_*(q^*\mathcal{Q} \otimes \mathcal{P})$ where $q : J \times \hat{J} \to \hat{J}$ and $\pi : J \times \hat{J} \to J$ are projections, it follows that $\hat{\mathcal{E}}|_{J_\infty}$ is determined by the parabolic structure. □

Corollary 6.9.16. *Let $D = P_1 + \cdots + P_n$ be parabolic points with zero-weights of multiplicity $m_{P_i}(0) = l - 1$ for all i, then*

$$\hat{\mathcal{E}}|_{J_\infty} \simeq L_{P_1} \oplus \cdots \oplus L_{P_n}.$$

Proof. From the assumptions, the cokernel sheaf of

$$0 \to E \xrightarrow{d\xi} \mathcal{S}K(D) \to \mathcal{Q} \to 0$$

is a skyscraper sheaf supported on D with length one at each point. Then

$$\pi_*(q^*Q \otimes \mathcal{P}) \simeq L_{P_1} \oplus \cdots \oplus L_{P_n}$$

proving the result. $\square$

Example 6.9.17. Let $D = P + (-P)$ where we use that $\hat{J}$ is an elliptic curve to say $-P \in \hat{J}$. We also assume P is not an order two point of $\hat{J}$. Define weights at $\pm P$ to be 0 with multiplicity $m_{\pm P}(0) = l - 1$ and $\alpha_\pm$ of multiplicity 1. Finally, assume $1 = \alpha_+ + \alpha_-$.

Let $\mathbb{E}$ be a stable parabolic Higgs bundle of parabolic degree zero with the above parabolic structure. Then the Nahm transform $(\hat{\mathbb{E}}, \hat{\nabla})$ has rank two and extends to a holomorphic bundle on $\hat{T} \times \mathbb{P}^1$ with Chern character $2 + h - lth$, where l is the rank of $\mathbb{E}$.

From Theorem 6.9.8 and Corollary 6.9.16, $\hat{\mathbb{E}}$ is an SU(2)-instanton with quadratic curvature decay. These instantons have been extensively studied by Jardim [46, 47] and Biquard and Jardim [10], where they obtain an equivalence between SU(2)-instantons on $J \times \mathbb{C}$ and *singular* Higgs bundles on $\hat{J}$. The equivalence also goes through a Nahm transform defined using L^2-theory. The singularities of the singular Higgs bundles are much like the ones for our parabolic Higgs bundle but with one difference being that the non-zero weights are $1 \pm \alpha$ with $0 \leq \alpha < \frac{1}{2}$ and topological degree -2. This is a singular Higgs bundle rather than a parabolic Higgs bundle as the difference between the weights is more than 1 at P. Another difference is that Jardim requires his residues of Higgs fields to have a non-zero eigenvalue, whereas we require our Higgs fields to have nilpotent residues.

The holomorphic structure of Jardim's bundles also extend to $J \times \mathbb{P}^1$ and due to the topological degree -2 the natural extension has Chern character $2 - lth$. It should be noted that if we do an elementary modification of our extended Nahm transform $\hat{\mathcal{E}}$ on J_∞ with the flat line bundle L_P, then the new holomorphic bundle $\hat{\mathcal{E}}'$ is also an extension of $\hat{\mathbb{E}}$ but now with Chern character $2 - lth$ and with $\hat{\mathcal{E}}'|_{J_\infty} \simeq L_P \oplus L_{-P}$.

7. Outlook

The results in this thesis raise several interesting questions which we discuss in this final chapter.

1. In Section 2.2 we discussed the shape of the L^2-solutions to the Dirac–Higgs equations for the global monomial rank one Higgs bundles $(\mathcal{O}, z^k dz)$ on $\mathbb{C}$. The space of L^2-solutions to the Dirac–Higgs equations is k-dimensional. Based on this we made Conjecture 2.2.4, claiming that if φ is a polynomial of degree k, then the space of L^2-solutions to the Dirac–Higgs equations for $(\mathcal{O}, \varphi dz)$ has dimension k.

