Modelling basket credit default swaps with default contagion

Helen Haworth

haworthh@maths.ox.ac.uk

OCIAM, Mathematical Institute

Oxford University

Outline

1. Some credit background
2. The structural model
3. Multiple firm model with default contagion
    - Methodology
    - Implementation
    - Results
    - Contagion with decay
4. Future work

What is Credit?

- Credit risk is the risk that an obligor does not honour its obligations
- It is often thought of as default risk
- Important to banks, corporations and investors
- Reason the multi-billion dollar credit derivatives market exists
    - Simplest: credit default swap (CDS)
    - Insurance against risk of default by a company

The Market Today

- iTraxx and CDX indicies were launched in Europe and the US, respectively, in 2004
- Each CDS index has 125 constituents which are updated every 6 months
- Created massive liquidity
- Standardized collateralized debt obligations (CDOs) and underlying tranches
- More esoteric products – CDO-squareds, options on CDO tranches etc.

Default Characterisation

Firms do not operate in isolation and a company default can be triggered in three main ways:

1. By factors directly impacting multiple companies
   - Cyclical
   - Market-wide shock
2. By company-specific incidents or situations
3. Due to inter-company ties
   - Physical
   - Perceived

Default Characterisation

- Default dependence then occurs primarily through two mechanisms
  1. As a direct consequence of a common factor driving default
  2. Due to inter-company ties: contagion
- The result is a complex network of non-symmetrical links between companies
- The impact of individual credit events can ripple through the market

The Structural Credit Model

- The original structural model dates back to the papers of Black & Scholes (1973) and Merton (1974)
- Based on economic fundamentals through modelling the dynamics of firm assets, default occurs if firm value, V, drops below the value of debt, K
- Firm assets are modelled as a geometrical Brownian motion, $dV = \mu V dt + \sigma V dW$
- Debt and equity valued as contingent claims on the firm assets,
  $C(V, T) = \min(K, V(T)) = K - \max(0, K - V(T))$

The Literature

In the single-firm case, there are many extensions to the basic model

- Black & Cox (1976) – allow default prior to maturity
- Longstaff & Schwarz (1995) – stochastic interest rates
- Leland & Toft (1996) – endogenous default trigger
- Zhou (1997) – jump diffusion model
- Duffie & Lando (2000), Giesecke (2003) – incomplete information
- Fouque, Sircar & Solna (2006) – stochastic volatility

The Literature

Existing work on multiple-firm structural models is limited

- Zhou (2001) calculates default correlations for two companies, modelled as correlated Brownian motions
- Hull, Predescu & White (2005) assume assets are driven by a common factor and price CDO tranches using Monte Carlo
- Luciano & Schoutens (2005), Moosbrucker (2006), Baxter (2006) model firms as Levy processes

Model Overview

- First passage model – firm values are modelled as correlated geometric Brownian motions with exponential default thresholds
- Idiosyncratic ties are incorporated through a jump in volatility on default of a related entity
- The framework is extremely flexible, incorporating default causality and allowing for asymmetrical links
- Value k^{th}-to-default CDS baskets in the presence of asset correlation and default contagion

Model Methodology

- We consider n companies, firm values V_i, with default as the first time that V_i hits a lower default barrier $b_i(t)$

$$dV_i(t) = (r_f - q_i)V_i(t)\,dt + \sigma_i V_i(t)\,dW_i(t)$$

- cov$(W_i(t), W_j(t)) = \rho_{ij}t$ for $i, j = 1, \ldots, n$
- We assume exponential default barriers, reflecting the existence of debt covenants,

$$b_i(t) = K_i e^{-\gamma_i(T-t)}$$

Model Methodology

Let Ω represent the default probability event of interest. For vector $\mathbf{V}$ of firm values, and infinitesimal generator $\mathcal{L}$, we solve

$$\mathcal{L}U = \frac{\partial U}{\partial t} + \sum_{i=1}^{n}(r_f - q_i)V_i\frac{\partial U}{\partial V_i} + \frac{1}{2}\sum_{i,j=1}^{n}\rho_{ij}\sigma_i\sigma_j V_i V_j\frac{\partial^2 U}{\partial V_i \partial V_j} = 0$$

$$U(\mathbf{V}, T) = \mathbb{I}_\Omega(\mathbf{V}(T))$$

for $U(\mathbf{V}, t)$ to calculate the probability of Ω by Feynman-Kac,

$$U(\mathbf{v}, t) = \mathbb{E}\left\{\mathbb{I}_\Omega(\mathbf{V}(T)) | \mathbf{V}(t) = \mathbf{v}\right\} = \mathbb{P}\left(\mathbf{V}(T) \in \Omega | \mathbf{V}(t) = \mathbf{v}\right)$$

