Modelling basket credit default swaps with default contagion

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What is Credit?

• Credit risk is the risk that an obligor does not honour its obligations
• It is often thought of as default risk
• Important to banks, corporations and investors
• Reason the multi-billion dollar credit derivatives market exists
  • Simplest: credit default swap (CDS)
  • Insurance against risk of default by a company
iTraxx and CDX indices were launched in Europe and the US, respectively, in 2004
Each CDS index has 125 constituents which are updated every 6 months
Created massive liquidity
Standardized collateralized debt obligations (CDOs) and underlying tranches
More esoteric products – CDO-squareds, options on CDO tranches etc.
Default Characterisation

Firms do not operate in isolation and a company default can be triggered in three main ways:

1. By factors directly impacting multiple companies
   - Cyclical
   - Market-wide shock

2. By company-specific incidents or situations

3. Due to inter-company ties
   - Physical
   - Perceived
Default Characterisation

- Default dependence then occurs primarily through two mechanisms
  1. As a direct consequence of a common factor driving default
  2. Due to inter-company ties: contagion
- The result is a complex network of non-symmetrical links between companies
- The impact of individual credit events can ripple through the market
The original structural model dates back to the papers of Black & Scholes (1973) and Merton (1974).

Based on economic fundamentals through modelling the dynamics of firm assets, default occurs if firm value, $V$, drops below the value of debt, $K$.

Firm assets are modelled as a geometrical Brownian motion,

$$dV = \mu V \, dt + \sigma V \, dW$$

Debt and equity valued as contingent claims on the firm assets,

$$C(V, T) = \min(K, V(T)) = K - \max(0, K - V(T))$$
In the single-firm case, there are many extensions to the basic model:

- Black & Cox (1976) – allow default prior to maturity
- Longstaff & Schwarz (1995) – stochastic interest rates
- Leland & Toft (1996) – endogenous default trigger
- Zhou (1997) – jump diffusion model
Existing work on multiple-firm structural models is limited

- Zhou (2001) calculates default correlations for two companies, modelled as correlated Brownian motions
- Hull, Predescu & White (2005) assume assets are driven by a common factor and price CDO tranches using Monte Carlo
Model Overview

- First passage model – firm values are modelled as correlated geometric Brownian motions with exponential default thresholds
- Idiosyncratic ties are incorporated through a jump in volatility on default of a related entity
- The framework is extremely flexible, incorporating default causality and allowing for asymmetrical links
- Value $k^{th}$-to-default CDS baskets in the presence of asset correlation and default contagion
We consider $n$ companies, firm values $V_i$, with default as the first time that $V_i$ hits a lower default barrier $b_i(t)$

$$dV_i(t) = (r_f - q_i)V_i(t)\,dt + \sigma_i V_i(t)\,dW_i(t)$$

$\text{cov}(W_i(t), W_j(t)) = \rho_{ij}t$ for $i, j = 1, \ldots, n$

We assume exponential default barriers, reflecting the existence of debt covenants,

$$b_i(t) = K_i e^{-\gamma_i(T-t)}$$
Let $\Omega$ represent the default probability event of interest. For vector $V$ of firm values, and infinitesimal generator $L$, we solve

$$LU = \frac{\partial U}{\partial t} + \sum_{i=1}^{n}(r_f - q_i)V_i \frac{\partial U}{\partial V_i} + \frac{1}{2} \sum_{i,j=1}^{n} \rho_{ij} \sigma_i \sigma_j V_i V_j \frac{\partial^2 U}{\partial V_i \partial V_j} = 0$$

$$U(V, T) = 1_{\Omega}(V(T))$$

for $U(V, t)$ to calculate the probability of $\Omega$ by Feynman-Kac,

$$U(v, t) = \mathbb{E} \{1_{\Omega}(V(T)) | V(t) = v\} = \mathbb{P}(V(T) \in \Omega | V(t) = v)$$
Implementation

• We solve backwards in time on $[0, T] \times \mathbb{R}^n_+$ using a finite-difference method with Crank Nicolson time-stepping and a multigrid solver

• Deal with boundary conditions by setting a firm’s drift and volatility to zero on its barrier

• Count the number of companies whose values are at or below their default barriers and define the initial condition accordingly
Introducing Contagion

- Dependence structure is currently driven just by correlation in firm values
- Incorporate default contagion by allowing company volatilities to jump on default of related entity
- For example, if company $i$ defaults, for $i \neq j$, we let

$$\sigma_j \rightarrow \sigma_j F^{\rho_{ij}}$$

for some constant $F \geq 1$
Introducing Contagion

• In other words,

\[ \sigma_j(V, t) = \begin{cases} 
\sigma_j & \text{if } V_i(t) > b_i(t), V_j(t) > b_j(t) \\
\sigma_j F^{\rho_{ij}} & \text{if } V_i(t) \leq b_i(t), V_j(t) > b_j(t) \\
0 & \text{if } V_j(t) \leq b_j(t) 
\end{cases} \]