   There are several possible avenues for proving the conjecture. Let $\mathcal{D}_\varphi$ denote the Dirac–Higgs operator for $(\mathcal{O}, \varphi dz)$. It is easy to prove that $\ker \mathcal{D}_\varphi = 0$. This follows directly from $\mathcal{D}_\varphi^* \mathcal{D}_\varphi$ being a real operator. If we were able to prove that the index of $\mathcal{D}_\varphi$ is $-k$, the conjecture would follow. To do this, we could use the conformal invariance of the Dirac–Higgs operator, and compactify to $\mathbb{P}^1$ by including the order $k+2$ pole of φdz. The right weighted L^2-space to consider should emerge from the analysis of Biquard and Boalch [9]. Having the right Sobolev spaces, we must prove that $\mathcal{D}_\varphi$ is Fredholm. The homotopy $t\varphi + (1-t)z^k$ gives a homotopy between $\mathcal{D}_\varphi$ and $\mathcal{D}_{z^k}$ showing that they have the same index. The conjecture then follows from the explicit calculations of Lemma 2.2.3.

   Alternatively, we could use the same approach as the proof of Theorem 6.4.1, giving an isomorphism between $\ker \mathcal{D}_\varphi$ and $\mathbb{H}^0(\mathcal{O} \xrightarrow{\varphi dz} \mathcal{O}(k))$. Here we again compactify to $\mathbb{P}^1$, and consider the Higgs field φdz as a section of $\mathcal{O}(k)$ as it is a meromorphic section of $\mathcal{O}(-2)$ with a pole of order $k+2$ at infinity. As the hypercohomology is supported at the zeros of φ the Conjecture 2.2.4 would immediately follow. To give a proof along the lines of Theorem 6.4.1, we could find an L^2-resolution of $\mathcal{O} \xrightarrow{\varphi dz} \mathcal{O}(k)$ on $\mathbb{P}^1$. We expect the following to be such a resolution

$$\begin{tikzcd} \mathcal{O} \arrow[r, hook] \arrow[d, "\varphi dz"] & L_1^2\Omega^0 \arrow[r, "\bar\partial"] \arrow[d, "\varphi dz"] & L^2\Omega^{0,1} \arrow[d, "\varphi dz"] \\ \mathcal{O}(k) \arrow[r, hook] & (1+r^{k+2})L_1^2\Omega^{1,0} \arrow[r, "\bar\partial"] & (1+r^{k+2})L^2\Omega^{1,1} \end{tikzcd}$$

   where $u \in (1+r^{k+2})L_i^2$ if $(1+r^{k+2})^{-1}u \in L_i^2$.

   The latter approach to Conjecture 2.2.4 lends itself well to the Nahm transform. At each point u the vector space $\ker \mathcal{D}_{\varphi+u}$ splits, as Lemma 2.4.3, in a sum of contributions from each zero of the z-polynomial $\varphi(z)+u$ on $\mathbb{C}$. This decomposition should be an eigenspace decomposition with respect to $\hat\varphi(u)$, and the zeros of $\varphi(z)+u$ are the eigenvalues of $\hat\varphi(u)$.

This description suggests that there is a spectral curve lurking behind the scenes, facilitating the Nahm transform. The spectral curve should be the solution set of $\varphi(z) + u$ on $\mathbb{C}^2$. Compactified to $\mathbb{P}^1 \times \mathbb{P}^1$ it is the zero locus of a section of $\mathcal{O}(k, 1)$. This makes for a $2k + 1$-dimensional space of divisors. But as the Higgs field is only singular at infinity, the intersection with $\{\infty\} \times \mathbb{P}^1$ or $\mathbb{P}^1 \times \{\infty\}$ is (∞, ∞). Furthermore, if the spectral curve is considered as a $k : 1$-covering of $\mathbb{P}^1$, the point $\infty \in \mathbb{P}^1$ is a branch point of order k. This cuts down the potential space of divisors to $k + 1$. These $k + 1$ parameters match the number of coefficients in φ. On the other hand, the spectral curve is the zero locus of $\det(\hat{\varphi}(u) + z \,\mathrm{Id})$. For fixed z this must be a degree-one polynomial in u in order to get a line bundle when doing the inverse Nahm transform. Expanding the characteristic polynomial for $\hat{\varphi}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ we see that for it to agree with the spectral curve of φ we must have $\det \hat{\varphi}$ be a degree-one polynomial, and all other coefficients of the characteristic polynomial be constants. This again leaves $k + 1$ parameters.