Implementation

- We solve backwards in time on $[0, T] \times \mathbb{R}_+^n$ using a finite-difference method with Crank Nicolson time-stepping and a multigrid solver
- Deal with boundary conditions by setting a firm's drift and volatility to zero on its barrier
- Count the number of companies whose values are at or below their default barriers and define the initial condition accordingly

Introducing Contagion

- Dependence structure is currently driven just by correlation in firm values
- Incorporate default contagion by allowing company volatilities to jump on default of related entity
- For example, if company i defaults, for $i \neq j$, we let

$$\sigma_j \longrightarrow \sigma_j F^{\rho_{ij}}$$

for some constant $F \geq 1$

Introducing Contagion

- In other words,

$$\sigma_j(\mathbf{V}, t) = \begin{cases} \sigma_j & \text{if} \quad V_i(t) > b_i(t), V_j(t) > b_j(t) \\ \sigma_j F^{\rho_{ij}} & \text{if} \quad V_i(t) \leq b_i(t), V_j(t) > b_j(t) \\ 0 & \text{if} \quad V_j(t) \leq b_j(t) \end{cases}$$

- Subsequent default by firm $k \notin \{i, j\}$ would give $\sigma_j \longrightarrow \sigma_j F^{\rho_{ij}} \longrightarrow \sigma_j F^{\rho_{ij}} F^{\rho_{kj}}$

$\mathbf{k}^{th}$-to-Default CDS Baskets

- Consider a contract par value K on a basket of n companies, with bond recovery on default of R and continuous spread payments, c
- The discounted default payment, paid to the CDS buyer in the event of the k^{th} company default, is

$$
\begin{aligned}
&(1-R)K\int_0^T e^{-r_f s}\mathbb{P}(s \leq \tau_k \leq s+ds)\,\mathrm{d}s \\
&= (1-R)K\int_0^T -e^{-r_f s}\frac{\partial}{\partial s}\mathbb{P}(\tau_k > s)\,\mathrm{d}s
\end{aligned}
$$

kth-to-Default CDS Baskets

- The discounted spread payment is

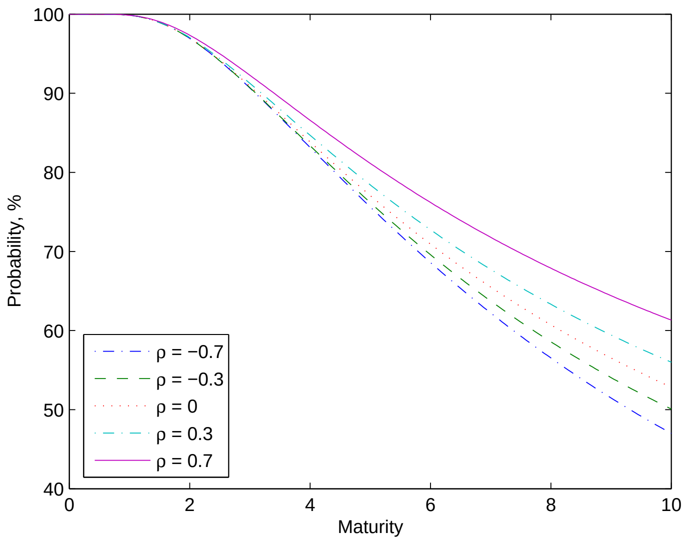
$$cK \int_0^T e^{-r_f s} \mathbb{P}(\tau_k > s)\, \mathrm{d}s,$$