• Subsequent default by firm \( k \notin \{i, j\} \) would give

\[ \sigma_j \rightarrow \sigma_j F^{\rho_{ij}} \rightarrow \sigma_j F^{\rho_{ij}} F^{\rho_{kj}} \]
Consider a contract par value $K$ on a basket of $n$ companies, with bond recovery on default of $R$ and continuous spread payments, $c$

The discounted default payment, paid to the CDS buyer in the event of the $k^{th}$ company default, is

\[
(1 - R)K \int_{0}^{T} e^{-r_{f}s} \mathbb{P}(s \leq \tau_k \leq s + ds) \, ds
\]

\[
= (1 - R)K \int_{0}^{T} -e^{-r_{f}s} \frac{\partial}{\partial s} \mathbb{P}(\tau_k > s) \, ds
\]
$k^{th}$-to-Default CDS Baskets

- The discounted spread payment is

$$cK \int_0^T e^{-rf_s} \mathbb{P}(\tau_k > s) \, ds,$$

- Equating these gives the $k^{th}$-to-default CDS spread

$$c_k = \frac{(1 - R) \left\{ 1 - e^{-rfT} \mathbb{P} (\tau_k > T) - \int_0^T r_f e^{-rf_s} \mathbb{P} (\tau_k > s) \, ds \right\}}{\int_0^T e^{-rf_s} \mathbb{P} (\tau_k > s) \, ds}$$
Two-firm Results

Joint survival probability

First-to-default CDS spread

\[ \sigma_i = 0.2, \ r_f = 0.05, \ q_i = 0, \ \gamma_i = 0.03, \ \text{initial credit quality} = 2, \ R = 0.5 \]
Two-firm Results

Probability of 1 default

Probability of 2 defaults

Ten-year default probabilities

\[ \sigma_i = 0.2, \quad r_f = 0.05, \quad q_i = 0, \quad \gamma_i = 0.03, \quad \text{initial credit quality} = 2 \]
Second-to-Default CDS Spreads

5-year 2nd-to-default CDS

10-year 2nd-to-default CDS

\[ \sigma_i = 0.2, \ r_f = 0.05, \ q_i = 0, \ \gamma_i = 0.03, \ \text{initial credit quality} = 2, \ R = 0.5 \]
Contagion with Decay

• More realistic for the spike in volatility on default to decay over time, \( \sigma_j \rightarrow \sigma_j (1 + \Delta_{ij} e^{-\zeta(t-\tau_i)}) \)

• Setting \( \Delta_{ij} = F^{\rho_{ij}} - 1 \) enables direct comparison of results with and without decay.

• After default by firm \( i \), time-dependence of \( \sigma_j(t) \) for \( j \neq i \) is removed by replacing \( \sigma_j^2(t) \) with its average over the remaining time-to-maturity, \( \bar{\sigma}_j^2 \)

\[
\bar{\sigma}_j^2 = \frac{1}{T - \tau_i} \int_0^{T-\tau_i} \sigma_j^2(s) \, ds
\]
Contagion with Decay

- This gives the new volatility

\[
\sigma_j^2 + \frac{2\Delta_{ij}\sigma_j^2}{\zeta(T - \tau_i)} (1 - e^{-\zeta(T - \tau_i)}) + \frac{\sigma_j^2 \Delta_{ij}^2}{2\zeta(T - \tau_i)} (1 - e^{-2\zeta(T - \tau_i)})
\]

- The problem decouples on the boundary, allowing us to solve the two-firm problem on \([0, \tau_i]\) and a one-company problem on \([\tau_i, T]\).
Contagion with Decay

**Probability of 1 default**

- $\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2

**Probability of 2 defaults**

**Ten-year default probabilities with decaying contagion**
Contagion with Decay

Additional spread on 2nd-to-default CDS due to contagion, $\rho = 0.75$

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2, $R = 0.5$
Three-firm Results

$1^{st}$-to-default CDS spread

$2^{nd}$-to-default CDS spread

$\sigma_i = 0.2$, $r_f = 0.05$, $q_i = 0$, $\gamma_i = 0.03$, initial credit quality = 2, $R = 0.5$

3D: Asymmetrical corresponds to $\rho_{12} = 0.5$, $\rho_{13} = -0.5$, $\rho_{23} = -0.25$

3D: $\rho_{12} = 0.5$ corresponds to $\rho_{12} = 0.5$, $\rho_{23} = \rho_{13} = 0$. 
Conclusion

- Structural approach - powerful tool for investigating impact of dependence assumptions on default probabilities and spreads
- Framework is extremely flexible, enabling calculation of many default probabilities and CDS spreads of interest
- Dependence structure incorporates both asset correlation and default contagion
- Default causality and asymmetric links possible
- Results reiterate need for credit models to account for full dependence structure
Future Work

• Extend current framework to include, for example,
  • stochastic volatility
  • stochastic correlation
  • random recovery rates

• Introduction of jump discontinuities to remove predictable nature of default

• Use of different numerical schemes to extend framework to higher dimensions