2. In Chapter 3 we investigated the Dirac–Higgs bundle on the family of Higgs pairs $(A, \Phi + t\alpha)$. Based on the localisation result in Theorem 3.1.1 and the model asymptotic behaviour in Proposition 3.2.1, we stated Conjecture 3.2.2 claiming that on a small neighbourhood of a zero of α a solution to the Dirac–Higgs equations converge as a distribution to a delta function. Based on the validity of this conjecture, we show that there is a relation between the distributional behaviour of a solution to the Dirac–Higgs equations and the limit of the corresponding sequence in the cokernel decomposition. An element in a cokernel gives a functional on the fibre of the vector bundle at the zeros of the Higgs field. Their limit is what we expect a solution to the Dirac–Higgs equations distributionally converge to. The same picture should be valid for higher rank Higgs bundles as well (Conjecture 3.2.6).

   If the two conjectures are true we give below a natural condition for a frame to have a unitary limit. We only consider the rank one case as the general case is similar. Let (A, Φ) be a Higgs pair solving the Higgs bundle equations on a Hermitian line bundle (L, h), and denote by $\hat{L}$ the Nahm transform.

   Recall from Lemma 2.4.3 the isomorphism

$$\rho' : \ker \mathcal{D}^*_{A,\Phi} \simeq \mathbb{H}^1(L, \Phi) \to \bigoplus_{i=1}^{2g-2} \operatorname{coker}(L_{z_i} \xrightarrow{\Phi_{z_i}} LK_{z_i})$$

   given by evaluating the $(1, 0)$-part of a 1-form in $\ker \mathcal{D}^*_{A,\Phi}$ at the zeros of Φ.

   Assume $\alpha \in H^0(K)$ only has simple zeros. On the line generated by α consider a frame $\hat{e}_t = (\hat{f}^t_1, \ldots, \hat{f}^t_{2g-2})$ for $\hat{L}$. The frame $\hat{e}_t$ is said to have the *limiting property* if

$$\lim_{t\to\infty} \sqrt{\frac{\pi}{t}}\rho'(\hat{f}_k^t)\cdot e_i = \delta_{ik} \quad \text{and} \quad \|\hat{f}_i^t\|_h \quad \text{constant in } t$$

where $\rho'(\hat{f}_k^t)\cdot e_i$ is the i'th component of the vector $\rho'(\hat{f}_k^t)$.

We define a frame by identifying each $\operatorname{coker}(L_{z_i} \xrightarrow{\Phi_{z_i}} LK_{z_i})$ with $\mathbb{C}$ and a section in the frame by specifying a complex number at each z_i. A frame with the k'th element of the frame being $\sqrt{\frac{t}{\pi}}$ at z_k and zero at the other z_i has the limiting property.

If Conjecture 3.2.2 is true we prove that frames with the limiting property are unitary in the limit.

Theorem 7.0.18. *Let $\hat{L}$ be the Nahm transform of a degree zero rank one Higgs bundle. Assume $\alpha \in H^0(K)$ only has simple zeros. On the line generated by α let $\hat{e}_t = (\hat{f}_1^t, \dots, \hat{f}_{2g-2}^t)$ be a frame with the limiting property. Then in the limit $t \to \infty$ the frame $\hat{e}_t$ is unitary with respect to the L^2-metric on $\hat{L}$.*

Proof. Let $x_1, \dots, x_{2g-2}$ be the zeros of α and let U_i be a disk around x_i. The zeros of $\Phi + t\alpha$ converge to the zeros of α as $t \to \infty$. For t sufficiently large there is one zero of $\Phi + t\alpha$ in each U_i. Let $C = \Sigma \setminus \cup_{i=1}^{2g-2} U_i$ be the complement of the U_i's. By definition

$$\left\langle \hat{f}_j^t, \hat{f}_k^t \right\rangle_h = \int_C h(\hat{f}_j^t, \hat{f}_k^t)\omega + \sum_{i=1}^{2g-2} \int_{U_i} h(\hat{f}_j^t, \hat{f}_k^t)\omega.$$

The first term vanishes in the limit by Cauchy–Schwartz and Theorem 3.1.1

$$\lim_{t\to\infty} \left| \int_C h(\hat{f}_j^t, \hat{f}_k^t)\omega \right| = 0.$$

The limiting property of the frame $\hat{e}$ and the local distributional behaviour of solutions to the Dirac–Higgs equations (Conjecture 3.2.2) shows that the element $\hat{f}_k^t$ concentrates at x_k and decays to zero at x_l for $l \neq k$. By Cauchy–Schwartz we get for $j \neq k$

$$\lim_{t\to\infty} \left| \int_{U_i} h(\hat{f}_j^t, \hat{f}_k^t)\omega \right| = 0 \quad \text{for all } i.$$