- Equating these gives the kth-to-default CDS spread

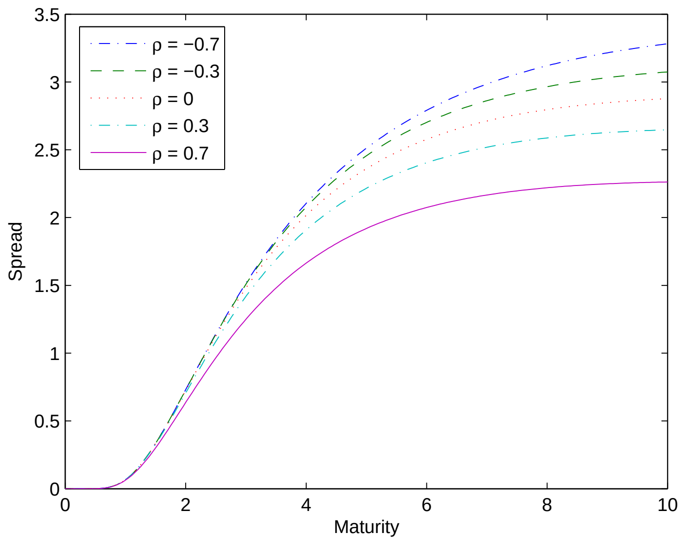
$$c_k = \frac{(1-R)\left\{1 - e^{-r_f T}\mathbb{P}(\tau_k > T) - \int_0^T r_f e^{-r_f s}\mathbb{P}(\tau_k > s)\, \mathrm{d}s\right\}}{\int_0^T e^{-r_f s}\mathbb{P}(\tau_k > s)\, \mathrm{d}s}$$

Two-firm Results

Joint survival probability



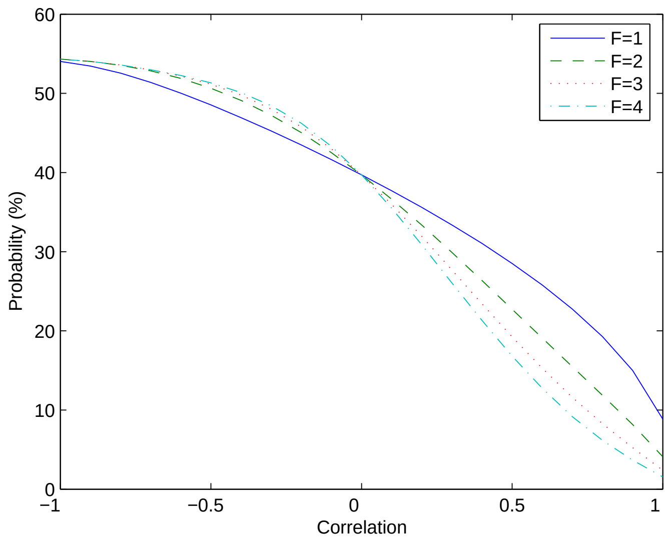
First-to-default CDS spread



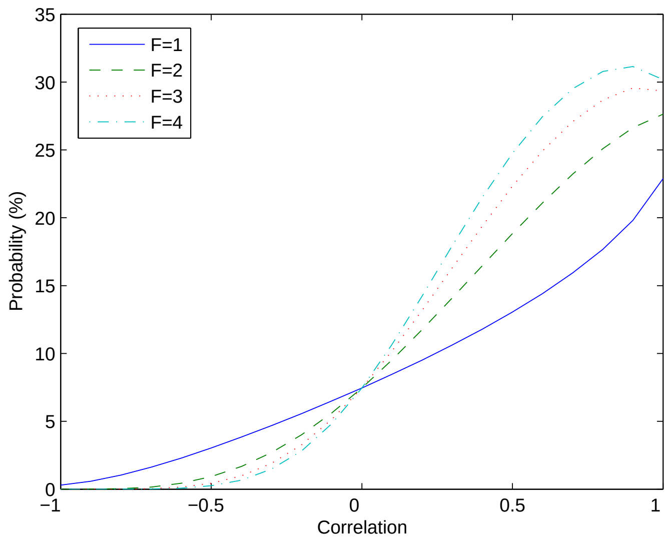
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2, R = 0.5

