Minimum Entropy Calibration of a Point Process Model for CDO Pricing

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Overview

- Model loss process as self-exciting point process (SEPP), in order to capture default correlation (top-down approach).

- Fit SEPP parameters to historical default data to obtain a Bayesian prior.

- Find an arbitrage-free point process close to the fitted one that matches market prices of liquid vanilla contracts and consistently prices certain exotic contracts.

- Numerical work in progress with real data.
Background on CDOs

Collateralized debt obligations (CDOs) repackage the credit risk of a pool of debt-like assets into multiple tranches, which are ranked in terms of seniority.

Creation of CDX, iTraxx indices in late 2003 has led to a liquid, standardized CDO market.

Abundance of pricing models in the literature, with recent push towards pricing more exotic contracts.

- common theme—need to capture default correlation
The CDX Index and Tranches

Dow Jones CDX.NA.IG
- liquid contracts with maturities 5, 7, and 10 years
- standardized tranches 0-3%, 3-7%, 7-10%, 10-15%, 15-30%

Cumulative loss $L(N_t) = \frac{1-\delta}{125} N_t$, where $N_t = \sum_{i=1}^{125} 1_{\{\tau_i \leq t\}}$.

Tranche loss for $K_i-K_{i+1}$% tranche is call spread

$$f_i(N_t) = \max(L(N_t) - K_i, 0) - \max(L(N_t) - K_{i+1}, 0).$$

Spread for tranche $i$ (quoted in bp) can be approximated by

$$s_i = \frac{\sum_{0 \leq t_k \leq T} p(0, t_k) \left\{ \mathbb{E}^Q f_i(N_{t_k}) - \mathbb{E}^Q f_i(N_{t_k-1}) \right\}}{\sum_{0 \leq t_k \leq T} p(0, t_k)(t_k - t_{k-1}) \mathbb{E}^Q \left\{ K_{i+1} - K_i - f_i(N_{t_k-1}) \right\}}.$$
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Self-Exciting Point Processes

A point process \( N \) is called self-exciting when its intensity process \( R_t \) is affected by the past of \( N \) over \((0, t)\).

Hawkes (1971) process \( N \) with intensity

\[
R_t = \mu e^{-ct} + \int_{(0,t)} ae^{-c(t-s)} dN_s, \quad t > 0.
\]

Has been used to model
- high-frequency data [Bowsher (2003)]
- trade arrivals [Salmon and McCulloch (2005)]
- credit derivatives [Das et. al. (2005); Giesecke and Goldberg (2005)], as alternative to doubly stochastic processes
Fitting an SEPP to Historical Default Data

We fit parameters of the Hawkes intensity by numerically maximizing the **log likelihood**

\[
\ell(\tau_1, \ldots, \tau_n) = \int_0^T \log R_t \, dN_t - \int_0^T R_t \, dt.
\]

Data from Moody’s US corporate, senior, unsecured rating and default database—spans 1970-2002 with roughly 300 default observations.

Fit is quite good, assessed either with AIC or by time-changing to homogeneous Poisson process, then running goodness-of-fit tests.
Model calibration problem: obtain a pricing rule consistent with market quotes.

In general, pricing constraints insufficient to yield unique pricing measure $Q$. We regularize via relative entropy in order to stay close to SEPP prior $P$.

Optimization problem over measures on path space:

$$\inf_{Q \ll P} \mathbb{E}^Q \left[ \log \frac{dQ}{dP} \right]$$

subject to

$$\mathbb{E}^Q f_i(N_{t_k}) = \pi_{ik} \quad \forall i, 0 \leq t_k \leq T.$$ 

Develop framework for pricing exotic and bespoke credit derivatives.

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Minimum Entropy Calibration
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What are the Pricing Constraints?