If $j = k$ the limiting property of $\hat{f}_j^t$ and Conjecture 3.2.2 gives

$$\lim_{t\to\infty} \int_{U_i} |\hat{f}_j^t|_h^2 \omega = \delta_{ij}.$$

Combining all of the above, the theorem follows. □

3. One of the main motivations for the study of asymptotics of solutions to the Dirac–Higgs equations is to identify the essential image of the Nahm transform for Higgs bundles constructed by Bonsdorff [14]. Bonsdorff shows that the Nahm transform is

injective. His proof only uses the fact that as a holomorphic bundle on $J \times H^0(K)$ the Nahm transform of a Higgs bundle extends to $J \times \mathbb{P}^g$. The connection plays no role in that story. It is therefore likely that studying the asymptotics of the connection on the transformed bundle will reveal boundary conditions under which the image can be identified.

If Conjecture 3.2.6 is true, frames with the limiting property should be helpful in revealing information about the asymptotic behaviour of the connection, e.g. it should be possible to show that the curvature vanish at infinity.

4. A Nahm transform should ultimately be an isometry between moduli spaces of solutions to different versions of the anti-self-duality equations. If this is also the case for Higgs bundles, we would expect to see a Hitchin fibration in this suggested 'moduli space' of Nahm transformed Higgs bundles. The spectral data construction of the Fourier–Mukai transform in Section 4.5 seems to suggest that an appropriate subspace of the rational maps $\mathbb{P}^g \to \mathbb{P}^{2r(g-1)-g}$ should constitue the base for the fibration. To determine the base we need to understand the locus of $\mathbb{P}^g$ on which the rational maps are fixed. Furthermore, the fixed locus would also give information about the deformation theory of the 'moduli space' of transformed Higgs bundles. A rational map in the base space is equivalent to a hypersurface Y from Section 4.5 as they both contain the information of the spectral curve (Theorem 4.5.10). In this light, it is natural to conjecture that deformations of $Y \subset S \times \mathbb{P}^g \subset T^*\Sigma \times \mathbb{P}^g$ are in one-to-one correspondence with deformations of the spectral curve $S \subset T^*\Sigma$ used to define Y. If this is so, it would indicate that the 'moduli space' of transformed Higgs bundles has the correct dimension and that the space of Y's or rational maps should indeed be the base space of the fibration.

   A similar type of Hitchin fibration is also conjectured by Biquard and Jardim [10] for doubly-periodic instantons.

5. In Section 6.9.1 we construct doubly-periodic instantons from parabolic Higgs bundles on a genus one Riemann surface. To define the Nahm transform we fix the topology of the underlying smooth bundle, the parabolic points, and the weights for the parabolic structure. On the instanton side we recover the parabolic points as the limiting holomorphic structure on the torus at infinity (Corollary 6.9.16). The rank of the parabolic Higgs bundle we expect to be the energy of the instanton and also the second Chern class of a bundle on the compactification extending the instanton, Proposition 6.9.4. We expect the weights to be the monodromy of the instanton around the torus at infinity.

The expectations are based on Example 6.9.17 where we construct rank two doubly-periodic instantons with quadratic curvature decay similar to those constructed by Jardim [46]. Using a good gauge at infinity Jardim [47] confirms the expectations above in regards to the rank and weights. His arguments solely build on the quadratic decay and therefore also applies to our situation. We expect it to be possible to extend Jardim's good gauge to the more general type of instantons.

Biquard and Jardim [10] identifies a third invariant μ determined by the asymptotic behaviour of the doubly-periodic instanton. This invariant corresponds to the one non-zero eigenvalue of the residue of the singular Higgs field. In our case, the residues are nilpotent. It is not that we construct instantons with $\mu = 0$, because if the parabolic point P is not of order two, then there are no doubly-periodic instantons with $\mu = 0$ by [10, Lemma 5.6].

If the invariants mentioned above are all fixed, Biquard and Jardim shows that the moduli space of quadratically decaying SU(2)-doubly-periodic instantons is a hyperkähler manifold of complex dimension $4k - 2$ where k is the instanton charge. Furthermore, they show that with these invariants fixed the Nahm transform is a hyperkähler isometry. If we consider the parabolic Higgs bundles in Example 6.9.17, the moduli space of these also has dimension $4k - 2$ where k is the rank of the parabolic Higgs bundle. In Example 6.9.17 we fix only the rank of the bundle, the parabolic points, and the weights. There is no extra continuous parameter as in the singular Higgs bundle case. Understanding the relation between the two Nahm transforms seems interesting.