Two-firm Results

Probability of 1 default



Probability of 2 defaults

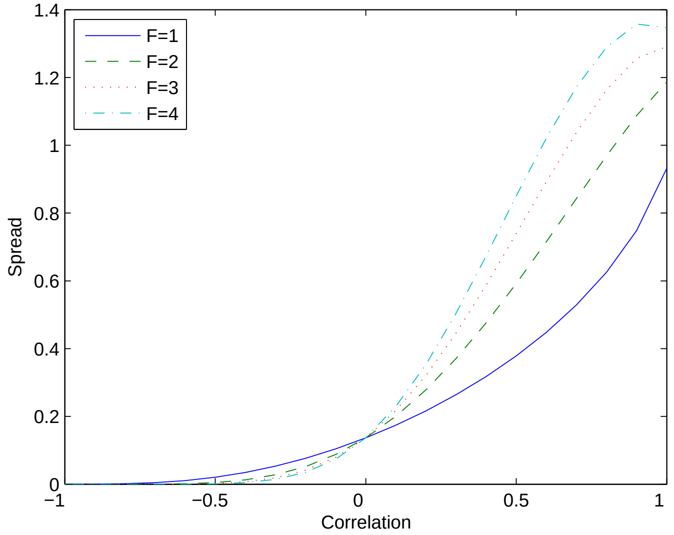


Ten-year default probabilities

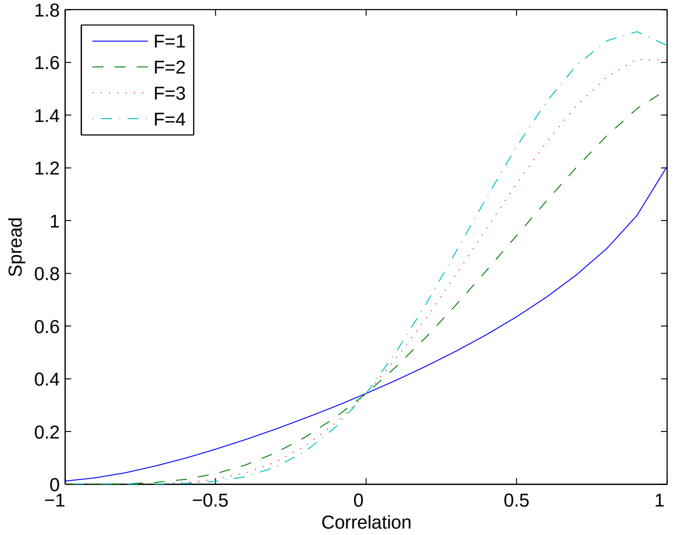
$\sigma_i = 0.2,\ r_f = 0.05,\ q_i = 0,\ \gamma_i = 0.03$, initial credit quality = 2

Second-to-Default CDS Spreads

5-year 2nd-to-default CDS



10-year 2nd-to-default CDS



$\sigma_i = 0.2,\ r_f = 0.05,\ q_i = 0,\ \gamma_i = 0.03$, initial credit quality = 2, R = 0.5

Contagion with Decay

- More realistic for the spike in volatility on default to decay over time, $\sigma_j \to \sigma_j(1 + \Delta_{ij} e^{-\zeta(t-\tau_i)})$
- Setting $\Delta_{ij} = F^{\rho_{ij}} - 1$ enables direct comparison of results with and without decay.
- After default by firm i, time-dependence of $\sigma_j(t)$ for $j \neq i$ is removed by replacing $\sigma_j^2(t)$ with its average over the remaining time-to-maturity, $\bar{\sigma}_j^2$

$$\bar{\sigma}_j^2 = \frac{1}{T - \tau_i} \int_0^{T-\tau_i} \sigma_j^2(s)\, \mathrm{d}s$$

Contagion with Decay

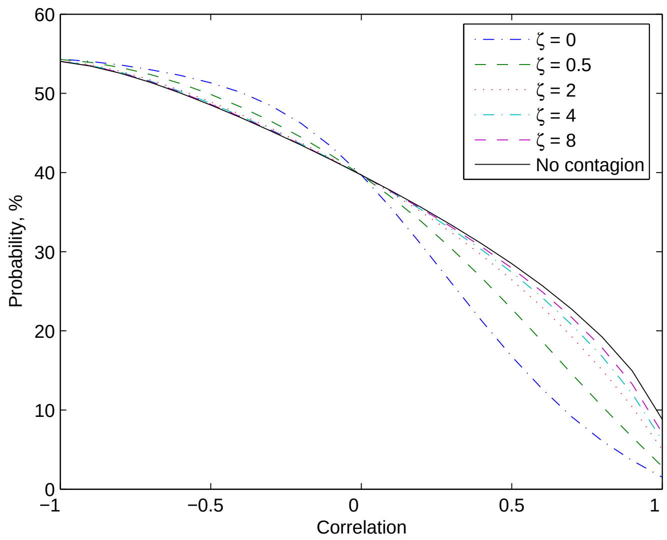
- This gives the new volatility

$$\sigma_j^2 + \frac{2\Delta_{ij}\sigma_j^2}{\zeta(T-\tau_i)}\left(1-e^{-\zeta(T-\tau_i)}\right) + \frac{\sigma_j^2\Delta_{ij}^2}{2\zeta(T-\tau_i)}\left(1-e^{-2\zeta(T-\tau_i)}\right)$$