Recall the spread formula

\[
s_i = \frac{\sum_{0 \leq t_k \leq T} p(0, t_k) \left\{ \mathbb{E}^Q f_i(N_{t_k}) - \mathbb{E}^Q f_i(N_{t_{k-1}}) \right\}}{\sum_{0 \leq t_k \leq T} p(0, t_k)(t_k - t_{k-1})\mathbb{E}^Q \left\{ K_{i+1} - K_i - f_i(N_{t_{k-1}}) \right\}}.
\]

Tranche spreads are nonlinear in \( Q \), so instead calibrate to expected tranche losses \( \pi_{ik} \). These quantities, however, are not market observables!

One approach: recover \( \pi_{ik} \) from observed tranche spreads using interpolation and constrained least-squares [Brigo, Pallavicini, Torresetti (2007)].
To reframe optimization problem as intensity control problem:

- scale $\mathbb{P}^0$-intensity $R^0_t$ with Markovian control $u_t = u(t, N_t, R^0_t)$; call scaled process $R^u_t = u_t R^0_t$
- to each control $u$ we associate a measure $\mathbb{P}^u$ under which $N_t$ has $(\mathbb{P}^u, \mathcal{F}_t)$-intensity $R^u_t$
- rewrite objective in Lagrangian form and define value function

$$V^\lambda(t, n, r^0) := \sup_{u_t \in \mathcal{U}} \mathbb{E}^u \left[ \int_t^T \left( - R^0_t u_t \log u_t + R^0_t (u_t - 1) + \sum_k \sum_i \lambda_{ik} \{ f_i(N_t) - \pi_{ik} \} \delta(t - t_k) \right) dt \mid \mathcal{F}_t \right].$$

Goal is to solve min-max problem $\inf_{\lambda} V^\lambda(0, N_0, R^0_0)$.

Similar to Avellaneda et. al. (1997) for diffusion models; Carmona and Xu (1997), Nayak and Papanicolaou (2006) for SV models
Stochastic Intensity Control Formulation

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HJB Equation

In using dynamic programming to derive the HJB equation, we establish the following Markovian structure:

1. The pair \((N, R^0)\) is Markov under \(P^0\).
2. The pair \((N, R^0)\) remains Markov under \(P^u\).

The HJB equation is

\[
- \sum_k \sum_i \lambda_{ik} \{ f_i(n) - \pi_{ik} \} \delta(t - t_k) = \frac{\partial V^\lambda}{\partial t}(t, n, r^0) - c r^0 \frac{\partial V^\lambda}{\partial R^0}(t, n, r^0) + \\
\sup_u \{- r^0 u \log u + r^0 (u - 1) + r^0 u [V^\lambda(t, n + 1, r^0 + a) - V^\lambda(t, n, r^0)]\},
\]

with terminal condition \(V^\lambda(T + 0, n, r^0) = 0\).

We prove a Verification theorem to show HJB equation acts as a sufficient condition for the intensity control problem.
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Numerical Algorithm

\( V^\lambda(t, n, r^0) \) is **convex** in the Lagrange multipliers \( \{\lambda_j\} \).

Check \( \frac{\partial V^\lambda}{\partial \lambda_j}(t, n, r^0) \) satisfies PDDE similar to that satisfied by \( V \).

**Computational scheme:**

1. For fixed \( \{\lambda_j\} \), use a finite-difference scheme to solve PDDEs for \( V(0, N_0, R_0^0), \frac{\partial V^\lambda}{\partial \lambda_1}(0, N_0, R_0^0), \ldots, \frac{\partial V^\lambda}{\partial \lambda_j}(0, N_0, R_0^0) \).
2. Update Lagrange multipliers using gradient-descent.

Once \( \{\lambda_j^*\} \) obtained, compute optimal control \( u^* \), then price by simulating from \( R^u^* \).
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Once $\{\lambda^*_j\}$ obtained, compute optimal control $u^*$, then price by simulating from $R^{u^*}$. 
Conclusion and Future Work

- Develop a framework for calibrating a point process model to a prescribed set of CDO prices.

- Calibration problem formulated as constrained optimal control problem.

- Results in an arbitrage-free point process that minimizes the relative entropy distance to a Hawkes process prior.

- Establish existence of solution to optimization problem.

- Examine resulting loss distributions, potential to price path-dependent credit derivatives, improvements upon base correlation framework used to price bespoke tranches.
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