6. In Section 6.8 we discussed the limiting configurations of [53] in the framework of parabolic Higgs bundles. We also mentioned that the construction of limiting configurations in [53] as limits of Higgs pairs (A_t, Φ_t) where $(A_t, t\Phi_t)$ satisfies the Higgs bundle equations. It would be interesting to examine the behaviour of L^2-solutions to the Dirac–Higgs equations for $(A_t, t\Phi_t)$ in the large t-limit. Because of the explicit expressions for the fiducial Higgs bundles it is possible to do a local study of the solutions to the Dirac–Higgs equations along the lines of Section 6.8.1. It is expected that the procedure for proving Conjecture 3.2.2 apply to this setting as well. If it is furthermore possible to prove Theorem 3.1.1 for the family of pairs $(A_t, t\Phi_t)$ we would be able to construct a limiting unitary frame discussed above.

The proof of Theorem 3.1.1 does not apply to the pairs $(A_t, t\Phi_t)$ as both the connection A_t and Higgs field Φ_t depend on t. In Remark 3.1.5 we stressed that it was important that the Hermitian metric was independent of t, but this was under the assumption that A and Φ are independent of t as well. This is not the case anymore. To make the same proof work we would need bounds on the operator norms of $G(-,\Phi_t)$, $H(-,\Phi_t)$, in the notation of Section 3.1. The analytical work in [53] might provide the bounds needed.

7. A similar limiting investigation of Higgs bundles is done by Collier and Li [18] for a special type of Higgs bundles in the so-called *Hitchin component*. They obtain interesting decay results for the Hermitian metric away from the zeros of the determinant of the Higgs field. It would be interesting to understand the asymptotic properties of solutions to the Dirac–Higgs equations in their setting.

Notational conventions

The following is a list of selected notation and notational conventions adopted throughout the thesis.

- Σ is a Riemann surface with canonical bundle K.
- E is either a complex or a holomorphic vector bundle.
- EK is shorthand for the tensor product $E \otimes K$.
- $\mathcal{E}$ is a coherent sheaf.
- $H^0(\mathcal{E})$ is an abbreviation for the cohomology group $H^0(X, \mathcal{E})$ when the underlying space X is apparent.
- $R^i f_*(\mathcal{F})$ is the higher direct image of sheaf $\mathcal{F}$ along a holomorphic map f.
- $\mathbb{R}^i f_*(\mathcal{F}^\bullet)$ is the higher direct image of complex of sheaves $\mathcal{F}^\bullet$ along a holomorphic map f.
- In general, bold face notation refers to sequences of coherent sheaves or objects derived from such, e.g. $\mathbb{E} = E \xrightarrow{\Phi} E \otimes K(D)$ is a parabolic Higgs bundle, $\mathbb{H}\mathrm{om}(\mathbb{E}, \mathbb{F})$ are homomorphisms between parabolic Higgs bundles, $\mathbb{H}^i(\mathbb{E})$ is hypercohomology.
- h is a Hermitian metric on E.
- A is a unitary connection on (E, h).
- ∂_A and $\bar{\partial}_A$ the $(1,0)$ and $(0,1)$-parts of the covariant derivative d_A of a connection A.
- (A, Φ) is a Higgs pair.
- $D'' = \bar{\partial}_A + \Phi$ and $D' = \partial_A + \Phi^*$ denote Higgs bundle differentials.
- $\mathcal{D}_{A,\Phi}$ is the Dirac–Higgs operator associated to a Higgs pair (A, Φ) and $\mathcal{D}^*_{A,\Phi}$ its adjoint.
- $\mathscr{A}$ is the affine space of unitary connections on (E, h).
- $\mathscr{C}$ is the affine space of $\bar{\partial}$-operators on E.
- $\mathscr{M}$ is the moduli space of polystable Higgs bundles of fixed rank and degree, and $\mathscr{M}^{st}$ is the stable locus.
- W^p_k is the weighted Sobolev space of functions with k derivatives in weighted L^p.
- D^p_k is the weighted Sobolev space of sections with the regular part of a section in unweighted Sobolev space L^p_k and singular part of a section in weighted Sobolev space W^p_k.

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