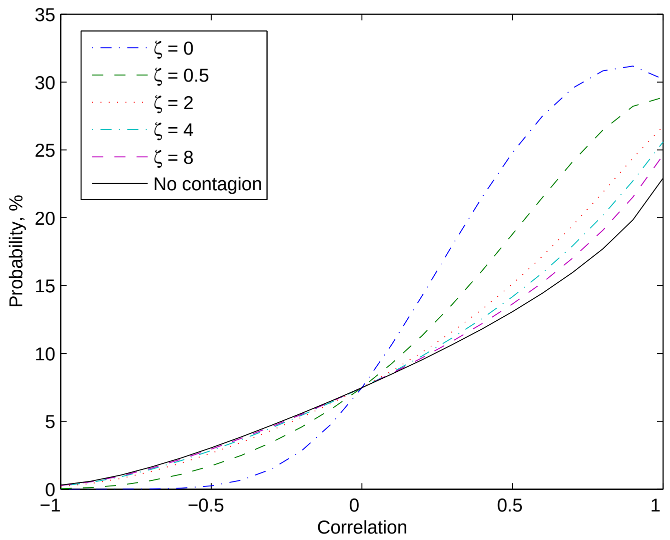
- The problem decouples on the boundary, allowing us to solve the two-firm problem on $[0, \tau_i]$ and a one-company problem on $[\tau_i, T]$.

Contagion with Decay

Probability of 1 default



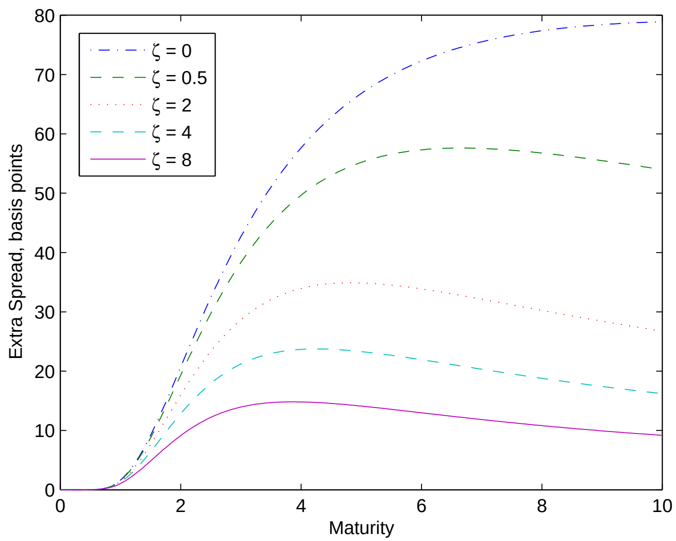
Probability of 2 defaults

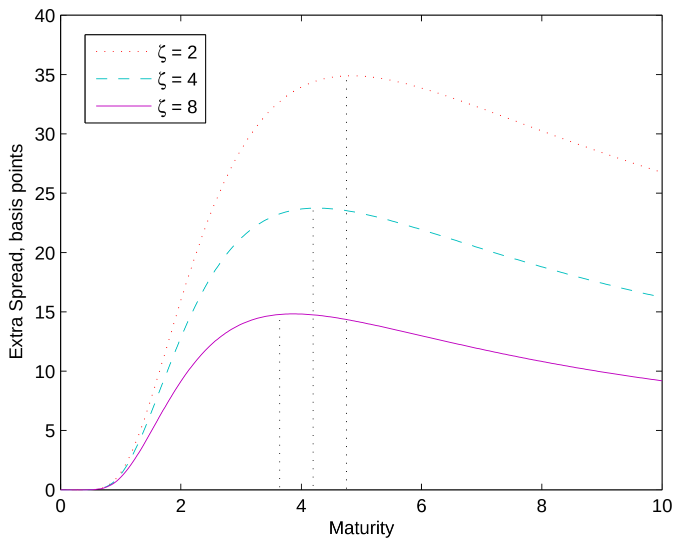


Ten-year default probabilities with decaying contagion

$\sigma_i = 0.2,\ r_f = 0.05,\ q_i = 0,\ \gamma_i = 0.03$, initial credit quality = 2

Contagion with Decay

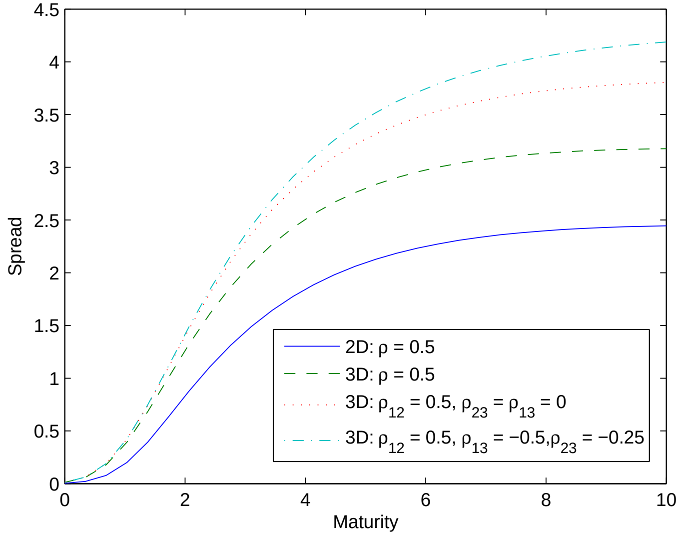




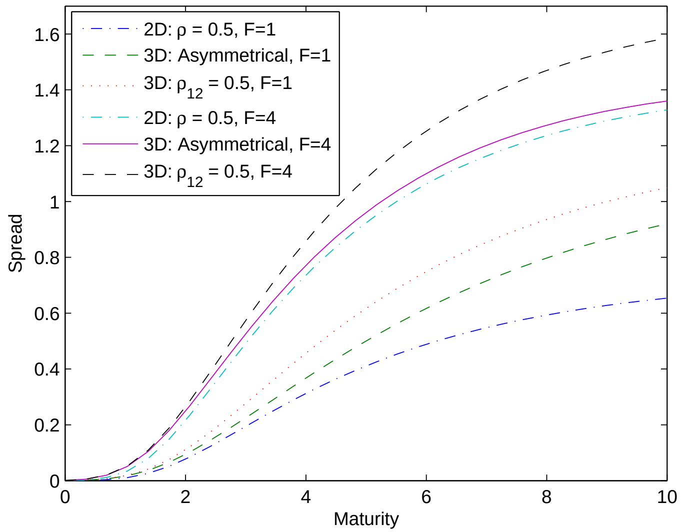
Additional spread on 2nd-to-default CDS due to contagion, $\rho = 0.75$
$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, *initial credit quality = 2, R = 0.5*

Three-firm Results

1^{st}-to-default CDS spread



2^{nd}-to-default CDS spread



$\sigma_i = 0.2,\ r_f = 0.05,\ q_i = 0,\ \gamma_i = 0.03$, initial credit quality = 2, R = 0.5

3D: Asymmetrical corresponds to $\rho_{12} = 0.5,\ \rho_{13} = -0.5,\ \rho_{23} = -0.25$

3D:$\rho_{12} = 0.5$ corresponds to $\rho_{12} = 0.5,\ \rho_{23} = \rho_{13} = 0.$

Conclusion

- Structural approach - powerful tool for investigating impact of dependence assumptions on default probabilities and spreads
- Framework is extremely flexible, enabling calculation of many default probabilities and CDS spreads of interest
- Dependence structure incorporates both asset correlation and default contagion
- Default causality and asymmetric links possible
- Results reiterate need for credit models to account for full dependence structure

Future Work

- Extend current framework to include, for example,
  - stochastic volatility
  - stochastic correlation
  - random recovery rates
- Introduction of jump discontinuities to remove predictable nature of default
- Use of different numerical schemes to extend framework to higher dimensions